We need a statistical approach when evaluating the strength of ceramic materials. The strength of ceramics and glasses depends upon the size and distribution of sizes of flaws. In these materials, flaws originate from the processing process as the ceramics are being manufactured. These can result during machining, grinding, etc. Glasses can also develop microcracks as a result of interaction with water vapor in air. If we test alumina or other ceramic components of different sizes and geometry, we often find a large scatter in the measured values—even if their nominal composition is same. Similarly, if we are testing the strength of glass fibers of a given composition, we find that, on average, shorter fibers are stronger than longer fibers. The strength of ceramics and glasses depends upon the probability of finding a flaw that exceeds a certain critical size. For larger components or larger fibers this probability increases. As a result, the strength of larger components or fibers is likely to be lower than that of smaller components or shorter fibers. In metallic or polymeric materials, which can exhibit relatively large plastic deformations, the effect of flaws and flaw size distribution is not felt to the extent it is in ceramics and glasses. In these materials, cracks initiating from flaws get blunted by plastic deformation. Thus, for ductile materials, the distribution of strength is narrow and close to a Gaussian distribution. The strength of ceramics and glasses, however, varies considerably (i.e., if we test a large number of identical samples of silica glass or alumina ceramic, the data will show a wide scatter owing to changes in distribution of flaw sizes). The strength of brittle materials, such as ceramics and glasses, is not Gaussian; it is given by the Weibull distribution.\cite{weibull} The Weibull distribution is an indicator of the variability of strength of materials resulting from a distribution of flaw sizes. This behavior results from critical sized flaws in materials with a distribution of flaw sizes (i.e., failure due to the weakest link of a chain).

The Weibull distribution shown in Figure 6-44 describes the fraction of samples that fail at different applied stresses. At low stresses, a small fraction of samples contain flaws large enough to cause fracture; most fail at an intermediate applied stress, and a few contain only small flaws and do not fail until large stresses are applied. To provide predictability, we prefer a very narrow distribution.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{weibull_distribution.png}
\caption{The Weibull distribution describes the fraction of the samples that fail at any given applied stress.}
\end{figure}
EXAMPLE 6-14

Figure 6-45  A cumulative plot (using special graph paper) of the probability that a sample will fail at any given stress yields the Weibull modulus or slope. Alumina produced by two different methods is compared with low carbon steel. Good reliability in design is obtained for a high Weibull modulus. (Source: From Mechanical Behavior of Materials, by M.A. Meyers and K.K. Chawla, 1999. Copyright © 1999 Prentice-Hall. Used with permission of Pearson Education, Inc., Upper Saddle River, N.J.)

SOLUTION

The failure probability and strength when plotted on a log-log scale result in data that can be fitted to a straight line. The slope of these lines provides us the measure of variability (i.e., the Weibull modulus).

For plain carbon steel the line is almost vertical (i.e., slope or \( m \) value is essentially approaching large values). This means that there is very little variation (5 to 10%) in the strength of different samples of the 0.2% C steel.

For alumina ceramics prepared using traditional processing, the variability is high (i.e., \( m \) is low \( \sim 4.7 \)).

For ceramics prepared using improved and controlled processing techniques the \( m \) is higher \( \sim 9.7 \) indicating a more uniform distribution of flaws. The average strength is also higher (\( \sim 578 \) MPa) suggesting lesser number of flaws that will lead to fracture.
Consider a body of volume $V$ with a distribution of flaws and subjected to a stress $\sigma$. If we assumed that the volume, $V$, was made up of $n$ elements with volume $V_0$ and each element had the same flaw-size distribution, it can be shown that the survival probability, $P(V_0)$, (i.e., the probability that a brittle material will not fracture under the applied stress $\sigma$) is given by:

$$P(V_0) = \exp \left[ -\left( \frac{\sigma - \sigma_u}{\sigma_0} \right)^m \right] \quad (6-24)$$

The probability of failure, $F(V_0)$, can be written as:

$$F(V_0) = 1 - P(V_0) = 1 - \exp \left[ -\left( \frac{\sigma - \sigma_u}{\sigma_0} \right)^m \right] \quad (6-25)$$

In Equations 6-24 and 6-25, $\sigma$ is the applied stress, $\sigma_0$ is characteristic strength (often assumed to be equal to the average strength), $\sigma_u$ is the stress level below which the probability of failure is zero (i.e., the probability of survival is 1.0). In these equations, $m$ is the Weibull modulus. In theory, Weibull modulus values can range from 0 to $\infty$. The Weibull modulus is a measure of the variability of the strength of the material.

