Matrix Algebra for Structural Analysis

A.1 Basic Definitions and Types of Matrices

With the recent accessibility of microcomputers, application of matrix algebra for the analysis of structures has become widespread. Matrix algebra provides an appropriate tool for this analysis, since it is relatively easy to formulate the solution in a concise form and then perform the actual matrix manipulations using a computer. For this reason it is important that the structural engineer be familiar with the fundamental operations of this type of mathematics.

**Matrix.** A matrix is a rectangular arrangement of numbers having $m$ rows and $n$ columns. The numbers, which are called elements, are assembled within brackets. For example, the $A$ matrix is written as:

$$
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

Such a matrix is said to have an order of $m \times n$ ($m$ by $n$). Notice that the first subscript for an element denotes its row position and the second subscript denotes its column position. In general, then, $a_{ij}$ is the element located in the $i$th row and $j$th column.
**Row Matrix.** If the matrix consists only of elements in a single row, it is called a row matrix. For example, a $1 \times n$ row matrix is written as

$$A = [a_1 \ a_2 \ \cdots \ a_n]$$

Here only a single subscript is used to denote an element, since the row subscript is always understood to be equal to 1, that is, $a_1 = a_{11}$, $a_2 = a_{12}$, and so on.

**Column Matrix.** A matrix with elements stacked in a single column is called a column matrix. The $m \times 1$ column matrix is

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

Here the subscript notation symbolizes $a_1 = a_{11}$, $a_2 = a_{21}$, and so on.

**Square Matrix.** When the number of rows in a matrix equals the number of columns, the matrix is referred to as a square matrix. An $n \times n$ square matrix would be

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

**Diagonal Matrix.** When all the elements of a square matrix are zero except along the main diagonal, running down from left to right, the matrix is called a diagonal matrix. For example,

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$
Unit or Identity Matrix. The unit or identity matrix is a diagonal matrix with all the diagonal elements equal to unity. For example,

\[
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Symmetric Matrix. A square matrix is symmetric provided \(a_{ij} = a_{ji}\). For example,

\[
A = \begin{bmatrix}
3 & 5 & 2 \\
5 & -1 & 4 \\
2 & 4 & 8
\end{bmatrix}
\]

A.2 Matrix Operations

Equality of Matrices. Matrices \(A\) and \(B\) are said to be equal if they are of the same order and each of their corresponding elements are equal, that is, \(a_{ij} = b_{ij}\). For example, if

\[
A = \begin{bmatrix}
2 & 6 \\
4 & -3
\end{bmatrix} \quad B = \begin{bmatrix}
2 & 6 \\
4 & -3
\end{bmatrix}
\]

then \(A = B\).

Addition and Subtraction of Matrices. Two matrices can be added together or subtracted from one another if they are of the same order. The result is obtained by adding or subtracting the corresponding elements. For example, if

\[
A = \begin{bmatrix}
6 & 7 \\
2 & -1
\end{bmatrix} \quad B = \begin{bmatrix}
-5 & 8 \\
1 & 4
\end{bmatrix}
\]

then

\[
A + B = \begin{bmatrix}
1 & 15 \\
3 & 3
\end{bmatrix} \quad A - B = \begin{bmatrix}
11 & -1 \\
1 & -5
\end{bmatrix}
\]

Multiplication by a Scalar. When a matrix is multiplied by a scalar, each element of the matrix is multiplied by the scalar. For example, if

\[
A = \begin{bmatrix}
4 & 1 \\
6 & -2
\end{bmatrix} \quad k = -6
\]

then

\[
kA = \begin{bmatrix}
-24 & -6 \\
-36 & 12
\end{bmatrix}
\]
Matrix Multiplication. Two matrices $A$ and $B$ can be multiplied together only if they are conformable. This condition is satisfied if the number of columns in $A$ equals the number of rows in $B$. For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

then $AB$ can be determined since $A$ has two columns and $B$ has two rows. Notice, however, that $BA$ is not possible. Why?

