1. **Problem.** Following closely the derivation of the rate for scattering with ionized impurities (see Lecture Notes, pages 65-68), derive the associated momentum relaxation rate:

\[
\frac{1}{\tau_{p, imp}(k)} = \frac{2\pi}{\hbar} \frac{e^4}{V\epsilon_s^2} \frac{1}{(2\pi)^3} \int d\mathbf{k}' \frac{1 - \cos \theta}{(|\mathbf{k}' - \mathbf{k}|^2 + \beta^2)^2} \delta[E(k') - E(k)] ,
\]

where \(\theta\) is the angle between the initial and final wavevectors \(\mathbf{k}\) and \(\mathbf{k}'\). (Note the factor \(1 - \cos \theta\) which is absent in the expression for the scattering rate, but denotes the change of momentum in the expression for the momentum relaxation rate). As done in class, assume a parabolic dispersion \(E(k) = \hbar^2 k^2/(2m^*)\). The relaxation rate will depend only on the energy \(E\) of the electron, not on the direction of the wavevector \(\mathbf{k}\). So, you should express the relaxation rate as a function of \(E\). You should get:

\[
\frac{1}{\tau_{p, imp}(E)} = \frac{2^{1/2} N_I e^4}{32\pi m^* 1/2 \epsilon_s^2 E^{3/2}} \left[ \ln \left( 1 + \frac{8m^*E}{\hbar^2 \beta^2} \right) + \left( \frac{1}{1 + 8m^*E/(\hbar^2 \beta^2)} - 1 \right) \right] ,
\]

where \(N_I\) is the number of ionized impurities per unit volume.

**Solution.** Let’s start from Eq. (1), having already summed over all impurities (so that the factor of \(V\) in the denominator is absorbed into the impurity concentration \(N_I\)), using polar coordinates with the polar axis aligned along the direction of \(\mathbf{k}\) and noticing that \(|\mathbf{k}' - \mathbf{k}|^2 = k'^2 + k^2 - 2k'k \cos \theta\):

\[
\frac{1}{\tau_{p, imp}(k)} = \frac{2\pi}{\hbar} \frac{e^4}{\epsilon_s^2} \frac{N_I}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \int_0^\infty dk' k'^2 (1 - \cos \theta) \frac{\delta[E(k') - E(k)]}{(k'^2 + k^2 - 2k'k \cos \theta + \beta^2)^2} .
\]

Now let’s employ the usual change of variable \(E' = (\hbar k')^2/(2m^*)\), which implies \(dk'k' = (m^*/\hbar^2)dE'\),
and \( k' = (2m^*E')^{1/2}/\hbar \), so that:

\[
\frac{1}{\tau_{p,\text{imp}}(k)} = \frac{N_I e^4 m^*^{3/2}}{2^{1/2} \pi \hbar^4 \epsilon_s^2} \int_0^\pi d\theta \sin \theta \int_0^\infty dE' \frac{E'^{1/2} (1 - \cos \theta) \delta(E' - E)}{[(2m^*/\hbar^2)(E' + E - 2E^{1/2}E'^{1/2} \cos \theta) + \beta^2]^2} =
\]

\[
= \frac{N_I m^*^{3/2} e^4 E^{1/2}}{2^{1/2} \pi \hbar^4 \epsilon_s^2} \int_0^\pi d\theta \frac{1 - \cos \theta}{[(4m^*/\hbar^2) E(1 - \cos \theta) + \beta^2]^2},
\]

(4)

having integrated over the delta-function in the last step and having set \( E = (\hbar k)^2/(2m^*) \). Now, using the variable \( \mu = \cos \theta \), and also recognizing that \( \tau_{p,\text{imp}} \) depends only on the magnitude of \( k \) (and so on \( E \)):

\[
\frac{1}{\tau_{p,\text{imp}}(E)} = \frac{N_I m^*^{3/2} e^4 E^{1/2}}{2^{1/2} \pi \hbar^4 \epsilon_s^2} \frac{\hbar^2}{4m^* E} \int_{-1}^{+1} d\mu \frac{(4m^* E/\hbar^2)(1 - \mu)}{[(4m^*/\hbar^2)(1 - \mu) + \beta^2]^2},
\]

