Random Fields:

Recall the sample space $\Omega$ and the fact that a random variable $X$ takes elements of $\Omega$ to the real line:

$$X(\omega) \in \mathbb{R} \quad X: \Omega \rightarrow \mathbb{R}$$

A random field $\psi$ maps elements of $\Omega \times \mathbb{R}$ into a class of functions:

$$\psi(\omega, x) \in \mathcal{C}^1$$

or can imagine a die with a picture of a function on each side.

We will usually write $H(x; \omega) = H(x) \in \mathbb{R}$

if $x \rightarrow x \in \mathbb{R}$ then we have a random process, a random field defined on a 1D domain.

We will deal here mostly with scalar random fields, but vector random fields $H(x) \in \mathbb{R}^n$ are possible as well.
We characterize a random field in the following ways.

1. Marginal distribution relates to the idea that \( H(x) \) is a random variable.

\[
f_{H, x}(h; x) = \frac{d}{dh} F_{H, x}(h; x)
\]

Note: here in position, and \( H(x) \) is a random variable.

\[
F_{H, x}(h; x) = P(H(x) \leq h)
\]

2. Finite dimensional distributions

\[
f_{H, x_1, \ldots, x_n}(h_1, x_1, \ldots, x_n) = \frac{\partial}{\partial h_1} \cdots \frac{\partial}{\partial h_n} F_{H, x_1, \ldots, x_n}(h_1, \ldots, h_n; x_1, \ldots, x_n)
\]

with \([H(x_1), \ldots, H(x_n)]\) a random vector in \(\mathbb{R}^n\).

\[
F_{H, x_1, \ldots, x_n}(h_1, \ldots, h_n; x_1, \ldots, x_n) = P(H(x) \leq h_1, \ldots, H(x_n) \leq h_n)
\]

The set of all fields is the only complete description of the random field, but is obviously impossible to obtain.

3. Mean function

\[
M_n(x) = E[H(x)]
\]
Covariance function (auto-covariance function)

\[ C_{HH}(x, y) = E[(H(x) - M_H(x))(H(y) - M_H(y))] \]

You can see that this is simply the covariance of the two random variables \( H(x), H(y) \).

Correlation \( R_{HH}(x, y) \) and coefficient of correlation \( \rho_{HH}(x, y) \) functions can be defined by incorporating the mean and variance appropriately.

**Exercise:** Write expression for \( R_{HH}(x, y) \) and \( \rho_{HH}(x, y) \) in terms of \( \mu_H(y) \) and \( \sigma_H^2(x) \).

A field is said to be homogeneous isotropic weakly or 2nd moment if

\[ C_{HH}(x, y) = C_{HH}(|x - y|) = C_{HH}(z) \]

If the field is also isotropic, then

\[ C_{HH}(x, y) = C_{HH}(|y - x|) \]

for a random process, isotropy has no meaning since \( z = y - x \) is scalar.
a field can be said to be stationary if

\[ f_{\mathbf{h}, \mathbf{x}_1 \ldots \mathbf{x}_n}(h_1, \ldots, h_n; x_1, \ldots, x_n) = f_{\mathbf{h}, \mathbf{x}_1 \ldots \mathbf{x}_n}(h_1, \ldots, h_n; x_1 + z, \ldots, x_n + z) \quad \forall z. \]

Example: Develop an example for the SCL/PSL data to show anisotropy correlation. (Data stored in the \texttt{data} directory)

Exercise: Consider

\[ X(t) = A \sin \omega t + B \cos \omega t \]

\[ A \sim N(0, 1) \quad B \sim N(0, 1) \]

1. Write MATLAB code to generate samples of the process.
2. Calculate the covariance function of \(X(t)\)
3. Prove that \(X(t)\) is mean square stationary.

Properties of \(C_{HH}\)

1. Symmetric: \(C_{HH}(x, y) = C_{HH}(y, x)\)

2. \(\mathbb{R}\) d.f. essentially

\[ C_{HH} \text{ with } [C_{HH}]_{ij} = C_{HH}(x_i, x_j) \]

\(\mathbb{P}\) d.f. for any \(\{x_1, \ldots, x_n\}\)
The spectral density contains the same information as the covariance and is defined for process $H(x)$

and the spectral density and correlation are Fourier pairs

$$R_{HH}(\tau) = \int_{-\infty}^{\infty} S_{HH}(\omega) e^{i\omega \tau} d\omega$$

$$S_{HH}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{HH}(\tau) e^{-i\omega \tau} d\tau$$

Note: $\mu_H \neq 0$

$$S_{HH}(\omega) = \mu_H^2 S(\omega) + \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{HH}(\tau) e^{i\omega \tau} d\tau$$

In engineering, we often use the one-sided spectral density

$$S_{HH}(\omega) = \mu_H^2 S(\omega) + 2\overline{S_{HH}(\omega)}$$

Here $\omega \geq 0$ is the frequency content.

