Infinite Array with Dielectric Cover

Recall the parallel plate waveguide array.

Since each mode in the exterior region is a plane wave and cover is an infinite dielectric slab of uniform thickness, the transmission line equivalent circuit is still valid with the following modification.

The exterior region is now a cascade connection of T-lines of different $Y_0$ & $b_2$ for each mode.
The new equivalent circuit is

\[ Y^e_n = \frac{w e_r e_0}{\varepsilon_{2n}} = \frac{w e_r e_0}{\sqrt{e_r e_0^2 - \varepsilon_{2n}^2}} \]

\[ Y^m_n = Y^e_n \frac{j Y^e + Y_{un} \cot(\varepsilon_{2n} \theta) \cot(\varepsilon_{2n} \theta + \phi) + j Y_{un}}{Y^e_n \cot(\varepsilon_{2n} \theta) + j Y_{un}} \]

\[ Y_a = j B_{int} + \sum_{n=-\infty}^{\infty} (T_n)^2 Y^m_n \]
Discussion of Results with Dielectric Case

1. $Y_a \to \infty$ whenever $Y_n^{th} \to \infty$ for any $n$, i.e.

$$Y_n^e \cot (k_{2n}^e t) + jY_n = 0$$

This can be satisfied only if $Y_n^e$ & $k_{2n}^e$ are real and $Y_n$ is imaginary. That is, the $n$-th mode propagates in the dielectric but not in air. This never happens for the $n=0$ mode, but for the $n=-1$ mode the condition is

$$\varepsilon_r \left| k_{2,1}^e \right| t \cot (k_{2,1}^e t) - k_{2,1}^e t = 0$$

where

$$\left| k_{2,1}^e \right| = \sqrt{k_{2,1}^2 - k_0^2}$$

This condition is eqn (4-56) in Harrington, i.e.,

$$\beta_x = k_{2,1}$$

"Forced Resonance" of surface wave

\[ Y_n^{th} = \infty \Rightarrow \text{Short Circuit} \]
Current Sheet Analysis of Planar Array

This is one way to analyze an infinite array of flat elements.

This analysis provides some very useful insights into array performance, and it can be used for some practical arrays.

To simplify our analysis, assume only $x$-directed electric currents at $z = 0$. This produces only $TM_x$ fields.
The array we will analyze is strip dipoles.

\[ \mathbf{J} = \sum J_x(x,y) \text{ at } z = 0 \]

specified on strips and zero elsewhere.

\[ J_x(x,y) \text{ is specified in the } (0,0) \text{ unit cell and is identical in other cells except for phase shift} \]

\[ J_x(x+mdx, y+ndy) = J_x(x,y) e^{-j \frac{2\pi}{mdx} mdx} e^{-j \frac{2\pi}{ndy} ndy} \]
Consider a Fourier Series representation for $J_x(x, y)$ in the $(0,0)$ unit cell.

$$J_x(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_{mn} e^{-j\pi mx} e^{-j\pi ny}$$

(1)

$J_{mn}$ will be determined below so that the B.C. at $z=0$ are satisfied.

Recognizing the periodicity of $J$ in the array, and accounting for the phase shift for beam steering,

$$J_x(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_{mn} e^{-j(\omega_0 + \frac{m\pi}{a_x})x} e^{-j(\omega_0 + \frac{n\pi}{a_y})y}$$

(2)

Eqn (2) is valid everywhere on the plane at $z=0$. This surface current will radiate fields,

$$E_x(x, y, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E_{mn} e^{-j2\pi \frac{m}{a_x}x} e^{-j2\pi \frac{n}{a_y}y} e^{-j\frac{\pi}{2}z mn^2}$$