(5)

where we have multiplied and divided by \( 4m^* E/\hbar^2 \) to render the next change of variable more obvious: In this integral let’s set \( x = (4m^* E/\hbar^2)(1 - \mu) \), so that \( d\mu = -[\hbar^2/(4m^* E)]dx \). We are left with:

\[
\frac{1}{\tau_{p,\text{imp}}(E)} = \frac{2^{1/2} N_I e^4}{32 \pi m^*^{1/2} \epsilon_s^2 E^{3/2}} \int_0^{K^2} dx \frac{x}{(x + \beta^2)^2},
\]

(6)

where \( K^2 = 8m^* E/\hbar^2 \). The integral can be computed by adding and subtracting \( \beta^2 \) in the numerator:

\[
\int_0^{K^2} dx \frac{x}{(x + \beta^2)^2} = \int_0^{K^2} dx \frac{x + \beta^2}{(x + \beta^2)^2} - \int_0^{K^2} dx \frac{\beta^2}{(x + \beta^2)^2} =
\]

\[
= \int_0^{K^2} dx \frac{1}{x + \beta^2} - \int_0^{K^2} dx \frac{\beta^2}{(x + \beta^2)^2} = \ln(x + \beta^2) \bigg|_0^{K^2} + \frac{\beta^2}{x + \beta^2} \bigg|_0^{K^2} =
\]
\[= \ln \left(1 + \frac{K^2}{\beta^2}\right) + \left(\frac{\beta^2}{K^2 + \beta^2} - 1\right). \tag{7}\]

Inserting the result of Eq. (7) into Eq. (6), we get

\[
\frac{1}{\tau_{p, imp}(E)} = \frac{2^{1/2} N_I e^4}{32 \pi m^{1/2} \epsilon_s^2 E^{3/2}} \left[ \ln \left(1 + \frac{8m^* E}{\hbar^2 \beta^2}\right) + \left(\frac{1}{1 + 8m^* E/(\hbar^2 \beta^2)} - 1\right) \right], \tag{8}\]

which is the desired Eq. (2).

2. **Problem.**

a. Using the result of Problem 1, Eq. (2), find the approximated expression valid to second-order in \(8m^* E/(\hbar^2 \beta^2)\). (Note that the first-order terms vanish). The following Taylor expansions are handy:

\[
\ln(1 + x) \approx x - \frac{x^2}{2},
\]

and

\[
\frac{1}{1 + x} \approx 1 - x + x^2.
\]

You should get:

\[
\frac{1}{\tau_{p, imp}(E)} \approx \frac{2m^*^{3/2} e^4 N_I E^{1/2}}{2^{1/2} \pi \epsilon_s^2 \hbar^4 \beta^4}. \tag{9}\]

b. In the opposite limit of weak screening in which \(8m^* E/(\hbar^2 \beta^2) >> 1\), derive the approximate result:

\[
\frac{1}{\tau_{p, imp}(E)} \approx \frac{2^{1/2} e^4 N_I}{32 \pi m^{1/2} \epsilon_s^2 E^{3/2}} \ln \left(\frac{8m^* E}{\hbar^2 \beta^2}\right). \tag{10}\]
Solution. a The last term of Eq. (7) can be written as:

\[ \ln \left( 1 + \frac{K^2}{\beta^2} \right) - \frac{1}{1 - K^2/\beta^2}. \]  

(11)

Setting \( x = K^2/\beta^2 \) in the Taylor expansions, we have

\[ \ln \left( 1 + \frac{K^2}{\beta^2} \right) - \frac{1}{1 - K^2/\beta^2} \approx \frac{1}{2} \frac{K^4}{\beta^4}, \]  

(12)

so that, in the limit \( K^2 \ll \beta^2 \) we have

\[ \frac{1}{\tau_{p,imp}(E)} \approx \frac{2^{1/2} N_I e^4}{32 \pi m^{1/2} \epsilon_s^2 E^{3/2}} \frac{1}{2} \frac{64 m^* E^2}{\hbar^4 \beta^4}, \]  

(13)

which, after a few simplifications, reduces to Eq. (9).

b. This is much simpler. All we have to do is to retain the leading term (that is, the highest-order term in \( K^2/\beta^2 \)) in Eq. (7). Thus:

\[ \ln \left( 1 + \frac{K^2}{\beta^2} \right) - \frac{1}{1 - K^2/\beta^2} = \ln \left( \frac{K^2}{\beta^2} \right) + o(1), \]  

(14)

which means that the terms we have neglected are of the order of 1 or even smaller. From Eqns. (8) and (14), Eq. (10) follows immediately.

