Transfer Functions – Part 2

1. Second-order systems
2. Poles and zeros
3. Time delays
4. Simulink example
Second-Order Systems

- General transfer function
  \[ \frac{Y(s)}{U(s)} = G(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} \]
  
  \( K = \text{steady state gain} \)
  \( \tau = \text{time constant} \)
  \( \xi = \text{damping coefficient} \)

- Overdamped system: \( \xi > 1 \)
  \[ G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \]

- Critically damped system: \( \xi = 1 \)
  \[ G(s) = \frac{K}{(\tau s + 1)^2} \]

- Underdamped system: \( 0 < \xi < 1 \)
  \[ G(s) = \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1} \]
Second-Order Example

- Isothermal, constant volume CSTR: $A \rightarrow B \rightarrow C$

$$\frac{d(VC_A)}{dt} = qC_{Af} - qC_A - k_1 C_A V$$

$$\frac{d(VC_B)}{dt} = -qC_B + k_1 C_A V - k_2 C_B V$$

- Deviation model ($q/V = 2$, $k_1 = 1$, $k_2 = 2$)

$$\frac{dC_A'}{dt} = \frac{q}{V} C_{Af}' - \frac{q}{V} C_A' - k_1 C_A' = 2C_{Af}' - 3C_A'$$

$$\frac{dC_B'}{dt} = -\frac{q}{V} C_B' + k_1 C_A' - k_2 C_B' = C_A' - 4C_B'$$

- Laplace transform of first equation

$$sC_A'(s) - C_A'(0) = 2C_{Af}'(s) - 3C_A'(s) \implies C_A'(s) = \frac{2}{s+3} C_{Af}'(s)$$
Second-Order Example cont.

- Laplace transform of second equation

\[
sC_B'(s) - C_B'(0) = C_A'(s) - 4C_B'(s) \quad \Rightarrow \quad C_B'(s) = \frac{1}{s+4} C_A'(s)
\]

- Substitute for \(C_A'(s)\)

\[
C_B'(s) = \frac{1}{s+4} C_A'(s) = \frac{2}{(s+3)(s+4)} C_{Af}'(s)
\]

- Rearrange to obtain standard form

\[
C_B'(s) = \frac{2}{(s+3)(s+4)} C_{Af}'(s) = \frac{1}{6} \left(\frac{1}{3} s + 1\right)\left(\frac{1}{4} s + 1\right) C_{Af}'(s)
\]
Linearization of Two-Dimensional System

- Nonlinear ODE model

\[
\frac{dy_1}{dt} = f_1(y_1, y_2, u) \quad \Rightarrow \quad f_1(y_1, y_2, u) = 0
\]
\[
\frac{dy_2}{dt} = f_2(y_1, y_2, u) \quad \Rightarrow \quad f_2(y_1, y_2, u) = 0
\]

- First-order Taylor series expansion

\[
\frac{dy_1}{dt} \approx f_1(\bar{y}_1, \bar{y}_2, u) + \left( \frac{\partial f_1}{\partial y_1} \right)_{(\bar{y}, \bar{u})} (y_1 - \bar{y}_1) + \left( \frac{\partial f_1}{\partial y_2} \right)_{(\bar{y}, \bar{u})} (y_2 - \bar{y}_2) + \left( \frac{\partial f_1}{\partial u} \right)_{(\bar{y}, \bar{u})} (u - \bar{u})
\]
\[
\frac{dy_2}{dt} \approx f_2(\bar{y}_1, \bar{y}_2, u) + \left( \frac{\partial f_2}{\partial y_1} \right)_{(\bar{y}, \bar{u})} (y_1 - \bar{y}_1) + \left( \frac{\partial f_2}{\partial y_2} \right)_{(\bar{y}, \bar{u})} (y_2 - \bar{y}_2) \left( \frac{\partial f_2}{\partial u} \right)_{(\bar{y}, \bar{u})} (u - \bar{u})
\]

- Linear ODE model

\[
\frac{dy_1'}{dt} \approx a_{11}y_1' + a_{12}y_2' + b_1u'
\]
\[
\frac{dy_2'}{dt} \approx a_{21}y_1' + a_{22}y_2' + b_2u'
\]
\[
\Rightarrow \quad \frac{dy'}{dt} = Ay' + bu'
\]
Overdamped & Critically Damped Systems

- Often results from two first-order systems in series

\[
\frac{Y(s)}{U(s)} = G(s) = \frac{K_1}{\tau_1 s + 1} \frac{K_2}{\tau_2 s + 1} = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}
\]

\[
G(s) = \frac{K}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2) s + 1} \equiv \frac{K}{\tau^2 s^2 + 2\xi \tau s + 1}
\]

\[
\tau = \sqrt{\tau_1 \tau_2} \quad \xi = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}} \geq 1
\]

- Step response

Overdamped \[y(t) = KM \left( 1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2} \right)\]

Critically damped \[y(t) = KM \left[ 1 - \left( \frac{t}{\tau} \right) e^{-t/\tau} \right] \]
Overdamped & Critically Damped Systems

Overdamped

\[ y(t) = KM \left( 1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2} \right) \]

Critically damped

\[ y(t) = KM \left[ 1 - \left( 1 + \frac{t}{\tau} \right) e^{-t/\tau} \right] \]
Underdamped Systems

- Result from inherently second-order systems

\[ G(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} \quad 0 < \xi < 1 \]