(3)
where

\[ \begin{align*}
\beta_{xm} &= \beta x_0 + m \frac{2\pi}{d_x} \\
\beta_{yn} &= \beta x_0 + n \frac{2\pi}{d_y} \\
\beta_{2mn} &= \beta x_0^2 - \beta_{xm} - \beta_{yn}
\end{align*} \]

\[ \text{Im} \beta_{2mn} \geq 0 \]

\[ H_y = \frac{i}{\eta} \frac{\beta x_0}{\beta x_0^2 - \beta x_0} \frac{d E_x}{d z} \]

\[ = \frac{1}{\eta} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \beta_{2mn} \frac{d E_{mn}}{d x_0^2 - \beta x_0^2} E_{mn} e^{i \beta_{xm} x} e^{-i \beta_{yn} y} e^{-i \beta_{2mn} z_1} \]

Apply B.C at \( z=0 \) to determine \( E_{mn} \). Note that the form of (3) guarantees \( E_{tan} \) is continuous.

\[ \hat{N} \times (\overline{H}_{tan}^+ - \overline{H}_{tan}^-) = \overline{J}_s \text{ at } z=0 \text{ yields} \]

\[ H_y(x, y, z=0^+) = -\frac{1}{2} \overline{J}_s(x, y) \]

Substitute (2) and (4) into (5) and use orthogonality of Floquet modes to get
\[ E_{mn} = -\frac{n}{2} J_{mn} \frac{\frac{b^2}{2} x_{mn} - \frac{b^2}{2}}{\frac{h^2}{2} \frac{h^2}{2} z_{mn}} \]  

(6)

Therefore,

\[ E_x(x, y) = -\frac{n}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_{mn} \frac{b^2 x_{mn}^2 - b^2}{h^2 z_{mn}^2} e^{j \frac{b x_{mn}}{h}} e^{j \frac{b y n}{h}} e^{j \frac{b z_{mn}}{h}} 1 \]  

(7)

The input impedance of an element in the array can be found from the (complex) power "radiated" by the antenna.

\[ Z_{in} = -2 \int_{cell}^{z=0+} \frac{\iint E \cdot \overline{J}^* \, ds}{|I|^2} \]  

(8)

where \( I \) is the terminal current and the factor of \( 2 \) accounts for power at \( z=0^+ \). Using (2) and (7),

\[ Z_{in} = \frac{n}{|I|^2} \int dxdy \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |J_{mn}|^2 \frac{b^2 - b^2 x_{mn}}{h^2 h^2 z_{mn}^2} \]  

(9)
Consider some special cases:

1. \( J_x(x, y) = \begin{cases} \frac{I_0}{b} \cos \frac{\pi x}{a} & \text{on (0,0) strip} \\ 0 & \text{elsewhere} \end{cases} \)

Equating this (1) and using orthogonality,

\[
\sum_m \sum_n J_{mn} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} e^{-j(\frac{\pi}{a}x + \frac{\pi}{b}y)} e^{-j(\frac{\pi}{a}x \pm \frac{\pi}{b}y)} \, dx \, dy
\]

\[
= \frac{I_0}{b} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \cos \frac{\pi x}{a} e^{j\frac{\pi x}{a}} e^{-j\frac{\pi y}{b}} \, dx \, dy
\]

\[
J_{mn} = \frac{2}{\pi} \frac{I_0}{b} \frac{\sin \frac{\pi y}{b}}{\frac{\pi y}{b}} \frac{\cos \frac{\pi x}{a}}{1 - \left( \frac{\frac{\pi x}{a}}{\frac{\pi y}{b}} \right)^2}
\]

2. \( J_x(x, y) = e^{-j\frac{\pi}{a}x} e^{-j\frac{\pi}{b}y} \)

This is the current corresponding to (0,0) Floquet mode.

\[
J_{mn} = \begin{cases} 1 & \text{if } m=n=0 \\ 0 & \text{otherwise} \end{cases}
\]
Then
\[ Z_{in} = \frac{\eta d\alpha d\beta (1-u_0^2)}{|I|^2 (1-u_0^2-V_0^2)^{1/2}} \]

