CHE 330, Fall 2009, mid-term exam

Numbers in parentheses on the right indicate the maximum grade for each question.

E.1.1 *Properties of a linear flow*

Consider a steady two-dimensional flow with velocity components

$$u_x = \xi\,(x+3y), \qquad u_y = \xi\,(2x-y). \tag{1}$$

(a) What are the dimensions of the constant ξ? (1)
(b) Compute the rate of rotation, Ω. (2)
(c) Compute the rate of expansion, α. (2)
(d) Compute the rates of stretching, G, and the angles of stretching, β. (2)
(e) Derive an analytical expression for the position of a point particle, $\mathbf{X}(t)$, which at $t=0$ is located at $X(0)=a$ and $Y(0)=0$. (6)

Solution

The velocity field can be expressed in vector form as in (1.6.38)

$$\begin{bmatrix} u_x & u_y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \xi & 2\xi \\ 3\xi & -\xi \end{bmatrix}. \tag{3}$$

(a)

$$\xi \quad [=] \quad \frac{\text{velocity}}{\text{length}} = \frac{\text{length/time}}{\text{length}} = \frac{1}{\text{time}}. \tag{4}$$

(b) Using (1.6.40), we find

$$\Omega = \frac{b-c}{2} = \frac{2\xi - 3\xi}{2} = -\frac{\xi}{2}. \tag{5}$$

(c) Using (1.6.41), we find

$$\alpha = a + d = \xi + (-\xi) = 0. \tag{6}$$

(d) Using (1.6.48) and (1.6.49), we compute

$$G = \lambda = \pm\frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2} = \pm\frac{1}{2}\sqrt{(2\xi)^2 + (5\xi)^2} = \pm\frac{\sqrt{29}}{2}\,\xi, \tag{7}$$

and then

$$(\mathbf{E} - \lambda\mathbf{I}) \cdot \mathbf{f} = \begin{bmatrix} \frac{a-d}{2} - \lambda & \frac{b+c}{2} \\ \frac{c+b}{2} & \frac{d-a}{2} - \lambda \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} \xi - \lambda & \frac{5}{2}\xi \\ \frac{5}{2}\xi & -\xi - \lambda \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \mathbf{0}, \tag{8}$$

and finally

$$\beta = \arctan(\frac{f_y}{f_x}) = \arctan(\frac{\xi - \lambda}{-\frac{5}{2}\xi}) = \arctan(\frac{\xi \pm \frac{\sqrt{29}\xi}{2}}{-\frac{5}{2}\xi}) = \arctan(\frac{2 \pm \sqrt{29}}{-5}). \tag{9}$$

(e) We need to solve the differential equations

$$\left[\begin{array}{c} u_x \\ u_y \end{array}\right] = \left[\begin{array}{c} \mathrm{d}X/\mathrm{d}t \\ \mathrm{d}Y/\mathrm{d}t \end{array}\right] = \xi \left[\begin{array}{cc} 1 & 3 \\ 2 & -1 \end{array}\right] \left[\begin{array}{c} X \\ Y \end{array}\right]. \tag{10}$$

We introduce the coefficient matrix and solution vector

$$\mathbf{A} = \xi \left[\begin{array}{cc} 1 & 3 \\ 2 & -1 \end{array}\right], \qquad \mathbf{X}(t) = \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right], \tag{11}$$

and find that the solution is

$$\mathbf{X}(t) = \alpha_1 \mathbf{c}_1 \, e^{\lambda_1 t} + \alpha_2 \mathbf{c}_2 \, e^{\lambda_2 t}, \tag{12}$$

where λ_1 and λ_2 are the eigevalues of $\mathbf{A}$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are the corresponding eigenvectors, and α_1 and α_2 are coefficients determined by the initial condition.

We formulate the linear system

$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{c} = \mathbf{0}, \tag{13}$$

and compute the roots of the chacteristic polynomial,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \left[\begin{array}{cc} \xi - \lambda & 3\xi \\ 2\xi & -\xi - \lambda \end{array}\right] = 0. \tag{14}$$

The solution is

$$\lambda_1, \lambda_2 = \pm\sqrt{7}\xi. \tag{15}$$

Next we compute the eigenvectors by solving the linear system

$$\left[\begin{array}{cc} \xi \mp \sqrt{7}\xi & 3\xi \\ 2\xi & -\xi \mp \sqrt{7}\xi \end{array}\right] \cdot \left[\begin{array}{c} c_x \\ c_y \end{array}\right] = 0 \tag{16}$$

yielding

$$(\xi \mp \sqrt{7}\xi) c_x + (3\xi) c_y = 0. \tag{17}$$

The solution is

$$\mathbf{c}_1, \mathbf{c_2} = \left[\begin{array}{c} 1 \\ \frac{-1 \pm \sqrt{7}}{3} \end{array}\right]. \tag{18}$$

The eigenvectors are

$$\mathbf{X}(t) = \alpha_1 \mathbf{c_1} e^{\lambda_1 t} + \alpha_2 \mathbf{c_2} e^{\lambda_2 t} = \left[\begin{array}{c} \alpha_1 c_{1x} e^{\lambda_1 t} + \alpha_2 c_{2x} e^{\lambda_2 t} \\ \alpha_1 c_{1y} e^{\lambda_1 t} + \alpha_2 c_{2y} e^{\lambda_2 t} \end{array}\right]. \tag{19}$$