- Step response

\[ y(t) = KM \left\{ 1 - e^{-\xi t/\tau} \left[ \cos\left(\frac{\sqrt{1-\xi^2}}{\tau} t\right) + \frac{\xi}{\sqrt{1-\xi^2}} \sin\left(\frac{\sqrt{1-\xi^2}}{\tau} t\right) \right] \right\} \]
Underdamped Systems

\[ y(t) = KM \left\{ 1 - e^{-\frac{\xi t}{\tau}} \left[ \cos \left( \frac{\sqrt{1-\xi^2}}{\tau} t \right) + \frac{\xi}{\sqrt{1-\xi^2}} \sin \left( \frac{\sqrt{1-\xi^2}}{\tau} t \right) \right] \right\} \]
Underdamped System Characteristics

- Time to first peak
  \[ t_p = \frac{\pi \tau}{\sqrt{1 - \xi^2}} \]

- Overshoot
  \[ OS = \frac{a}{b} = \exp\left(-\frac{\pi \xi}{\sqrt{1 - \xi^2}}\right) \]

- Decay ratio
  \[ DR = \frac{c}{a} = (OS)^2 = \exp\left(-\frac{2\pi \xi}{\sqrt{1 - \xi^2}}\right) \]

- Period
  \[ P = \frac{2\pi \tau}{\sqrt{1 - \xi^2}} \]
Poles

- General transfer function form

\[ G(s) = \frac{N(s)}{D(s)} e^{-\theta s} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} e^{-\theta s} \]

- Poles

\[ D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad \Rightarrow \quad s = p_1, p_2, \ldots, p_n \]

- Dynamic response for distinct poles

\[ y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots + \alpha_n e^{p_n t} + \text{input terms} \]

- Stability

  **Negative real root**
  \[ e^{p_1 t} \rightarrow 0 \]

  **Positive real root**
  \[ e^{p_1 t} \rightarrow \infty \]

  **Complex root**
  \[ e^{p_1 t} = e^{(\alpha+j\beta)t} = e^{\alpha t} \sin(\beta t) \quad \alpha < 0 \quad e^{p_1 t} \rightarrow 0 \]

  \[ \alpha > 0 \quad e^{p_1 t} \rightarrow \infty \]
Zeros

- Zeros

\[ N(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 = 0 \quad \Rightarrow \quad s = z_1, z_2, \ldots, z_m \]

- Second-order system with numerator dynamics

\[
G(s) = K \frac{\tau_a s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}
\]

\[
y(t) = KM \left( 1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2} \right)
\]
Inverse response if $\tau_a < 0$

$$y(t) = KM \left(1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2} \right)$$
Time Delays

- Transportation delays
  - Fluid flow to process (input delay)
  - Sample delivery to analyzer (output delay)
  - Treated equivalently in Laplace domain

- Pure time delay

\[
y(t) = \begin{cases} 
0 & t < \theta \\
x(t - \theta) & t \geq \theta 
\end{cases} \quad \Rightarrow \quad \frac{Y(s)}{X(s)} = G(s) = e^{-\theta s}
\]

- First-order-plus-time-delay (FOPTD) model

\[
G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}
\]
Pure Time Delay

\[ y(t) = \begin{cases} 
0 & t < \theta \\
 x(t - \theta) & t \geq \theta 
\end{cases} \quad \Rightarrow \quad \frac{Y(s)}{X(s)} = G(s) = e^{-\theta s} \]
Pade Approximation

- Taylor series expansion

\[
e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2!} - \frac{\theta^3 s^3}{3!} + \frac{\theta^4 s^4}{4!} - \frac{\theta^5 s^5}{5!} + O(s^6)
\]

- Pade approximation – rational approximation of time delay

1/1 approximation

\[
e^{-\theta s} \approx G_1(s) = \frac{1 - \frac{\theta}{2} s}{1 + \frac{\theta}{2} s} = 1 - \theta s + \frac{\theta^2 s^2}{2} - \frac{\theta^3 s^3}{4} + \cdots \text{ error is } O(s^3)
\]

2/2 approximation

\[
e^{-\theta s} \approx G_2(s) = \frac{1 - \frac{\theta}{2} s + \frac{\theta^2}{12} s^2}{1 + \frac{\theta}{2} s + \frac{\theta^2}{12} s^2} \quad \text{error is } O(s^5)
\]
Transfer Function Approximations

- **Taylor series method**
  - Retain the largest (dominant) time constant
  - Approximate the other time constants as time delays: \( \frac{1}{\tau s + 1} \approx e^{-\tau s} \)

- **Skogestad’s method**
  - Approximate the smallest time constants as time delays
  - Add half the second largest time constant to the dominant constant
  - Add half the second largest time constant to the time delay

- **Example**

\[
G(s) = \frac{K(-0.1s + 1)}{(5s + 1)(3s + 1)(0.5s + 1)}
\]

\[
G_{TS}(s) = \frac{Ke^{-3.6s}}{5s + 1}
\]

\[
G_{Sk}(s) = \frac{Ke^{-2.1s}}{6.5s + 1}
\]
Transfer Function Approximations cont.

\[
G(s) = \frac{K(-0.1s + 1)}{(5s + 1)(3s + 1)(0.5s + 1)}
\]

\[
G_{TS}(s) = \frac{Ke^{-3.6s}}{5s + 1}
\]

\[
G_{Sk}(s) = \frac{Ke^{-2.1s}}{6.5s + 1}
\]
Simulink Example: tdexample.m

\[
G(s) = \frac{-0.1s + 1}{(5s + 1)(3s + 1)(0.5s + 1)} = \frac{-0.1s + 1}{7.5s^3 + 19s^2 + 8.5s + 1}
\]

\[
G_{TS}(s) = \frac{e^{-3.6s}}{5s + 1}
\]

\[
G_{Sk}(s) = \frac{e^{-2.1s}}{6.5s + 1}
\]