1 Preliminary

1.1 LU Factorization

Write the step-by-step in-place LU factorization of $A$ and then report the matrices $L$ and $U$:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 6 & 4 & 2 \\ 0 & 4 & 10 & 4 \\ 1 & 2 & 4 & 10 \end{pmatrix}$$

For example if $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 0.5 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 0.5 & 1.5 \end{pmatrix}$$

then $L = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}$, and $U = \begin{pmatrix} 2 & 1 \\ 0 & 1.5 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 6 & 4 & 2 \\ 0 & 4 & 10 & 4 \\ 1 & 2 & 4 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \\ 0 & 4 & 10 & 4 \\ 1 & 0 & 4 & 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \end{pmatrix}$$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

$U = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 4 & 0 \\ 2 & 4 & 1 \end{pmatrix}$
2 Coding hw2.f90

You need to complete and run the code. It creates a diagonally dominant matrix and solve a linear system. It contains 3 parts: (i) LAPACK-blas3 routines; (ii) a BLAS-2 implementation of LU and solve (to be completed); (iii) a BLAS-2 implementation of UL and solve (to be completed). Fill-up the following table report both residual and time inside each case (using 1 decimal digit).

<table>
<thead>
<tr>
<th>N</th>
<th>Lapack-BLAS3</th>
<th>Residual / TIME [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>with LU-BLAS2</td>
</tr>
<tr>
<td>400</td>
<td>1.6 $10^{-14}$</td>
<td>1.4 $10^{-2}$</td>
</tr>
<tr>
<td>800</td>
<td>3.1 $10^{-14}$</td>
<td>4.5 $10^{-2}$</td>
</tr>
<tr>
<td>1600</td>
<td>6.6 $10^{-14}$</td>
<td>3.3 $10^{-1}$</td>
</tr>
<tr>
<td>3200</td>
<td>1.2 $10^{-13}$</td>
<td>2.4 $10^{-13}$</td>
</tr>
</tbody>
</table>

- Looking at the time for LAPACK going from $N=800$ to $N=1600$ or $N=1600$ to $N=3200$, what can you say about the increase in computing time? Was it expected?

  when $N$ increase by a factor 2, we get $N \times 7$ increase in time
  
  $\Rightarrow O\left(N^3\right)$ behavior as expected.

- Report the LAPACK-Blas3 speed-up [time Lu-blas2 / time lapack] for all $N=400, 800, 1600, 3200$. Comments.
1.2 UL Factorization

Write the step-by-step in-place UL factorization of $B$ and then report the matrices $\tilde{U}$ and $L$.

\[
\begin{pmatrix}
10 & 4 & 2 & 1 \\
4 & 10 & 4 & 0 \\
2 & 4 & 6 & 2 \\
1 & 0 & 2 & 0
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
9 & 4 & 0 & 1 \\
4 & 10 & 4 & 0 \\
0 & 4 & 2 & 2 \\
1 & 0 & 2 & 1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
9 & 4 & 0 & 1 \\
4 & 2 & 2 & 0 \\
0 & 4 & 2 & 2 \\
1 & 0 & 2 & 1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
4 & 2 & 0 & 0 \\
0 & 4 & 2 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

We note that $A = PBP$ where $P = \{e_N, e_{N-1}, \ldots, e_1\}$, is there a way to obtain the UL factorization of $B$ knowing only the LU factorization of $A$?

\[
A = LU
\]

\[
B = PAP = PLPPUP = \frac{P \tilde{L}}{U} \frac{P \tilde{U}}{U}
\]

In general to find the UL of matrix $B$

1. Permute matrix elements $PBP$
2. Perform LU $PBP = LU$
3. $\tilde{U} = P \tilde{L}$, $\tilde{L} = PUP$, $B = \tilde{U} \tilde{L}$
3 Problem- LU with iterative refinement

We consider the matrix $A$ of Problem 1.1. We propose to solve the linear system $Ax = b$, where $b_i = 1 i = 1,...,4$. However, we suppose that the LU factorization of $A$ is not performed accurately (for some reasons). Stated otherwise, we suppose that we can obtain an accurate LU factorization of a slightly perturbed matrix $\tilde{A} = A + D$ where $D$ is a diagonal matrix with entries $10^{-8} * ||A||_1$.

Solve the system and report the residual (give also the value of $||A||_1$); then perform two iterative refinements and report the new residuals. Remark: solve the problem in double precision- include a copy of your code with this Homework.

\[ ||A||_1 = 18 \]

\[ \text{Solve } \tilde{A}x_0 = b \Rightarrow x \]

\[ \text{Compute residual } r_0 = \frac{||b - A x_0||_\infty}{||b||_\infty} \approx 1.16 \times 10^{-5} \]

\[ \begin{align*}
&\text{Solve } \tilde{A} (x_1 - x_0) = r_0 \Rightarrow x_1 \\
&\text{obtain } x_1 = x_0 + x_1 \\
&\text{Residual } r_1 = \frac{||b - A x_1||_\infty}{||b||_\infty} \approx 2.32 \times 10^{-10} \\
&\text{Solve } \tilde{A} (x_2 - x_1) = r_1 \\
&\text{obtain } x_2 = x_1 + x_2 \\
&\text{Residual } r_2 = \frac{||b - A x_2||_\infty}{||b||_\infty} \approx 8.68 \times 10^{-15}
\end{align*} \]