ChE 330 Homework 2 Solution

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1.3.2 *Acceleration*

Derive the plane polar components of the acceleration given in (1.3.56).

Solution (25 points)

The coordinates of a moving molecule are functions of t, denoted by

$$r = R(t), \qquad \theta = \Theta(t). \tag{1}$$

The position vector is given by

$$\mathbf{X} = R(t)\mathbf{e_r}. \tag{2}$$

Take the time derivative of the position vector to derive the velocity vector,

$$\frac{d\mathbf{X}}{dt} = \frac{d}{dt}\left(R\mathbf{e_r}\right) = \frac{dR}{dt}\mathbf{e_r} + R\frac{d\mathbf{e_r}}{dt}, \tag{3}$$

and use the first relation in (1.3.54) to eliminate the time derivative of the radial unit vector,

$$\frac{d\mathbf{X}}{dt} = \frac{dR}{dt}\mathbf{e_r} + R\left(\frac{d\Theta}{dt}\mathbf{e}_\theta\right). \tag{4}$$

Take the time derivative of the velocity vector to derive the acceleration components,

$$\frac{d^2\mathbf{X}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{X}}{dt}\right) = \frac{d^2R}{dt^2}\mathbf{e_r} + \frac{dR}{dt}\frac{d\mathbf{e_r}}{dt} + \frac{dR}{dt}\frac{d\Theta}{dt}\mathbf{e}_\theta + R\frac{d^2\Theta}{dt^2}\mathbf{e}_\theta + R\frac{d\Theta}{dt}\frac{d\mathbf{e}_\theta}{dt}, \tag{5}$$

and use the relations in (1.3.54) to eliminate the time derivatives of radial and polar unit vectors,

$$\frac{d^2\mathbf{X}}{dt^2} = \frac{d^2R}{dt^2}\mathbf{e_r} + \frac{dR}{dt}\left(\frac{d\Theta}{dt}\mathbf{e}_\theta\right) + \frac{dR}{dt}\frac{d\Theta}{dt}\mathbf{e}_\theta + R\frac{d^2\Theta}{dt^2}\mathbf{e}_\theta + R\frac{d\Theta}{dt}\left(-\frac{d\Theta}{dt}\mathbf{e_r}\right), \tag{6}$$

$$= \left(\frac{d^2R}{dt^2} - R\left(\frac{d\Theta}{dt}\right)^2\right)\mathbf{e_r} + \left(2\frac{dR}{dt}\frac{d\Theta}{dt} + R\frac{d^2\Theta}{dt^2}\right)\mathbf{e}_\theta = a_r\mathbf{e_r} + a_\theta\mathbf{e}_\theta. \tag{7}$$

The plane polar components of the acceleration are therefore

$$a_r = \frac{d^2R}{dt^2} - R\left(\frac{d\Theta}{dt}\right)^2 \tag{8}$$

and

$$a_\theta = R\frac{d^2\Theta}{dt^2} + 2\frac{dR}{dt}\frac{d\Theta}{dt} = \frac{1}{R}\frac{d}{dt}\left(R^2\frac{d\Theta}{dt}\right). \tag{9}$$

1.6.1 *Material lines*

A collection of point particles distributed along a line in a flow defines a material line. Explain why, if the flow is steady, a material line that lies on a streamline at a certain time will remain on the streamline at all times.

Solution (25 points)

In a steady flow, the fluid velocity is independent of time. Since a streamline is tangential to the velocity vector at each point, the streamlines of a steady flow are also independent of time. A material line on a streamline therefore moves along the streamline; that is, it remains on the streamline at all times.

1.6.2 *Rotation of coordinates*

Derive two equations that relate the old coordinates, (x', y'), to the new coordinates, (x, y), and then express them in vector form similar to that shown in equation (1.6.25).

Solution (25 points)

Method 1. Using trigonometry in Fig. 1.6.2, we find that the old coordinates are related to the new coordinates by

$$x' = x\cos\beta + y\sin\beta, \qquad y' = -x\sin\beta + y\cos\beta. \tag{10}$$

The old coordinates are related to the new coordinates in vector form as

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}. \tag{11}$$

Method 2. Start with Eqn. (1.6.25), expression of $\mathbf{x}$ in terms of $\mathbf{x}'$, $\mathbf{x} = \mathbf{x}' \cdot \mathbf{R}$,

$$\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}. \tag{12}$$

To get the expression of $\mathbf{x}'$ in terms of $\mathbf{x}$, multiply both sides of this equation by the inverse matrix $\mathbf{R^{-1}}$,

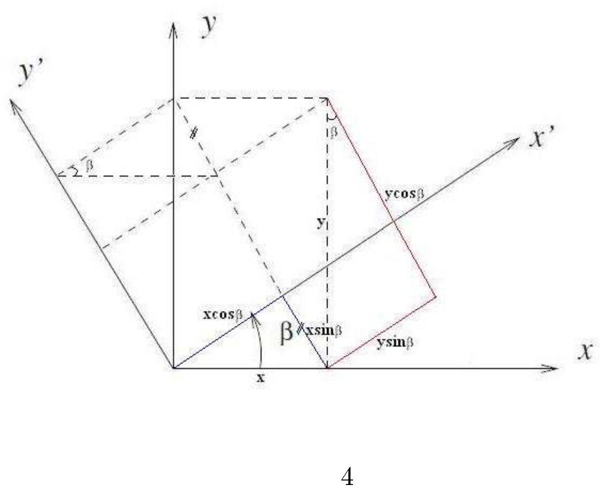
$$\mathbf{x} \cdot \mathbf{R^{-1}} = \mathbf{x}' \cdot \mathbf{R} \cdot \mathbf{R^{-1}} = \mathbf{x}' \cdot \left(\mathbf{R} \cdot \mathbf{R^{-1}}\right) = \mathbf{x}' \cdot \mathbf{I} = \mathbf{x}'. \tag{13}$$

The inverse matrix is

$$\begin{aligned} \mathbf{R^{-1}} &= \frac{1}{\cos\beta\cos\beta - (-\sin\beta)\sin\beta} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}. \end{aligned} \tag{14}$$

Therefore,

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}. \tag{15}$$



(5)

1.6.2 *Flow decomposition*

Carry out the decomposition of a two-dimensional flow with velocity components $u_x(x, y, t) = w(t)(2x + 3y)$ and $u_y(x, y, t) = w(t)(-x-2y)$, where $w(t)$ is a given function of time.

Solution

The velocity field can be expressed in vector form as in Eqn. (1.6.38),

$$\begin{bmatrix} u_x & u_y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{16}$$

The four parameters are

$$a = 2w(t), \quad b = -w(t), \quad c = 3w(t), \quad d = -2w(t). \tag{17}$$

The matrix can be decomposed into three constituents as in Eqn. (1.6.39)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a-d & b+c \\ c+b & d-a \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a+d & 0 \\ 0 & a+d \end{bmatrix}. \tag{18}$$

Therefore,

$$\begin{bmatrix} 2w(t) & -w(t) \\ 3w(t) & -2w(t) \end{bmatrix} = \frac{1}{2}w(t) \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} + \frac{1}{2}w(t) \begin{bmatrix} 4 & 2 \\ 2 & -4 \end{bmatrix} + \frac{1}{2}w(t) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{19}$$