Example: Flip coin 100 times: significance test

R.V. $X$ of subexperiment: $X = \begin{cases} 0 & \text{if tail up} \\ 1 & \text{if head up} \end{cases}$

$E[X] = \frac{1}{2}$, $\text{Var}[X] = \frac{1}{4}$, $\sigma_X = \frac{1}{2}$

Estimation error: $Y = M_u(x) - E[X] = M_u(x) - \mu_x$

Central limit theorem approximation:

$M_u$ is a Gaussian $(\mu_x, \frac{\sigma_x^2}{n})$ R.V.

Then, according to T 7.13, a confidence interval estimate of $\mu_x$ of the form

$M_u(x) - c \leq \mu_x \leq M_u(x) + c$

has the confidence coefficient $1 - \alpha$ where

$\frac{\alpha}{2} = Q\left(\frac{c \mu_u}{\sigma_x}\right) = 1 - \Phi\left(\frac{c \mu_u}{\sigma_x}\right)$.

Let's say our experiment provided $M_u(100) = 0.45$ for $n = 100$, $\sigma_x = \frac{1}{2}$.

$\Rightarrow \alpha = 2 \cdot Q\left(\frac{10}{\frac{1}{2}}\right) = 2 \cdot Q\left(2\right) = 2 \left(1 - \phi(2)\right)$

$\Rightarrow \alpha = 2 \left(1 - 0.84\right) = 2 \cdot 0.16 = 32\%$

$\Rightarrow$ The probability of making a false rejection of $H_0$ (i.e., that the coin is unfair) on the basis of the observation $|M_{100} - \mu_x| = 0.05$ is 32%.
Now, we design a significance test:

**Question:** How large must n be to achieve a significance level of \( \alpha = 0.01, 0.05, 0.01 \) for \( c = 0.05 \)?

In other words, how many times do we have to flip a coin in order to reduce the false-rejection probability for \( c = 0.05 \) to \( \alpha = 0.01 \) (10%), \( \alpha = 0.05 \) (5%), or \( \alpha = 0.01 \) (1%)?

**Given:** \( G_x = \frac{1}{2}, \ c = 0.05 \)

\[
\Rightarrow \ \frac{\alpha}{2} = Q \left( \frac{c \sqrt{n}}{G_x} \right) = Q \left( 0.1 \ \sqrt{n} \right)
\]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \frac{\alpha}{2} )</th>
<th>( 0.1 \ \sqrt{n} )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.016</td>
<td>1</td>
<td>100</td>
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<tr>
<td>0.05</td>
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<tr>
<td>0.01</td>
<td>0.0025</td>
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<tr>
<td>0.01</td>
<td>0.005</td>
<td>2.58</td>
<td>666</td>
</tr>
</tbody>
</table>

**Application:** Test whether a random-number generator is biased.

\[
0.1 \ \sqrt{n} = 1.65 
\Rightarrow n = \left( \frac{1.65}{0.1} \right)^2 = 16.5^2
\]

The probability of making a false rejection of \( H_0 \) (i.e., that the coin is fair) on the basis of the distribution \( \text{Max}(\text{bias}) = 0.05 \) is \( 8\% \).