If matrix $A$ having an order of $(m \times n)$ is multiplied by matrix $B$ having an order of $(n \times q)$ it will yield a matrix $C$ having an order of $(m \times q)$, that is,

$$A \quad B = \quad C \quad \quad (A-1)$$

$$(m \times n)(n \times q) \quad (m \times q)$$

The elements of matrix $C$ are found using the elements $a_{ij}$ of $A$ and $b_{ij}$ of $B$ as follows:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad \quad (A-2)$$

The methodology of this formula can be explained by a few simple examples. Consider

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -1 & 6 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix}$$

By inspection, the product $C = AB$ is possible since the matrices are conformable, that is, $A$ has three columns and $B$ has three rows. By Eq. A-1, the multiplication will yield matrix $C$ having two rows and one column. The results are obtained as follows:

$c_{11}$: Multiply the elements in the first row of $A$ by corresponding elements in the column of $B$ and add the results; that is,

$$c_{11} = c_1 = 2(2) + 4(6) + 3(7) = 49$$

$c_{21}$: Multiply the elements in the second row of $A$ by corresponding elements in the column of $B$ and add the results; that is,

$$c_{21} = c_2 = -1(2) + 6(6) + 1(7) = 41$$

Thus

$$C = \begin{bmatrix} 49 \\ 41 \end{bmatrix}$$
As a second example, consider

\[
A = \begin{bmatrix}
5 & 3 \\
4 & 1 \\
-2 & 8
\end{bmatrix}
\quad B = \begin{bmatrix}
2 & 7 \\
-3 & 4
\end{bmatrix}
\]

Here again the product \( C = AB \) can be found since \( A \) has two columns and \( B \) has two rows. The resulting matrix \( C \) will have three rows and two columns. The elements are obtained as follows:

- \( c_{11} = 5(2) + 3(-3) = 1 \) (first row of \( A \) times first column of \( B \))
- \( c_{12} = 5(7) + 3(4) = 47 \) (first row of \( A \) times second column of \( B \))
- \( c_{21} = 4(2) + 1(-3) = 5 \) (second row of \( A \) times first column of \( B \))
- \( c_{22} = 4(7) + 1(4) = 32 \) (second row of \( A \) times second column of \( B \))
- \( c_{31} = -2(2) + 8(-3) = -28 \) (third row of \( A \) times first column of \( B \))
- \( c_{32} = -2(7) + 8(4) = 18 \) (third row of \( A \) times second column of \( B \))

The scheme for multiplication follows application of Eq. A–2. Thus,

\[
C = \begin{bmatrix}
1 & 47 \\
5 & 32 \\
-28 & 18
\end{bmatrix}
\]

Notice also that \( BA \) does not exist, since written in this manner the matrices are nonconformable.

The following rules apply to matrix multiplication.

1. In general the product of two matrices is not commutative:

\[
AB \neq BA
\]  \hspace{1cm} (A–3)

2. The distributive law is valid:

\[
A(B + C) = AB + AC
\]  \hspace{1cm} (A–4)

3. The associative law is valid:

\[
A(BC) = (AB)C
\]  \hspace{1cm} (A–5)
**Transposed Matrix.** A matrix may be transposed by interchanging its rows and columns. For example, if

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix} \quad B = [b_1 \ b_2 \ b_3]
\]

Then

\[
A^T = \begin{bmatrix}
    a_{11} & a_{21} & a_{31} \\
    a_{12} & a_{22} & a_{32} \\
    a_{13} & a_{23} & a_{33}
\end{bmatrix} \quad B^T = \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix}
\]

Notice that \(AB\) is nonconformable and so the product does not exist. (\(A\) has three columns and \(B\) has one row.) Alternatively, multiplication \(AB^T\) is possible since here the matrices are conformable (\(A\) has three columns and \(B^T\) has three rows). The following properties for transposed matrices hold:

\[
(A + B)^T = A^T + B^T \quad (A-6)
\]
\[
(kA)^T = kA^T \quad (A-7)
\]
\[
(AB)^T = B^TA^T \quad (A-8)
\]

This last identity will be illustrated by example. If

\[
A = \begin{bmatrix}
    6 \\
    1 \\
    -3
\end{bmatrix} \quad B = \begin{bmatrix}
    4 & 3 \\
    2 & 5
\end{bmatrix}
\]