Setting

$$\left[\begin{array}{c} X(0) \\ Y(0) \end{array}\right] = \left[\begin{array}{c} \alpha_1 + \alpha_2 \\ \alpha_1 \frac{-1+\sqrt{7}}{3} + \alpha_2 \frac{-1-\sqrt{7}}{3} \end{array}\right] = \left[\begin{array}{c} a \\ 0 \end{array}\right], \tag{20}$$

we find

$$\alpha_1 = \frac{1 + \sqrt{7}}{2\sqrt{7}} a, \qquad \alpha_2 = \frac{-1 + \sqrt{7}}{2\sqrt{7}} a. \tag{21}$$

E.1.2 *Properties of a quadratic flow*

Consider a three-dimensional flow with velocity components

$$u_x = 3\xi x^2 z + \eta(3y + z), \qquad u_y = -\xi xyz, \qquad u_z = \xi(x^2 y - xz^2). \tag{1}$$

(a) What are the dimensions of the constants ξ and η? (2)
(b) Compute the rate of expansion, α. (2)
(c) Compute the vorticity vector field, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. (3)
(d) Compute the velocity gradient tensor, $\nabla\mathbf{u}$. (2)
(e) Compute the vorticity tensor, $\boldsymbol{\Xi}$. (2)
(f) Compute the rate of deformation tensor, $\mathbf{E}$. (2)

Solution

(a) The dimensions are:

$$\xi \quad [=] \quad \frac{\text{velocity}}{\text{length}^3} = \frac{\text{length/time}}{\text{length}^3} = \frac{1}{\text{time} \cdot \text{distance}^2}, \qquad \eta \quad [=] \quad \frac{1}{\text{time}}. \tag{2}$$

(b) Using (2.1.29), we find

$$\alpha = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = (6\xi xz) + (-\xi xz) + (-2\xi xz) = 3\xi xz. \tag{3}$$

(c) From (2.3.6)

$$\boldsymbol{\omega} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{e}_z. \tag{4}$$

Substituting, we find

$$\boldsymbol{\omega} = \left[\xi x^2 - (-\xi xy)\right]\mathbf{e}_x + \left[(3\xi x^2 + \eta) - \xi(2xy - z^2)\right]\mathbf{e}_y + \left[-\xi yz - 3\eta\right]\mathbf{e}_z, \tag{5}$$

and then

$$\boldsymbol{\omega} = \xi x\,(x + y)\,\mathbf{e}_x + \left(\xi(3x^2 - 2xy + z^2) + \eta\right)\mathbf{e}_y - (\xi yz + 3\eta)\,\mathbf{e}_z. \tag{6}$$

(d) From (2.1.7) and (2.1.13)

$$\nabla\mathbf{u} = \left[\begin{array}{ccc} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_Y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial Z} \end{array}\right]. \tag{7}$$

Substituting, we find

$$\nabla\mathbf{u} = \left[\begin{array}{ccc} 6\xi xz & -\xi yz & \xi(2xy - z^2) \\ 3\eta & -\xi xz & \xi x^2 \\ 3\xi x^2 + \eta & -\xi xy & -2\xi xz \end{array}\right]. \tag{8}$$

(e) From (2.1.27)

$$\boldsymbol{\Xi} = \frac{1}{2}\left[\begin{array}{ccc} 0 & \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} & \frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z} \\ \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} & 0 & \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \\ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} & \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} & 0 \end{array}\right] = \frac{1}{2}\left[\begin{array}{ccc} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{array}\right]. \tag{9}$$

Substituting, we find

$$\boldsymbol{\Xi} = \frac{1}{2}\left[\begin{array}{ccc} 0 & -\left(\xi yz + 3\eta\right) & -\left(\xi(3x^2 - 2xy + z^2) + \eta\right) \\ (\xi yz + 3\eta) & 0 & \xi x(x+y) \\ (\xi(3x^2 - 2xy + z^2) + \eta) & -\xi x(x+y) & 0 \end{array}\right]. \tag{10}$$

(*f*) From (2.1.28)

$$\mathbf{E} = \left[\begin{array}{ccc} \frac{\partial u_x}{\partial x} - \frac{1}{3}\alpha & \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} - \frac{1}{3}\alpha & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & \frac{\partial u_z}{\partial z} - \frac{1}{3}\alpha \end{array}\right]. \tag{11}$$

Substituting, we find

$$\mathbf{E} = \left[\begin{array}{ccc} (6\xi xz) - \frac{1}{3}(3\xi xz) & \frac{1}{2}\left(-\xi yz + 3\eta\right) & \frac{1}{2}\left(\xi(2xy - z^2) + (3\xi x^2 + \eta)\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & (-\xi xz) - \frac{1}{3}(3\xi xz) & \frac{1}{2}\left(\xi x^2 + (-\xi xy)\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & (-2\xi xz) - \frac{1}{3}(3\xi xz) \end{array}\right]. \tag{12}$$

Simplifying, we obtain

$$\mathbf{E} = \left[\begin{array}{ccc} 5\xi xz & \frac{1}{2}\left(-\xi yz + 3\eta\right) & \frac{1}{2}\left(\xi(3x^2 + 2xy - z^2) + \eta)\right) \\ \frac{1}{2}\left(-\xi yz + 3\eta\right) & -2\xi xz & \frac{1}{2}\xi x\left(x - y\right) \\ \frac{1}{2}\left(\xi(3x^2 + 2xy - z^2) + \eta)\right) & \frac{1}{2}\xi x\left(x - y\right) & -3\xi xz \end{array}\right]. \tag{13}$$