1.7.1 *Quadratic interpolation*

Solve the linear system of three equations (1.7.5) and (1.7.6) to derive formulas (1.7.7). *Hint*: Compute first the coefficient $c^{(k)}$ using the first of equations (1.7.5).

Solutions

The three equations read

$$
\begin{aligned}
&P_2^{(k)}(x_k) = c^{(k)} = f_k,\\
&P_2^{(k)}(x_{k+1}) = a^{(k)}(x_{k+1} - x_k)^2 + b^{(k)}(x_{k+1} - x_k) + c^{(k)} = f_{k+1},\\
&P_2^{(k)}(x_{k-1}) = a^{(k)}(x_{k-1} - x_k)^2 + b^{(k)}(x_{k-1} - x_k) + c^{(k)} = f_{k-1}.
\end{aligned}
$$

We substitute the value of $c^{(k)}$ from the first equation in the second and third equations, multiply the second equation by $(x_{k-1} - x_k)$ and the third equation by $(x_{k+1} - x_k)$, subtract the resulting expressions to eliminate $b^{(k)}$, and solve the emerging equation for $a^{(k)}$.

To compute $b^{(k)}$. we multiply the second equation by $(x_{k-1} - x_k)^2$ and the third equation by $(x_{k+1} - x_k)^2$, and subtract the resulting expressions to eliminate $a^{(k)}$. The result is

$$
\begin{aligned}
&a^{(k)} = \frac{\frac{f_{k+1}-f_k}{h_k} - \frac{f_k-f_{k-1}}{h_{k-1}}}{h_k + h_{k-1}}, \qquad b^{(k)} = \frac{h_{k-1}\frac{f_{k+1}-f_k}{h_k} + h_k\frac{f_k-f_{k-1}}{h_{k-1}}}{h_k + h_{k-1}},\\
&c^{(k)} = f_k,
\end{aligned}
$$

where $h_{k-1} = x_k - x_{k-1}$ and $h_k = x_{k+1} - x_k$

1.7.2 *Forward-point parabolic interpolation*

Consider the parabolic interpolation of a function of one variable, $f(x)$, as discussed in the text. Forward interpolation uses the interpolation condition

$$P_2^{(k)}(x_{k+2}) = a^{(k)}(x_{k+2} - x_k)^2 + b^{(k)}(x_{k+2} - x_k) + c^{(k)} = f_{k+2},$$

in place of (1.7.6) Derive expressions for the coefficients $a^{(k)}$, $b^{(k)}$, and $c^{(k)}$ in terms of the grid values f_k, f_{k+1}, and f_{k+2}, and the interval sizes h_k and h_{k+1}. Then derive simplified expressions when h_k and h_{k+1} are both equal to h.

Solutions

The three equations to be solved read

$$\begin{aligned}
&P_2^{(k)}(x_k) = c^{(k)} = f_k,\\
&P_2^{(k)}(x_{k+1}) = a^{(k)}(x_{k+1} - x_k)^2 + b^{(k)}(x_{k+1} - x_k) + c^{(k)} = f_{k+1},\\
&P_2^{(k)}(x_{k+2}) = a^{(k)}(x_{k+2} - x_k)^2 + b^{(k)}(x_{k+2} - x_k) + c^{(k)} = f_{k+2}.
\end{aligned}$$

Working as in problem 1.7.1, we derive the expressions

$$a^{(k)} = \frac{\frac{f_{k+2}-f_{k+1}}{h_{k+1}} - \frac{f_{k+1}-f_k}{h_k}}{h_{k+1} + h_k}, \qquad b^{(k)} = \frac{h_k \frac{f_{k+2}-f_{k+1}}{h_{k+1}} + h_{k+1} \frac{f_{k+1}-f_k}{h_k}}{h_{k+1} + h_k},$$

$$c^{(k)} = f_k,$$

where $h_k = x_{k+1} - x_k$ and $h_{k+1} = x_{k+2} - x_{k+1}$. When $h_{k+1} = h_k = h$, we find that

$$a^{(k)} = \frac{f_{k+2} - 2f_{k+1} + f_k}{2h^2}, \qquad b^{(k)} = \frac{f_{k+2} - f_k}{2h}.$$

1.7.5 *Streamlines by interpolation*

Run the code *rec_2d_strml* for a velocity field of your choice. Generate, plot, and discuss the streamline pattern.

Solutions

This is a straightforward numerical exercise.

2.1.2 *Decomposition of a linearized flow*

(a) Linearize the velocity described by equations (1.5.2) around the origin of the y axis, and then decompose the velocity gradient tensor of the linearized flow into the three modes shown on the right-hand side of (2.1.26).

(b) Decompose the velocity gradient tensor of the linearized flow expressed by equations (2.1.16) into the three modes shown on the right-hand side of (2.1.26).

Solutions

(a) The velocity field is

$$u_x = a\,y^2 + b\,y + c, \qquad u_y = 0, \qquad u_z = 0.$$

Linearizing about $y = 0$, we obtain

$$u_x \simeq b\,y + c, \qquad u_y = 0, \qquad u_z = 0,$$

which can be written is

$$\mathbf{u} = [c, 0, 0] + \mathbf{x} \cdot \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix on the right-hand side is the velocity gradient tensor. Decomposition into an antisymmetric component, a symmetric component with zero trace, and a diagonal component yields

$$\begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case $\alpha = 0$.

(b) The linearized equations (2.1.16) read

$$\begin{aligned} u_x(x, y, z) &\simeq u_x(1, 0, 0) + cdt\, e^{d\,t}\,(x - 1), \\ u_y(x, y, z) &\simeq u_y(1, 0, 0) + 2a\,(x - 1) + cdty, \\ u_z(x, y, z) &\simeq u_z(1, 0, 0) + 2a\,(x - 1) + cdtz. \end{aligned}$$

In vector notation,

$$\mathbf{u} = \mathbf{u}(1, 0, 0) + [x - 1, y, z] \cdot \begin{bmatrix} cdt\, e^{dt} & 2a & 2a \\ 0 & cdt & 0 \\ 0 & 0 & cdt \end{bmatrix}.$$

The matrix on the right-hand side is the velocity gradient tensor. Decomposition yields

$$\begin{aligned}
&\begin{bmatrix} cdt\,e^{dt} & 2a & 2a \\ 0 & cdt & 0 \\ 0 & 0 & cdt \end{bmatrix} \\
&\qquad = \begin{bmatrix} 0 & a & a \\ -a & 0 & 0 \\ -a & 0 & 0 \end{bmatrix} \\
&\qquad\quad + \begin{bmatrix} \frac{2}{3}\,cdt\,(-1+e^{dt}) & a & a \\ a & \frac{1}{3}\,cdt\,(1-e^{dt}) & 0 \\ a & 0 & \frac{1}{3}\,cdt\,(1-e^{dt}) \end{bmatrix} \\
&\qquad\qquad + \frac{1}{3}\,cdt\,(2+e^{dt}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$