3. Problem. a. Using the result of Problem 2a, Eq. (9), calculate the mobility limited by this scattering process. Use the Kubo-Greenwood formula derived in class from the second moment of the Boltzmann transport equation:

\[ \mu_{imp} = \frac{2e}{3nm^* k_B T} \int_0^\infty dE \rho(E) E \tau_{p,imp}(E) f_{FD}(E)[1 - f_{FD}(E)], \]  

(15)
where \( \rho(E) = \frac{2^{1/2} m^{3/2} E^{1/2}}{(\pi^2 \hbar^3)} \) is the density of states. Use the Boltzmann approximations:

\[
f_{FD}(E) \approx \exp\left(-\frac{E - E_F}{k_B T}\right) \quad \text{and} \quad 1 - f_{FD}(E) \approx 1.
\]

The following integrals are useful:

\[
\int_0^\infty dx \ x^{1/2} e^{-x} = \frac{\pi^{1/2}}{2}, \quad \int_0^\infty dx \ x e^{-x} = 1.
\]

You should get:

\[
\mu_{imp} \approx \frac{2(2\pi)^{1/2} \varepsilon_s^2 \hbar^4 \beta^4}{3 m^{5/2} e^3 N_I (k_B T)^{1/2}}.
\] (16)

b. Using now the result Eq. (10) of Problem 2b, calculate the electron mobility limited by impurity scattering in the limit of weak screening. Since the presence of the logarithmic term is troublesome, approximate it to the constant \( \ln\left[8 m^* E_{th} / (\hbar^2 \beta^2)\right] \), where the variable \( E \) has been replaced by its thermal value \( E_{th} = (3/2) k_B T \). You should get:

\[
\mu_{imp} \approx \frac{2^{15/2} \pi^{1/2} \varepsilon_s^2 (k_B T)^{3/2}}{3 m^{1/2} e^3 N_I} \ln^{-1} \left(\frac{12 m^* k_B T}{\hbar^2 \beta^2}\right).
\] (17)

**Solution. a.** From Eq. (15) and recalling that (see Lecture Notes, page 37, first equation):

\[
\rho(E) = \frac{2^{1/2} m^{3/2} E^{1/2}}{\pi^2 \hbar^3}
\]

and that in the Boltzmann limit:

\[
n = \int_0^\infty \rho(E) f_{FD}(E) dE \approx \frac{2^{1/2} m^{3/2}}{\pi^2 \hbar^3} \int_0^\infty E^{1/2} \exp\left(\frac{E_F - E}{k_B T}\right),
\] (18)
we have, always in the Boltzmann limit:

\[
\mu_{\text{imp}} \approx \frac{2e}{3nm^*k_B T} \frac{2^{1/2}m^*3/2eE_F/(k_B T)}{\pi^2\hbar^3} \int_0^\infty E^{3/2} \tau_{p,\text{imp}}(E) e^{-E/(k_B T)} dE \approx \\
\approx \frac{2e}{3m^*k_B T} \int_0^{\infty} \frac{E^{3/2} \tau_{p,\text{imp}}(E) e^{-E/(k_B T)}}{E^{1/2} e^{-E/(k_B T)}} dE .
\]

(19)

Inserting Eq. (9) into this equation, we get:

\[
\mu_{\text{imp}} \approx \frac{2e}{3m^*k_B T} \frac{2^{1/2}m^*3/2e^4N_I}{2m^*3/2e^4N_I} \int_0^\infty E \tau_{p,\text{imp}}(E) e^{-E/(k_B T)} dE = \\
= \frac{2e}{3m^*k_B T} \frac{2^{1/2}m^*3/2e^4N_I}{2m^*3/2e^4N_I} \frac{2}{\pi^{1/2}} (k_B T)^{1/2} ,
\]

(20)

(having used the values of the integrals given above) from which Eq. (16) follows.

