Final Exam (2 hours)

**Problem 1**: The maximum contact stress and radius of the contact area between two spheres brought into contact by a force $P$ are given by

$$\sigma_c = 1.5 \frac{P}{\pi a_c^2}, \quad a_c = 0.88 \left[ \frac{P(E_1 + E_2)r_1r_2}{E_1E_2(r_1 + r_2)} \right]^{1/3}.$$  

What is the contact stress between two rigid ($E_1 = E_2 = \infty$) elastic moduluspheres of finite radius? Show your work and explain your answer.

**Problem 2**: Propose a displacement field $v(x)$ that includes two trigonometric terms that could be used in a Rayleigh-Ritz solution. Show that your proposed $v(x)$, with appropriate choice of constants, can provide a reasonable displaced shape for the cases $k \to \infty$ and $k \to 0$

**Problem 3**: Consider a thin hollow steel sphere with initial radius $r_s$, thickness $t \ll r_s$, elastic modulus $E$ and Poisson’s ratio $\nu = 0.25$. This sphere is submerged in a fluid of density $\rho$ subject to gravitational acceleration $g$. The state of stress in the sphere can be approximated by

$$\sigma_1 = \sigma_2 = \frac{pr_s}{2t}, \quad \sigma_3 = 0 \quad (1)$$

where $p = \rho g z$ is the pressure at depth $z$ and $\sigma_3$ is in the through thickness direction.

(a) What is the strain energy in the sphere as a function of depth?
(b) Write an expression for the work done by the pressure $p$ in deforming the sphere from initial radius $r_s$ to final radius $r_f$. How would you use this expression to calculate the final radius $r_f$.

**Problem 4**: Consider a bar of steel loaded uniaxially with $E = 29000ksi$, $\sigma_{yp} = 50ksi$, and $E_h = 300ksi$. Assume a bilinear hardening model.

(a) The bar is loaded to 55ksi. What are the elastic and plastic parts of the strain?
(b) Draw the von Mises yield surface in plane stress principal coordinates before and after loading. Assume isotropic hardening.

**Problem 5**: Consider an infinite plate with $K_{Ic} = 100ksi \sqrt{in}$, a center crack, and an applied tensile stress of $\sigma_0$. At what crack length does the crack propagate?

**Problem 6**: Given two plates with identical geometry, loading, and boundary conditions and elastic moduli, but different Poisson’s ratios $\nu_1 = 0, \nu_2 = 0.3$.

(a) Which plate deflects less?
(b) What is the ratio of the smaller to the larger deflection?

**Useful expressions**:

- $K_{\text{center crack}} = \sigma_0 \sqrt{\pi a}$
- von Mises: $(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) = 2\sigma_{yp}^2$
- surface area of a sphere $4\pi r^2$
- Plate bending rigidity $D = Eh^3/(12(1-\nu^2))$
- Strain energy density $U_0 = \frac{1}{2G} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{2G} (\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z) + \frac{1}{2G} (\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)$.