Projection:

$$S_{HH}(\omega), G_{HH}(\omega) \geq 0$$

$$S_{HH}(-\omega) = S_{HH}(\omega)$$
An alternative form of the Fourier transform:

\[ R_{HH}(\xi) = \int_0^\infty G_{HH}(\omega) \cos(\omega \xi) \, d\omega \]

\[ G_{HH}(\xi) = \frac{1}{\pi} \int_0^\infty R_{HH}(\xi) \cos(\omega \xi) \, d\xi \]

It follows from either Fourier transform expression that

\[ R_{HH}(0) = \int_0^\infty S_{HH}(\omega) \, d\omega \]

\[ = \int_0^\infty G_{HH}(\omega) \, d\omega \]

\[ \therefore m_{HH} = 0 \quad R_{HH}(0) = \text{Var}[H(x)] \]

Exercise: Prove that the two versions of the FT are equivalent. Hint: use Euler's rule for \( e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \).
Exercise: Write math code to numerically integrate to do $R_{hh}(\tau) \rightarrow G_{hh}(\omega)$.

See pair of $G_{hh}(\omega)$, $R_{mm}(\omega)$.

\[
\begin{align*}
\Re & \quad \Im \\
\prod_{\omega} \mathcal{F}(\tau) & \quad q_0 \sin(\omega \tau) \\
\int_0^\infty \frac{\sin(\omega \tau)}{\tau} & \quad \begin{cases} 
q_0 & 0 < \omega \leq \omega_c \\
0 & \omega_c < \omega \\
0 & \omega > \omega_b
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\frac{q_0(\omega_b - \omega_e) \sin(\rho)(\cos(\delta))}{\rho} & \quad \begin{cases} 
q_0 & \omega_b < \omega \leq \omega_b \\
0 & \omega > \omega_b
\end{cases}
\end{align*}
\]

\[
\begin{align*}
s^2 (1 - \frac{1}{\Delta}) & \quad \begin{cases} 
s^2 \Delta \sin \frac{\omega \Delta}{2} & 0 < \Delta \\
0 & \Delta > \Delta
\end{cases}
\end{align*}
\]

\[
\sigma^2 \exp(-\lambda |\xi|) & \quad \frac{2\sigma^2 \lambda}{\pi (\omega^2 + \lambda^2)}
\]

\[
\sigma^2 \exp(-\lambda |\xi|)(1 + \lambda |\xi|) & \quad \frac{4\sigma^2 \lambda^3}{\pi (\omega^2 + \lambda^2)^2}
\]

Exercise: Prove, $X(t) = \sqrt{1-p} W_1(t) + \sqrt{p} W_b(t)$.

$W_1 \sim (0, C_{W_1}(\tau))$  $W_b \sim (0, C_{W_b}(\tau))$  $\tau \rightarrow \tau$

$C_{XX}(t, s) \rightarrow C_{XX}(\tau) = \mathbb{E} (1-p) C_{W_1}(\tau) + p C_{W_b}(\tau)$
Simulation by Spectral Representation

Consider \( X(t) \sim (0, \mathbf{C}_x, \mathbf{S}_x) \)

let \( \omega^* > 0 \) be an upper cut-off frequency

let \( \omega_r = \frac{(r - \frac{R}{2}) \omega^*}{n} \) be a set of equally spaced \((\Delta \omega)\) frequencies.

also, let \( A_r \sim N(0, \frac{\sigma^2}{\sigma^2_{\omega}} G_{xx}(\omega_r) \Delta \omega) \) if \( \Delta \omega \) small

\( B_r \sim N(0, G_{xx}(\omega_r) \Delta \omega) \)

\( A_r, B_r \) independent.

\[
X_n(t) = \sum_{r=1}^{n} \frac{1}{\sigma_r} \left[ A_r \cos \omega_r t + B_r \sin \omega_r t \right]
\]

\[
= \sum_{r=1}^{n} \frac{1}{\sigma_r} \left[ \tilde{A}_r \cos \omega_r t + \tilde{B}_r \sin \omega_r t \right]
\]

\[
\tilde{A}_r = \frac{A_r}{\sigma_r} \quad \tilde{B}_r = \frac{B_r}{\sigma_r}
\]

as the project \( \lim_{\omega^* \to \infty} X_n(t) = X(t) \).

Note: 1. \( X_n(t) \) is periodic: \( T = \frac{2\pi}{\omega} \), which is undesirable.

MATLAB Examples.