Then, by Eq. A-8,

\[
\begin{bmatrix}
    6 & 2 \\
    1 & -3
\end{bmatrix}^T \begin{bmatrix}
    4 & 3 \\
    2 & 5
\end{bmatrix} = \begin{bmatrix}
    6 & 1 \\
    3 & 5 \\
    2 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
    28 & 28 \\
    -2 & -12
\end{bmatrix}^T = \begin{bmatrix}
    28 & -2 \\
    28 & -12
\end{bmatrix}
\]

\[
\begin{bmatrix}
    28 & -2 \\
    28 & -12
\end{bmatrix} = \begin{bmatrix}
    28 & -2 \\
    28 & -12
\end{bmatrix}
\]

**Matrix Partitioning.** A matrix can be subdivided into submatrices by partitioning. For example,

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix} = \begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}
\]

Here the submatrices are

\[
A_{11} = [a_{11}] \quad A_{12} = [a_{12} \ a_{13} \ a_{14}]
\]

\[
A_{21} = \begin{bmatrix}
    a_{21} \\
    a_{31}
\end{bmatrix} \quad A_{22} = \begin{bmatrix}
    a_{22} & a_{23} & a_{24} \\
    a_{32} & a_{33} & a_{34}
\end{bmatrix}
\]
The rules of matrix algebra apply to partitioned matrices provided the partitioning is conformable. For example, corresponding submatrices of $A$ and $B$ can be added or subtracted provided they have an equal number of rows and columns. Likewise, matrix multiplication is possible provided the respective number of columns and rows of both $A$ and $B$ and their submatrices are equal. For instance, if

$$A = \begin{bmatrix} 4 & 1 & -1 \\ -2 & 0 & -5 \\ 6 & 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 8 \\ 7 & 4 \end{bmatrix}$$

then the product $AB$ exists, since the number of columns of $A$ equals the number of rows of $B$ (three). Likewise, the partitioned matrices are conformable for multiplication since $A$ is partitioned into two columns and $B$ is partitioned into two rows, that is,

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

Multiplication of the submatrices yields

$$A_{11}B_{11} = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -4 & 2 \end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix} -1 \\ -5 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ -35 & -20 \end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix} 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 18 \end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix} 8 & 4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 & 4 \end{bmatrix} = \begin{bmatrix} 56 & 32 \end{bmatrix}$$

Thus,

$$AB = \begin{bmatrix} 8 & 4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -7 & -4 \\ -35 & -20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -39 & -18 \end{bmatrix}$$

$$[12 & 18] + [56 & 32]$$
A.3 Determinants

In the next section we will discuss how to invert a matrix. Since this operation requires an evaluation of the determinant of the matrix, we will now discuss some of the basic properties of determinants.

A determinant is a square array of numbers enclosed within vertical bars. For example, an $n$th-order determinant, having $n$ rows and $n$ columns, is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (A-9)$$

Evaluation of this determinant leads to a single numerical value which can be determined using Laplace's expansion. This method makes use of the determinant's minors and cofactors. Specifically, each element $a_{ij}$ of a determinant of $n$th order has a minor $M_{ij}$ which is a determinant of order $n-1$. This determinant (minor) remains when the $i$th row and $j$th column in which the $a_{ij}$ element is contained is canceled out. If the minor is multiplied by $(-1)^{i+j}$ it is called the cofactor of $a_{ij}$ and is denoted as

$$C_{ij} = (-1)^{i+j}M_{ij} \quad (A-10)$$

For example, consider the third-order determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The cofactors for the elements in the first row are

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Laplace's expansion for a determinant of order $n$, Eq. A-9, states that the numerical value represented by the determinant is equal to the sum of the products of the elements of any row or column and their respective cofactors, i.e.,

$$D = a_{1i}C_{i1} + a_{12}C_{12} + \cdots + a_{1n}C_{in} \quad (i = 1, 2, \ldots, n)$$

or

$$D = a_{ij}C_{ij} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (j = 1, 2, \ldots, n)$$
For application, it is seen that due to the cofactors the number \( D \) is defined in terms of \( n \) determinants (cofactors) of order \( n - 1 \) each. These determinants can each be reevaluated using the same formula, whereby one must then evaluate \((n - 1)\) determinants of order \((n - 2)\), and so on. The process of evaluation continues until the remaining determinants to be evaluated reduce to the second order, whereby the cofactors of the elements are single elements of \( D \). Consider, for example, the following second-order determinant

\[
D = \begin{vmatrix}
3 & 5 \\
-1 & 2
\end{vmatrix}
\]