The Weibull modulus $m$ indicates the strength variability. For metals and alloys, the Weibull modulus is $\sim 100$. For traditional ceramics (e.g., bricks, pottery, etc.), the Weibull modulus is less than 3. Engineered ceramics, in which the processing is better controlled and hence the number of flaws is expected to be less, have a Weibull modulus of $5$ to $10$.

Note that for ceramics and other brittle solids, we can assume $\sigma_u = 0$. This is because there is no nonzero stress level for which we can claim a brittle material will not fail. For brittle materials, Equations 6-24 and 6-25 can be rewritten as follows:

$$P(V_0) = \exp \left[ -\left( \frac{\sigma}{\sigma_0} \right)^m \right] \quad (6-26)$$

and

$$F(V_0) = 1 - P(V_0) = 1 - \exp \left[ -\left( \frac{\sigma}{\sigma_0} \right)^m \right] \quad (6-27)$$

From Equation 6-26, for an applied stress $\sigma$ of zero, the probability of survival is 1. As the applied stress $\sigma$ increases, $P(V_0)$ decreases, approaching zero at very high values of applied stresses. We can also describe another meaning of the parameter $\sigma_0$. In Equation 6-26, when $\sigma = \sigma_0$, the probability of survival becomes $1/e \approx 0.37$. Therefore, $\sigma_0$ is the stress level for which the survival probability is $\approx 0.37$ or 37%. We can also state that $\sigma_0$ is the stress level for which the failure probability is $\approx 0.63$ or 63%.

Equations 6-26 and 6-27 can be modified to account for samples with different volumes. It can be shown that for an equal probability of survival samples with larger volumes will have lower strengths. This is what we mentioned before (e.g., longer glass fibers will be weaker than shorter glass fibers).

The following examples illustrate how the Weibull plots can be used for analysis of mechanical properties of materials and designing of components.
EXAMPLE 6-15

Stress and Ductility and Probability of Failure

A ceramic having Weibull modulus $m = 9$. The failure strength is observed at a probability of failure $F = 0.4$. What is the required strength (or failure strength) if the probability of failure has to be 0.1?

SOLUTION

We assume all samples tested had the same volume, thus the size of the sample volume is a factor in this case. We can use the symbol $V$ for sample volume instead of $V_0$. We are dealing with a brittle material, so we begin with Equation 6-27:

$$ F(V) = 1 - F(V) = 1 - \exp \left[-\left(\frac{V}{V_0}\right)^m\right] $$

or

$$ 1 - F(V) = \exp \left[-\left(\frac{V}{V_0}\right)^m\right] $$

Take the logarithm of both sides to get

$$ \ln(1 - F(V)) = \left(\frac{V}{V_0}\right)^m $$

Take logarithm of both sides again,

$$ \ln(\ln(1 - F(V))) = -m(\ln \sigma - \ln \sigma_0) \quad (6-28) $$

We can eliminate the minus sign on the right-hand side of Equation 6-28 by rewriting it as:

$$ \ln \left[\ln \left(\frac{1}{1 - F(V)}\right)\right] = m(\ln \sigma - \ln \sigma_0) \quad (6-29) $$

For $F = 0.4$, $\sigma = 250$ MPa, and $m = 9$, so from Equation 6-29, we have

$$ \ln \left[\ln \left(\frac{1}{1 - 0.4}\right)\right] = 9(\ln 250 - \ln \sigma_0) \quad (6-30) $$

Therefore, $\ln \{\ln 1.0/0.6\} = \ln \left(\ln 1.66667\right) = \ln (0.510826) = -0.67173 = 9(5.52146 - \ln \sigma_0)$.

Therefore, $\ln \sigma_0 = 5.52146 + 0.07464 = 5.5961$. This gives us a value of $\sigma_0 = 269.4$ MPa. This is the characteristic strength of the ceramic, often taken as the average strength of the ceramic. For a stress level of 269.4 MPa, the probability of survival is 0.37 (or the probability of failure is 0.63). As the required probability of failure ($F$) goes down, the stress level to which the ceramic can be subjected (or) also goes down.