Let
\[ R_0 = Z_{in}(u_0=0, v_0=0) = \frac{\eta d\alpha d\beta}{|I|^2} \]

In E-plane, \( v_0 = 0 \).
\[ Z_{in}(\theta, \phi=0^\circ)/R_0 = (1-u_0^2)^{1/2} = \cos \theta \]

In H-plane, \( u_0 = 0 \).
\[ Z_{in}(\theta, \phi=90^\circ)/R_0 = (1-v_0^2)^{-1/2} = \frac{1}{\cos \theta} \]

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To visualize the behavior of the fields, a graph is shown with axes labeled as follows:
- The horizontal axis is labeled \( \theta \) from 0 to 90 degrees.
- The vertical axis is labeled \( \Gamma \) ranging from -1 to 1.
- The diagram indicates the transition from E-plane to H-plane.

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Notes:
1. For grid spacing < \( \lambda/2 \), only (0,0) Floquet
   mode propagates. That is, only (0,0) mode
   contributes to \( R_{in} \) (real radiated power).
2. More generally, only propagating
   modes contribute to \( R_{in} \). Evanescent modes
   contribute to \( X_{in} \) only.
3. Scan impedance in D-plane often
   resembles an average of E-plane and H-plane.
   (See Fig. 30)
4. H. A. Wheeler (T-AP, July 1965) used
   current sheet models to gain insight into
   phased array performance.
   Any (equivalent) current distribution
   at the array aperture can be decomposed
   in a summation of Floquet current sheets.
   Keeping only (0,0) mode captures main
   beam effects. If grating lobes
   propagate, must include those Floquet
   modes to observe radiation effects.
   If there is a dielectric cover or
   substrate, that supports surface wave,
   you must include Floquet mode (e.g., \( m=-1 \))
   to observe blindness. (See Fig. 6 & 7)
5. If you have a planar array (e.g., dipoles) over a ground plane at $z = -h$, use image theory.

$$E_x(x, y, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j\frac{\pi}{d} x_m x - j\frac{\pi}{d} y_n y}$$

$$\begin{cases} E_{+m} e^{-j\frac{\pi}{d} z_{mn}} , & z \geq 0 \\ E_{-m} [e^{j\frac{\pi}{d} z_{mn}} - e^{-j\frac{\pi}{d} (z+2h)}] , & -h \leq z \leq 0 \end{cases}$$

The B.C. that must be enforced are

$$E_x(x, y, z = 0^+) = E_x(x, y, z = 0^-)$$

$$H_y(x, y, z = 0^+) - H_y(x, y, z = 0^-) = -J_x(x, y)$$
Fig. 30. Variation of active resistance, normalized to its value at broadside, as a function of scan angle for three different scan planes, for an array of infinitesimal dipoles and an array of $\lambda/2$ dipoles, both in free space (from Diamond, 1965a).

Fig. 6. Input reflection coefficient for an infinite array on a grounded dielectric substrate with $\varepsilon_r = 12.8$ and $d = 0.06 \lambda_0$. The simple theory uses the $m = n = 0$ and $m = -1, n = 0$ Floquet modes in the $E$-plane and the $m = n = 0$ mode in the $H$-plane. Comparison is made with the results of [2], for $a = b = 0.5 \lambda_0$, dipole length $= 0.156 \lambda_0$, and dipole width $= 0.002 \lambda_0$.

Fig. 7. Input reflection coefficient for an infinite array on a grounded dielectric substrate with $\varepsilon_r = 2.55$ and $d = 0.1\,\lambda_0$. The simple theory uses the $m = n = 0$ and $m = -1, n = 0$ Floquet modes in the E-plane and the $m = n = 0$ mode in the H-plane. Comparison is made with the results of [3], for $a = b = 0.5\,\lambda_0$, patch length $= 0.29\,\lambda_0$, and patch width $= 0.3\,\lambda_0$.