(A-9)

We can evaluate \( D \) along the top row of elements, which yields

\[
D = 3(-1)^{1+1}(2) + 5(-1)^{1+2}(-1) = 11
\]

Or, for example, using the second column of elements, we have

\[
D = 5(-1)^{1+2}(-1) + 2(-1)^{2+2}(3) = 11
\]

Rather than using Eqs. A–11, it is perhaps easier to realize that the evaluation of a second-order determinant can be performed by multiplying the elements of the diagonal, from top left down to right, and subtract from this the product of the elements from top right down to left, i.e., follow the arrow,

\[
D = \begin{vmatrix}
3 & 5 \\
-1 & 2
\end{vmatrix} = 3(2) - 5(-1) = 11
\]

Consider next the third-order determinant

\[
|D| = \begin{vmatrix}
1 & 3 & -1 \\
4 & 2 & 6 \\
-1 & 0 & 2
\end{vmatrix}
\]

(A–10)

Using Eq. A–11, we can evaluate \(|D|\) using the elements along the top row, which yields

\[
D = (1)(-1)^{1+1}[2 & 6 \\
0 & 2] + (3)(-1)^{1+2}[4 & 6 \\
-1 & 2] + (-1)(-1)^{1+3}[4 & 2 \\
-1 & 0]
\]

\[
= 1(4 - 0) - 3(8 + 6) - 1(0 + 2) = -40
\]

It is also possible to evaluate \(|D|\) using the elements along the first column, i.e.,

\[
D = 1(-1)^{1+1}[2 & 6 \\
0 & 2] + 4(-1)^{2+1}[3 & -1 \\
0 & 2] + (-1)(-1)^{3+1}[3 & -1 \\
2 & 6]
\]

\[
= 1(4 - 0) - 4(6 - 0) - 1(18 + 2) = -40
\]

As an exercise try to evaluate \(|D|\) using the elements along the second row.
A.4 Inverse of a Matrix

Consider the following set of three linear equations:
\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3
\end{align*}
\]
which can be written in matrix form as
\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
= \begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{bmatrix} \tag{A-12}
\]
or
\[
Ax = C \tag{A-13}
\]
One would think that a solution for \( x \) could be determined by dividing \( C \) by \( A \); however, division is not possible in matrix algebra. Instead, one multiplies by the inverse of the matrix. The inverse of the matrix \( A \) is another matrix of the same order and symbolically written as \( A^{-1} \). It has the following property,
\[
AA^{-1} = A^{-1}A = I
\]
where \( I \) is an identity matrix. Multiplying both sides of Eq. A–13 by \( A^{-1} \), we obtain
\[
A^{-1}Ax = A^{-1}C
\]
Since \( A^{-1}Ax = Ix = x \), we have
\[
x = A^{-1}C \tag{A-14}
\]
Provided \( A^{-1} \) can be obtained, a solution for \( x \) is possible.

For hand calculation the method used to formulate \( A^{-1} \) can be developed using Cramer's rule. The development will not be given here; instead, only the results are given.* In this regard, the elements in the matrices of Eq. A–14 can be written as
\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
= \frac{1}{|A|}
\begin{bmatrix}
    C_{11} & C_{21} & C_{31} \\
    C_{12} & C_{22} & C_{32} \\
    C_{13} & C_{23} & C_{33}
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{bmatrix} \tag{A-15}
\]

Here $|A|$ is an evaluation of the determinant of the coefficients of $A$, which is determined using the Laplace expansion discussed in Sec. A.3. The square matrix containing the cofactors $C_{ij}$ is called the adjoint matrix. By comparison it can be seen that the inverse matrix $A^{-1}$ is obtained from $A$ by first replacing each element $a_{ij}$ by its cofactor $|C_{ij}|$, then transposing the resulting matrix, yielding the adjoint matrix, and finally multiplying the adjoint matrix by $1/|A|$.

