A.M.P. Boelens

October 7, 2009

1.1.1 *Nature of a liquid/solid suspension*

Fluids containing suspended particles abound in nature, physiology, and technology. Examples include blood consisting of a dense suspension of red, white, and other blood cells; slurry used in the petroleum industry for the hydrodynamic transport of particulates; toothpaste and dough. Discuss whether a suspension should be classified as a fluid or solid with reference to the volume fraction of the suspended solid phase.

Solution

A suspension is a two-phase system where small particles are dispersed in a liquid. Examples of suspensions are paint and muddy/sandy water. When the volume fraction of the particles is small, the suspension behaves like a liquid. However, as the volume fraction tends to 100%, the suspension behaves more like a solid. For example, it is possible to build a sand castle from a suspension with a high volume fraction of sand particles in water. The water makes the sand grains stick together so that the do not flow.

1.2.1 *A rolling sphere*

A sphere is rolling over a plane following a rectilinear path. How is the velocity at the center of the sphere related to the angular velocity of rotation about the center of the sphere?

Solution

If V is the velocity of the center of the sphere, the velocity at the point of contact is $V_L = V - \Omega a$, where a is the radius. In the absence of slip, $V_L = 0$ and $V = \Omega a$.

In the case of a spinning and slipping stationary sphere, $V = 0$ and $V_L = -\Omega a$.

1.3.1 *Spherical polar coordinates*

Derive the inverse transformation rules shown in equations (1.3.27); that is, express the Cartesian components in terms of the spherical polar components of the velocity.

Solution

We begin by calculating

$$v_y = v_\sigma \cos\varphi - v_\phi \sin\varphi, \tag{1}$$

and

$$v_z = v_\sigma \sin\phi + v_\phi \cos\varphi. \tag{2}$$

The σ component of the velocity can be derived working in the $x\sigma$ plane where the vectors v_x and v_σ are orthogonal. The x component of the velocity is

$$v_x = v_r \cos\theta - v_\theta \sin\theta, \tag{3}$$

and the σ component of the velocity is

$$v_\sigma = v_r \sin\theta + v_\theta \cos\theta. \tag{4}$$

Substituting the expression for v_σ in the equations for v_y and v_z gives the following expressions for the Cartesian components in terms of spherical components,

$$v_x = v_r \cos\theta - v_\theta \sin\theta, \tag{5}$$

$$v_y = (v_r \sin\theta + v_\theta \cos\theta)\cos\varphi - v_\varphi \sin\varphi, \tag{6}$$

$$v_z = (v_r \sin\theta + v_\theta \cos\theta)\sin\varphi + v_\varphi \cos\varphi. \tag{7}$$

1.4.2 *Streamline patterns*

Sketch streamline patterns for (a) a two-dimensional flow, (b) a swirling flow, (c) an axisymmetric flow, and (d) an axisymmetric flow with swirling motion.

Solution

(a) Any streamline will do assuming that it satisfies the conditions discussed in section 1.4.1.
(b) In a swirling flow, all velocity components are zero except v_φ, and the streamlines are circles.
(c) In an axisymmetric flow, the velocity component v_φ is zero and the streamlines lie in meridional planes.
(d) The streamlines of a swirling axisymmetric flow spiral around the x axis.

1.5.1 *Streamlines by analytical integration*

Consider a steady two-dimensional flow with velocity components

$$u_x = ax + by, \qquad u_y = bx - ay. \tag{8}$$

Deduce the dimensions of the constants a and b. Derive analytical expressions for the position of a point particle.

Solution

Since ax and by represent velocity and x and y are distances, the dimension of a and b is inverse time.

To describe particle paths, the following two first-order ordinary differential equations must be solved,

$$\dot{X}_1 = aX_1 + bX_2, \qquad \dot{X}_2 = bX_1 - aX_2, \tag{9}$$

where a dot denotes a derivative with respect to time, $X_1 = X$ and $Y_1 = Y$. In matrix notation,

$$\dot{\mathbf{X}} = \mathbf{A} \cdot \mathbf{X}. \tag{10}$$

Assuming an exponential solution,

$$\mathbf{X} = \mathbf{c} \exp(\lambda t), \tag{11}$$

we find

$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{c} = \mathbf{0}, \tag{12}$$

where $\mathbf{c}$ is a constant vector. This shows that λ is an eigenvalue of the matrix $\mathbf{A}$, and $\mathbf{c}$ is the corresponding eigenvector. To find the eigenvalues, we set the determinant of the matrix $\mathbf{A} - \lambda \mathbf{I}$ to zero,

$$\det \begin{bmatrix} a - \lambda & b \\ b & -(a + \lambda) \end{bmatrix} = 0, \tag{13}$$

and compute

$$\lambda_1 = \sqrt{a^2 + b^2}, \qquad \lambda_2 = -\sqrt{a^2 + b^2}. \tag{14}$$

Substituting these values in the equation

$$\begin{bmatrix} a-\lambda & b \\ b & -(a+\lambda) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{15}$$

we obtain the eigenvectors

$$\mathbf{c}^1 = \begin{bmatrix} 1 \\ \frac{\sqrt{a^2+b^2}-a}{b} \end{bmatrix} \tag{16}$$

and

$$\mathbf{c}^2 = \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}. \tag{17}$$

The general solution is

$$\mathbf{X} = \alpha_1 \mathbf{c}^1 e^{\lambda_1 t} + \alpha_2 \mathbf{c}^2 e^{\lambda_2 t}. \tag{18}$$

At $t = 0$,

$$\mathbf{X}_0 = \alpha_1 \mathbf{c}^1 + \alpha_2 \mathbf{c}^2. \tag{19}$$

This equation provides us with a system of two equations for the two unknowns α_1 and α_2.