b. Once again, using Eq. (15) in the Boltzmann limit we have:

\[
\mu_{\text{imp}} \approx \frac{2e}{3nm^*k_B T} \frac{32\pi m^*1/2\epsilon_s^2\hbar^4\beta^4}{2^{1/2}e^4N_I} \frac{2^{1/2}m^*3/2}{\pi^2\hbar^3} \int_0^\infty E^3 \ln^{-1} \left( \frac{8m^*E}{\hbar^2\beta^2} \right) e^{(E_F-E)/(k_B T)} .
\]

(21)

Now let’s approximate the logarithmic term inside the integral as a constant, \(\ln^{-1}[8m^*E_{th}/(\hbar^2\beta^2)]\), so we can take it outside the integral. Recalling also once more Eq. (18), recalling that \(E_{th} = 3k_B T/2\), and noticing that

\[
\int_0^{\infty} E^3 e^{-E/(k_B T)} = (k_B T)^4 \int_0^{\infty} dx x^3 e^{-x} = (k_B T)^4 \Gamma(4) = 6 (k_B T)^4 ,
\]

we obtain Eq. (17)... up to a factor of 3!
4. **Problem.** A useful rule for estimating the total mobility due to several scattering processes is the so-called Matthiessen’s rule: If we have a total of $N$ scattering processes (for example: with optical phonons, acoustic phonons, longitudinal and transverse, emission and absorption, with ionized impurities, etc.), each labeled by an index $n$ and each yielding a mobility $\mu_n$, the total mobility due to the combined effect of all of these processes will be:

$$\frac{1}{\mu_{tot}} \approx \sum_{n=1}^{N} \frac{1}{\mu_n}, \quad (22)$$

that is: The inverse of the total mobility is the sum of the inverse of the mobilities limited by each scattering process. While this rule is exact only for elastic processes, often it is a useful ‘rule of thumb’ to estimate the total mobility.

Using Matthiessen’s rule, the expression for the impurity-scattering-limited mobility, Eq. (17) of Problem 3b, and the expression for mobility limited by acoustic phonons (page 81 of the Lecture Notes),

$$\mu_{ac} = \frac{2^{3/2} \pi^{1/2} e h^4 \rho c_s^2}{3m^{5/2} \Delta_{ac}^2 (k_B T)^{3/2}} ,$$

sketch the total mobility $\mu_{tot} = (1/\mu_{imp} + 1/\mu_{ac})^{-1}$ as a function of impurity concentration $N_I$ for $10^{15}$ cm$^{-3} \leq N_I \leq 10^{20}$ cm$^{-3}$. Use the following expression for the screening parameter (assuming $n$-type doping, $n = N_I$ by charge neutrality, and – for simplicity – the non-degenerate limit, even though it isn’t correct for $n > 10^{19}$ cm$^{-3}$):

$$\beta^2 = \frac{e^2 N_I}{\epsilon_s k_B T} ,$$
and use the following parameters (note that I had given the electron mass in grams instead of kg!):

\[ m^* = 0.32 \, m_0 \]

\[ T = 300K \]

\[ \Delta_{ac} = 11 \, (\text{eV}) \]

\[ c_s = 9 \times 10^5 \, \text{cm/s} \]

\[ \rho = 2.33 \, \text{gr/cm}^3 \]

\[ e = 1.602 \times 10^{-19} \, \text{(Coulomb)} \]

\[ \hbar = 1.05 \times 10^{-34} \, \text{(joule sec)} \]

\[ k_B = 1.38 \times 10^{-23} \, \text{(joule/Kelvin)} \]

\[ \epsilon_s = 11.7 \, \epsilon_0 \]

where \( m_0 \approx 9.1 \times 10^{-31} \, (\text{kgrams}) \) is the free electron mass

room temperature

an average sound velocity

crystal density

electron charge

reduced Planck constant

Boltzmann constant

where \( \epsilon_0 = 8.85 \times 10^{-12} \, \text{(Farad/m)} \) is the vacuum permittivity

\textbf{Solution.} Using the parameters given, we get

\[ \mu_{imp} \approx \frac{4.14 \times 10^{21}}{N_I} \frac{1}{\ln \left( \frac{2.2 \times 10^{19}}{N_I} \right)} \]  \hspace{1cm} (23)

where \( N_I \) is expressed in \( \text{cm}^{-3} \) and \( \mu_{imp} \) in \( \text{cm}^2/\text{Vs} \), and

\[ \mu_{ac} \approx 1645.3 \, \text{cm}^2/\text{Vs} \]  \hspace{1cm} (24)

