ChE 631 Assignment 1 Solution

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Problem 1.1

Let A be a real and symmetric matrix, λ_i be an eigenvalue, and u be the corresponding eigenvector. The eigenvalue equation is

$$Au = \lambda_i u \tag{1}$$

Taking the conjugate transpose of both sides we find that

$$\Rightarrow (Au)^* = u^* A^* = (\lambda_i u)^* = u^* \lambda_i^*$$
$$\Rightarrow u^* A^* = u^* \lambda_i^*$$

Since $A^* = A$, we have

$$\Rightarrow u^* A = u^* \lambda_i^* \tag{2}$$

Left multiply Eq. (1) by u^*

$$\Rightarrow u^* A u = \lambda_i u^* u \tag{3}$$

Right multiply Eq. (2) by u

$$\Rightarrow u^* A u = \lambda_i^* u^* u \tag{4}$$

Subtracting Eq. (4) from (3)

$$\Rightarrow 0 = (\lambda_i - \lambda_i^*) u^* u$$

Since $u^* u$ is a real number greater than 0

$$\Rightarrow \lambda_i = \lambda_i^*$$

Hence, λ_i is a real number

Now let v be another eigenvector and $\lambda_j (\neq \lambda_i)$ the corresponding eigenvalue,

$$\Rightarrow Av = \lambda_j v \tag{5}$$

Right multiply Eq. (2) by v and since λ's are real

$$\Rightarrow u^* A v = \lambda_i u^* v \tag{6}$$

Left multiply Eq. (5) by u^*

$$\Rightarrow u^* A v = \lambda_j u^* v \tag{7}$$

Subtracting Eq. (7) from (6)

$$\Rightarrow (\lambda_i - \lambda_j) u^* v = 0$$

Since $\lambda_i \neq \lambda_j$

$$\Rightarrow u^* v = 0$$

Hence, u and v are orthogonal

Problem 1.2

We have

$$B = P^{-1} \cdot A \cdot P$$

Let u and λ_i be a pair of eigenvalues and eigenvectors for A

$$\Rightarrow Au = \lambda_i u \tag{1}$$

Let v and λ_j be a pair of eigenvalues and vectors for B

$$\Rightarrow Bv = \lambda_j v$$

$$\Rightarrow P^{-1}APu = \lambda_j v$$

Left multiplying by P

$$\Rightarrow APu = \lambda_j Pv$$

Let $z = P \cdot u$ which is an $n \times 1$ vector

$$\Rightarrow Az = \lambda_j z$$

Hence λ_j is also an eigenvalue of A

Problem 1.3

Let

$$T_{ij} = \rho u_i u_j \tag{1}$$

where u_i,u_j are components of u(x) Let u'_k and u'_l be components of u(x')

$$u_i = u^{'}_k A_{ki} \tag{2}$$

$$u_j = u^{'}_l A_{lj} \tag{3}$$

Substituting Eq. (2) and (3) in Eq. (1)

$$\Rightarrow T_{ij} = \rho(u^{'}_k A_{ki})(u^{'}_l A_{lj})$$

$$\Rightarrow T_{ij} = (\rho u^{'}_k u^{'}_l)(A_{ki} A_{lj})$$

$$\Rightarrow T_{ij} = T_{kl} A_{ki} A_{lj}$$

We conclude that T is a second order tensor

Problem 1.4

For plane polar coordinates, the velocity gradient tensor components are

$$L_{rr} = \frac{\partial u_r}{\partial r}$$

$$L_{r\theta} = \frac{\partial u_\theta}{\partial r}$$

$$L_{\theta r} = \frac{1 \partial u_r}{r \partial \theta} - \frac{u_\theta}{r}$$

$$L_{\theta\theta} = \frac{1 \partial u_\theta}{r \partial \theta} + \frac{u_r}{r}$$

Here,

$$u_r = 0, \qquad u_\theta = \Omega r$$

$$L_{rr} = 0 \qquad L_{r\theta} = \Omega \qquad L_{\theta r} = -\Omega \qquad L_{\theta\theta} = 0$$

The vorticity in the z direction is

$$\omega_z = L_{r\theta} - L_{\theta r} = 2\Omega$$

The vorticity tensor is

$$V = \frac{1}{2} \begin{bmatrix} 0 & 2\Omega \\ -2\Omega & 0 \end{bmatrix}$$

Now,

$$L = V + E + \frac{1}{3} trace(L) I$$

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 1.5

The velocity in the frame of reference fixed on the body with respect is

$$\vec{V} = \vec{U} + \Omega \times \vec{X}$$

where $\vec{X}$ is a point on the body. Now consider the motion in the frame of reference fixed with the rigid body. In this frame, the flow is steady. If $\vec{u}$ is the velocity of a point on the fluid then,

$$\frac{D\vec{u}}{Dt} = 0$$

$$\Rightarrow \frac{\partial \vec{u}}{\partial t} + (\vec{U} + \Omega \times \vec{X}) \cdot \nabla \vec{u} = 0$$

$$\Rightarrow \frac{\partial \vec{u}}{\partial t} = -(\vec{U} + \Omega \times \vec{X}) \cdot \nabla \vec{u}$$