To illustrate how to obtain $A^{-1}$ numerically, we will consider the solution of the following set of linear equations:

\[
\begin{align*}
    x_1 - x_2 + x_3 &= -1 \\
    -x_1 + x_2 + x_3 &= -1 \\
    x_1 + 2x_2 - 2x_3 &= 5
\end{align*}
\]  

(A-16)

Here

\[
A = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & 1 \\
1 & 2 & -2
\end{bmatrix}
\]

The cofactor matrix for $A$ is

\[
C = \begin{bmatrix}
1 & 1 & -1 & 1 \\
2 & -2 & 1 & -2 \\
-1 & 1 & 1 & -1 \\
2 & -2 & 1 & 2 \\
-1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

Evaluating the determinants the adjoint matrix is

\[
C^T = \begin{bmatrix}
-4 & 0 & -2 \\
-1 & -3 & -2 \\
-3 & -3 & 0
\end{bmatrix}
\]

Since

\[
A = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & 1 \\
1 & 2 & -2
\end{bmatrix} = -6
\]

The inverse of $A$ is, therefore,

\[
A^{-1} = -\frac{1}{6} \begin{bmatrix}
-4 & 0 & -2 \\
-1 & -3 & -2 \\
-3 & -3 & 0
\end{bmatrix}
\]
Solution of Eqs. A–16 yields

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = -\frac{1}{6} \begin{bmatrix}
  -4 & 0 & -2 \\
  -1 & -3 & -2 \\
  -3 & -3 & 0
\end{bmatrix} \begin{bmatrix}
  -1 \\
  -1 \\
  5
\end{bmatrix}
\]

\[
x_1 = \frac{1}{6} \left[ (-4)(-1) + 0(-1) + (-2)(5) \right] = 1
\]

\[
x_2 = \frac{1}{6} \left[ (-1)(-1) + (-3)(-1) + (-2)(5) \right] = 1
\]

\[
x_3 = \frac{1}{6} \left[ (-3)(-1) + (-3)(-1) + (0)(5) \right] = -1
\]

Obviously, the numerical calculations are quite expanded for larger sets of equations. For this reason, computers are used in structural analysis to determine the inverse of matrices.

### A.5 The Gauss Method for Solving Simultaneous Equations

When many simultaneous linear equations have to be solved, the Gauss elimination method may be used because of its numerical efficiency. Application of this method requires solving one of a set of \( n \) equations for an unknown, say \( x_1 \), in terms of all the other unknowns, \( x_2, x_3, \ldots, x_n \). Substituting this so-called pivotal equation into the remaining equations leaves a set of \( n-1 \) equations with \( n-1 \) unknowns. Repeating the process by solving one of these equations for \( x_2 \) in terms of the \( n-2 \) remaining unknowns \( x_3, x_4, \ldots, x_n \) forms the second pivotal equation. This equation is then substituted into the other equations, leaving a set of \( n-3 \) equations with \( n-3 \) unknowns. The process is repeated until one is left with a pivotal equation having one unknown, which is then solved. The other unknowns are then determined by successive back substitution into the other pivotal equations. To improve the accuracy of solution, when developing each pivotal equation one should always select the equation of the set having the largest numerical coefficient for the unknown one is trying to eliminate. The process will now be illustrated by an example.

Solve the following set of equations using Gauss elimination:

\[
-2x_1 + 8x_2 + 2x_3 = 2 \quad \text{(A–17)}
\]

\[
2x_1 - x_2 + x_3 = 2 \quad \text{(A–18)}
\]

\[
4x_1 - 5x_2 + 3x_3 = 4 \quad \text{(A–19)}
\]

We will begin by eliminating \( x_1 \). The largest coefficient of \( x_1 \) is in Eq. A–19; hence, we will take it to be the pivotal equation. Solving for \( x_1 \), we have

\[
x_1 = 1 + 1.25x_2 - 0.75x_3 \quad \text{(A–20)}
\]

Substituting into Eqs. A–17 and A–18 and simplifying yields

\[
2.75x_2 + 1.75x_3 = 2 \quad \text{(A–21)}
\]

\[
1.5x_2 - 0.5x_3 = 0 \quad \text{(A–22)}
\]
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Eigenvalue problems

General definitions:

Consider the system of algebraic equations expressed by

\[ Ax = bx \]  \hspace{1cm} (1)

where \( A \) is an \( n \times n \) square matrix, \( x \) is an \( n \times 1 \) column vector, and \( b \) is a scalar. This system of equations can be rearranged to the form

\[ (A - bI)x = 0 \]  \hspace{1cm} (2)

where \( I \) is the identity matrix and \( 0 \) is the zero vector. For this system of equations to have a non-trivial solution, that is, a solution other than \( x = 0 \), the condition

\[ |A - bI| = 0 \]  \hspace{1cm} (3)

where \( | \cdot | \) is the determinant operator. A matrix for which the determinant is zero is called singular, and does not have an inverse. Thus, \( (A - bI)^{-1} \) does not exist. One way to see that this condition must be met is to see what happens if \( (A - bI)^{-1} \) does exist. If it does, then we can write

\[ (A - bI)^{-1}(A - bI)x = (A - bI)^{-1}0 \]  \hspace{1cm} (4)

\[ x = 0 \]  \hspace{1cm} (5)

implying that if the inverse of \( (A - bI) \) exists, then the trivial solution to Eq. 1 exists. We have established that if Eq. 1 is to have a non-trivial solution, then Eq. 3 must be satisfied. Expanding the determinant in the left hand side of Eq. 3 yields an \( n^{th} \) order polynomial in the scalar \( b \). This polynomial has \( n \) roots \( \{b_1, \ldots, b_n\} \), and these roots are called the eigenvalues of \( A \). The eigenvectors of \( A \) are obtained by making the substitution

\[ (A - b_iI)x_i = 0 \]  \hspace{1cm} (6)

and solving the resulting system of linear equations for the components of the eigenvector \( x_i \). The system of equations in Eq. 6 are underdetermined by one degree, meaning that the eigenvector \( x_i \) can be calculated only up to a constant factor. That is, if \( x_i \) is an eigenvector of \( A \) then so is \( cx_i \) where \( c \) is a constant. It is customary to normalized the eigenvectors so the the length, magnitude, or Euclidean norm of \( x \) is one, that is, the eigenvectors should be written as unit vectors. There are many special cases in which the structure of \( A \) makes it possible to determine some of the solution by inspection, when some of the eigenvalues and eigenvectors coincide, and so on. Here, I provide one simple example to take you through the mechanics of solving an eigenvalue problem.
Example: Let
\[
A = \frac{1}{4} \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}
\]  
(7)
which gives
\[
(A - bI)x = \frac{1}{4} \begin{bmatrix} 5 - 4b & \sqrt{3} \\ \sqrt{3} & 7 - 4b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  
(8)
and
\[
|A - bI| = b^2 - 3b + \left(35 \frac{3}{16} - \frac{3}{16}\right) = 0
\]  
(9)
with roots
\[
b_1 = 2
\]  
(10)
\[
b_2 = 1
\]  
(11)
which are the eigenvalues of $A$. To find the eigenvectors, first substitute $b_1 = 2$ into Eq. 9 to get the pair of equations
\[
\frac{3}{4} x_{1,1} + \frac{\sqrt{3}}{4} x_{1,2} = 0
\]  
(12)
\[
\frac{\sqrt{3}}{4} x_{1,1} - \frac{1}{4} x_{1,2} = 0
\]  
(13)
which have the solution
\[
x_{1,1} = \frac{x_{1,2}}{\sqrt{3}}.
\]  
(14)
Note that both equations yield the same solution. They are linear combinations of one another, and so, this system of two equations in two variables is underdetermined by one degree. Since the system is underdetermined, we are free to choose either of $x_{1,1}$ or $x_{1,2}$ freely. Here we choose $x_{1,2} = \sqrt{3}/4$ so that
\[
x_1 = \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}
\]  
(15)
is a unit vector. To find the second eigenvector, substitute $b_2 = 1$ into Eq. 9 to obtain
\[
\frac{1}{4} x_{2,1} + \frac{\sqrt{3}}{4} x_{2,2} = 0
\]  
(16)
\[
\frac{\sqrt{3}}{4} x_{2,1} - \frac{3}{4} x_{2,2} = 0
\]  
(17)
which again are linear combinations of one another and have the solution
\[
x_{2,1} = -\sqrt{3} x_{2,2}
\]  
(18)
in which we set $x_{2,2} = \frac{1}{2}$ so that

$$x_2 = \frac{1}{2} \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

(19)

is a unit vector.