From Eq. (23) we see that at small \( N_I \) \( \mu_{imp} \) is much larger than \( \mu_{ac} \) (as expected: when we have a small number of impurities, the mobility is mainly limited by scattering with phonons). Actually, as long \( N_I < 10^{16} \, \text{cm}^{-3} \) the value of the total mobility is barely affected by impurity-scattering. Indeed, for \( N_I = 10^{16} \, \text{cm}^{-3} \) we have \( \mu_{imp} \approx 5.38 \times 10^4 \, \text{cm}^2/\text{Vs} \), so, by Matthiessen’s rule, \( \mu_{tot} \approx 1596 \, \text{cm}^2/\text{Vs} \). For \( N_I = 10^{18} \, \text{cm}^{-3} \) we have \( \mu_{imp} \approx 1.34 \times 10^3 \, \text{cm}^2/\text{Vs} \) and \( \mu_{tot} \approx 73.98 \, \text{cm}^2/\text{Vs} \). By the time the doping density reaches the value of \( 10^{19} \, \text{cm}^{-3} \), the total mobility begins to be significantly affected by impurity scattering, since now
\( \mu_{\text{imp}} \approx 5.25 \times 10^2 \text{ cm}^2/\text{Vs} \) and \( \mu_{\text{tot}} \approx 390 \text{ cm}^2/\text{Vs} \). However, as we reach \( N_I \sim 10^{19} \text{ cm}^{-3} \), while it is true that \( \mu_{\text{imp}} \) is beginning to affect \( \mu_{\text{tot}} \), the initial assumption of weak screening, \( 8m^*E/\hbar^2 >> \beta^2 \) breaks down. Indeed, at sufficiently large \( N_I \) the logarithmic term becomes negative, showing that our calculations are inconsistent. At very large values of \( N_I \) the strong screening approximation, and so the expression given by Eq. (16) for \( \mu_{\text{imp}} \), would be more appropriate. Unfortunately, at large impurity concentrations several additional complications enter the picture: The Fermi Golden rule (equivalent to what’s know as the ‘Born approximation’ for Coulomb scattering) breaks down; scattering with several impurities cannot be considered as a succession of independent scattering events (the impurities are so close, that the electron wavelength is longer that the average separation of impurities), etc.

Note that the phonon-limited mobility in Si is about 1,450 cm²/Vs at 300K. The value for \( \mu_{\text{ac}} \) we have obtained is significantly different for several reasons: We have assumed only acoustic phonons and only emission (or
absorption) processes (this effect is usually ‘lumped’ into the values of $\Delta_{ac}$ and $c_s$); we have ignored inter-valley processes (not very important as far as the ohmic mobility is concerned); we have ignored the fact that there are transverse and longitudinal phonons (this effect is also usually lumped into the value of $\Delta_{ac}$); most notably, while the value of 0.32 for the effective mass is a realistic value for the density-of-states mass in one valley, Si has 6 equivalent valleys and some of them exhibit a smaller conductivity mass (0.19, and this makes a significant difference since $m$ appears raised to the 2.5 power in the denominator); we have assumed ‘parabolic’ bands. Some of these factors contribute to depressing the phonon-limited mobility, some (mainly the use of the correct DOS and conductivity masses in each valley) to boosting it.

5. **Problem.** An alternative way to derive the Einstein’s ‘closure’ relation $\mu = (k_B T/e)D$ is the following. From the current continuity equation in 1D along $z$,

$$ j_{z,n} = en\mu_n F_z + eD_n \frac{dn}{dz} ,$$

at equilibrium, when $j_{z,n} = 0$, we have:

$$ en\mu_n F_z = -eD_n \frac{dn}{dz} .$$

(25)

(26)

Since $dn/dz = d/dz\{n_i \exp[(E_F - E_i)/(k_B T)]\} = n/(k_B T)(dE_F/dz)$, we have:

$$ en\mu_n F_z = -e \frac{n}{k_B T} D_n \frac{dE_F}{dz} .$$

(27)

But at equilibrium we must also have $dE_F/dz = -dE_c/dz = eF_z$, so that:

$$ en\mu_n F_z = n \frac{e^2}{k_B T} D_n F_z \rightarrow \mu_n = \frac{e}{k_B T} D_n ,$$

