Definition of inner product space

An inner product space \((V, \langle \cdot, \cdot \rangle)\) is a space containing a vector space \(V\) and an inner product \(\langle \cdot, \cdot \rangle\). The inner product is a scalar (real or complex valued) function defined on the Cartesian product of \(V \times V\). These are the mathematical spaces that signal processing relies on.

If \(x, y\) and \(z\) are elements of \(V\) (which can be defined on either the real or complex numbers), then \(\langle \cdot, \cdot \rangle\) is an inner product if:

1. \(\langle x, x \rangle \geq 0\) with equality if and only if \(x = 0\), and
2. \(\langle x, y \rangle = \langle y, x \rangle\) for real \(V\), and \(\langle x, y \rangle = \langle y, x \rangle^\dagger\) for complex \(V\), where \(^\dagger\) indicates complex conjugate
3. \(\langle \alpha x, y \rangle = \alpha \langle x, y \rangle\) where \(\alpha\) is a scalar (linearity of the first argument), and
4. \(\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle\)

Notes: If \((V, \langle \cdot, \cdot \rangle)\) is an IPS and \(x\) an element of \(V\), then \(\|x\| \equiv +\sqrt{\langle x, x \rangle}\) induces (defines) a norm on \(V\). Since the norm, in turn, induces a metric, an inner product also induces a metric, \(d(x, y) \equiv +\sqrt{\langle x, y \rangle}\). An inner product space which is complete (i.e., every Cauchy sequence in the space converges to an element of the space; see notes on metric spaces) is called a Hilbert space. Hilbert spaces are fundamental to signal processing (and, hence, image processing).

The Cauchy-Schwarz inequality: \(|\langle x, y \rangle| \leq \|x\| \|y\|\), where the norm is that induced by the inner product. The Cauchy-Schwarz inequality appears in many forms, but this is the most general.

Two elements \(x, y\) of \(V\) are orthogonal if \(\langle x, y \rangle = 0\), where \(x, y \neq 0\).

Examples of inner product spaces

1. \(V = \mathbb{R}^n\) (real vectors)
   \(\langle x, y \rangle = x^T y\) the usual dot product (or scalar product) of two real vectors \(x \cdot y\)
   \(\langle x, y \rangle = x^T A y\) where \(A\) is a positive definite matrix. \(A\) is positive definite if and only if \(x^T A x > 0\) for all \(x \neq 0\) in \(V\).

2. \(V = \mathbb{C}^n\) (complex vectors)
   \(\langle x, y \rangle = \overline{y}^T x \equiv y^H x\) where \(H\) indicates the transpose of the complex conjugate.
   A basic property of the complex vector spaces, for all \(x, y \in \mathbb{C}^n\):
   \(\langle Ax, y \rangle = \langle x, A^H y \rangle\) where \(A\) is any real or complex matrix
   A square complex matrix \(A\) is called Hermitian if \(A^H = A\); this is the complex version of a real symmetric matrix.

Inner Product Spaces
(3) \( V = \mathbb{C}^\infty \) a continuous stream of complex values, or discrete signal space
\[
\langle f, g \rangle = \sum_{n=-\infty}^{\infty} |f_n g_n|^2
\]

(4) \( V = C[a, b] \) (the set of continuous real functions on a closed interval, or \( V \) can be over the open interval \( C(-\infty, \infty) \), which is common in analog signal processing.)

If \( f(t), g(t) \in V \), then
\[
\langle f, g \rangle = \int_{a}^{b} f(t) g(t) dt
\]
If \( V \) is over the field of complex numbers, then we must take the complex conjugate of \( g(t) \) in the integral.
\[
\langle f, g \rangle = \int_{a}^{b} f(t) g(t) \overline{\psi(t)} dt \text{ where } \psi(t) \text{ is a weight function}
\]

**Links**
- Vector Spaces
- Metric Spaces
- Normed Linear Spaces