Now, we want to determine the value of $\sigma$ for $F = 0.1$. We know that $m = 9$ and $\sigma_0 = 269.4$ MPa, so we need to get the value of $\sigma$. We substitute these values into Equation 6-29:

$$ \ln \left[\ln \left(\frac{1}{1 - 0.1}\right)\right] = 9(\ln \sigma - \ln 269.4) $$

$$ \ln \left[\ln \left(\frac{1}{0.9}\right)\right] = 9(\ln \sigma - \ln 269.4) $$
\[
\begin{align*}
\ln(\ln 1.1111) &= \ln(0.105361) = -2.25037 = 9(\ln \sigma - 5.596097), \\
\therefore \quad -0.25037 &= \ln \sigma - 5.596097, \quad \text{or} \\
\ln \sigma &= 5.346056
\end{align*}
\]

or \(\sigma = 209.8\) MPa. As expected, as we lowered the probability of failure to 0.1, we also decreased the level of stress that can be supported.

**EXAMPLE 6-16**

![Figure 6-46](image)

Plot of cumulative probability of failure versus fracture stress. Note the fracture strength is plotted on a log scale.
The total number of specimens is \( n \) (in our case, 7). The probability of failure \( F \) is then the numerical rank divided by \( n + 1 \) (in our case, 8). We can then plot \( \ln[\ln(1 - F(V_0))] \) versus \( \sigma \). The following table and Figure 6-46 show the results of these calculations. Note that \( \sigma \) is plotted on a log scale.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>( \sigma ) (MPa)</th>
<th>( F(V_0) )</th>
<th>( \ln[\ln(1 - F(V_0))] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23</td>
<td>1/8 = 0.125</td>
<td>-2.013</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>2/8 = 0.250</td>
<td>-1.246</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>3/8 = 0.375</td>
<td>-0.755</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>4/8 = 0.500</td>
<td>-0.367</td>
</tr>
<tr>
<td>5</td>
<td>43</td>
<td>5/8 = 0.625</td>
<td>-0.019</td>
</tr>
<tr>
<td>6</td>
<td>49</td>
<td>6/8 = 0.750</td>
<td>+0.327</td>
</tr>
<tr>
<td>7</td>
<td>55</td>
<td>7/8 = 0.875</td>
<td>+0.732</td>
</tr>
</tbody>
</table>

The slope of the fitted line, or the Weibull modulus \( m \), is (using the two points indicated on the curve):

\[
m = \frac{0.5 - (-2.0)}{\ln(52) - \ln(23.5)} = \frac{2.5}{3.951 - 3.157} = 3.15
\]

This low Weibull modulus of 3.15 suggests that the ceramic has a highly variable fracture strength, making it difficult to use reliably in high load-bearing applications.

---

**Fatigue**

Fatigue is the lowering of strength or failure of a material due to repetitive stress which may be above or below the yield strength.[13,14] It is a common phenomenon in load-bearing components in cars and airplanes, turbine blades, springs, crankshafts and other machinery, biomedical implants, and consumer products, such as shoes, that are subjected constantly to repetitive stresses in the form of tension, compression, bending, vibration, thermal expansion and contraction, or other stresses. These stresses are often below the yield strength of the material! However, when the stress occurs a sufficient number of times, it causes failure by fatigue! Quite a large fraction of components found in an automobile junkyard belongs to those that failed by fatigue. The possibility of a fatigue failure is the main reason why aircraft components have a finite life. Fatigue is an interesting phenomenon in that load-bearing components can fail while the overall stress applied may not exceed the yield stress! Fatigue can occur even if the components are subjected to stress above the yield strength. A component is often subjected to the repeated application of a stress below the yield strength of the material.

There are examples of situations in which certain materials, such as thin films of ferroelectric materials, also show fatigue. This fatigue is electrical in nature and it is linked to the eventual inability of materials to show changes in electrical properties in response to the applied electric field. A detailed discussion of this is outside the scope of this book. The point is anytime we have a component that is going to be subjected to mechanical, electrical, thermal, and magnetic or other forces that are likely to be cyclical, we need to look at the effect of these external factors over a long period of time.