(28)

which is indeed the Einstein relation.
This derivation can be extended to a degenerate situation. In this case (see Lecture Notes, page 43):

\[ n = N_c \mathcal{F}_{1/2}(\eta_{Fc}) , \]  

(29)

where

\[ \eta_{Fc} = \frac{E_F - E_c}{k_B T} \]

and the Fermi integral of order \( n \) is defined as:

\[ \mathcal{F}_n(\eta) = \int_0^\infty dx \frac{x^n}{1 + e^{x-\eta}} , \]

and

\[ N_c = \frac{1}{2\pi^2} \left( \frac{2m^*k_B T}{\hbar^2} \right)^{3/2} . \]

Using the derivation above – Eqns. (25)-(28) – and noticing that

\[ \frac{d\mathcal{F}_{1/2}(\eta)}{d\eta} = \frac{1}{2} \mathcal{F}_{-1/2}(\eta) , \]

(30)

derive the form of Eq. (28) valid in the degenerate case.

**Solution.** From Eq. (29) we have:

\[ \frac{dn}{dz} = \frac{d}{dz} \left[ N_c \mathcal{F}_{1/2}(\eta_{Fc}) \right] = N_c \frac{d\mathcal{F}_{1/2}(\eta_{Fc})}{d\eta_{Fc}} \frac{d\eta_{Fc}}{dz} = N_c \frac{d\mathcal{F}_{1/2}(\eta_{Fc})}{d\eta_{Fc}} \frac{1}{k_B T} \frac{dE_F}{dz} . \]

(31)

Now:

\[ \frac{d\mathcal{F}_{1/2}(\eta_{Fc})}{d\eta_{Fc}} = \frac{d}{d\eta_{Fc}} \int_0^\infty dx \frac{x^{1/2}}{1 + e^{x-\eta_{Fc}}} = \int_0^\infty dx \frac{x^{1/2}}{1 + e^{x-\eta_{Fc}}} \frac{d}{d\eta_{Fc}} \frac{1}{1 + e^{x-\eta_{Fc}}} = \]

---

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\[ = - \int_0^\infty dx \, x^{1/2} \frac{d}{dx} \left( \frac{1}{1 + e^{-\eta F_c}} \right) = x^{1/2} \left. \frac{1}{1 + e^{-\eta F_c}} \right|_0^\infty + \frac{1}{2} \int_0^\infty dx \, x^{-1/2} \frac{1}{1 + e^{-\eta F_c}} = \frac{1}{2} \mathcal{F}_{-1/2}(\eta), \]

which is Eq. (30). Inserting it into Eq. (31) we get:

\[
\frac{dn}{dz} = \frac{N_c}{2} \mathcal{F}_{-1/2}(\eta F_c) \frac{1}{k_B T} \frac{dE_F}{dz} = \frac{n}{2} \frac{\mathcal{F}_{-1/2}(\eta F_c)}{\mathcal{F}_{1/2}(\eta F_c)} \frac{1}{k_B T} \frac{dE_F}{dz}.
\]

(32)

Therefore, instead of Eq. (28) valid in the non-degenerate limit, we now get:

\[
en\mu_n F_z = e \frac{n}{k_B T} D_n \frac{\mathcal{F}_{-1/2}(\eta F_c)}{2\mathcal{F}_{1/2}(\eta F_c)} F_z.
\]

(33)

Therefore, Einstein’s closure relation in the degenerate limit becomes:

\[
\mu_n = \frac{e}{k_B T} \frac{\mathcal{F}_{-1/2}(\eta F_c)}{2\mathcal{F}_{1/2}(\eta F_c)} D_n.
\]

(34)

Note: You may have found in other tests this expression but without the factor of 2 in the denominator. This is the result of a different definition of the Fermi integral \( \mathcal{F}_n(\eta) \) as:

\[
\mathcal{F}_n(\eta) = \frac{1}{\Gamma(n + 1)} \int_0^\infty dx \, \frac{x^n}{1 + e^{x-\eta}},
\]

so that, with this alternate definition, one has

\[
\frac{d\mathcal{F}_{1/2}(\eta)}{d\eta} = \mathcal{F}_{-1/2}(\eta),
\]

(35)

without the factor of 1/2 present in Eq. (30). Often the notation \( F_n(\eta) \) instead of \( \mathcal{F}_n(\eta) \) is used for the definition we have employed.