A NEAR WALL MODEL FOR THE DISSIPATION TENSOR

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ABSTRACT

A new model for the tensor dissipation term of the Reynolds stress equations is derived. This model extends classical dissipation models into the near wall region in a mathematically sound and physically appropriate way. Comparisons with direct numerical simulation data confirm the efficacy of the model.

INTRODUCTION

The majority of Reynolds stress equation models are implicitly or explicitly based on the assumption that turbulence is quasi-homogeneous. Many also assume some sort of quasi-isotropy. These assumptions break down dramatically in the presence of walls where turbulence is known to be highly inhomogeneous and anisotropic. Since the near wall region is vital to almost all engineering calculations this situation must be remedied, and the history of turbulence modeling is full of various proposals on how best to alter existing models so that they will perform reasonably in the near wall region.

This brief paper will add to that debate, but focus on the near wall modeling of only one term of the Reynolds stress equations, the tensor dissipation. Using a simple mathematical decomposition an equation will be developed that decomposes the dissipation into a homogeneous term, an inhomogeneous term, and some redistribution terms. It will then be shown that this equation can be used to derive models for the dissipation. These models satisfy a number of important physical and mathematical constraints and give very good agreement with simulation data, particularly in the near wall region.

DISSIPATION TENSOR

The dissipation term of the Reynolds stress equations is responsible for the damping of turbulent intensities. For an incompressible, constant property fluid, it is written in cartesian tensor notation as,

$$\epsilon_{ij} \equiv 2\nu\overline{u_{i,k}u_{j,k}} \tag{1}$$

where u is the fluctuating velocity, ν is the kinematic viscosity, an overbar denotes an ensemble average, and repeated indices imply summation. In the context of Reynolds stress modeling the average quantity on the right hand side is unknown and must be modeled in terms of the mean velocities U_i, and Reynolds stresses $R_{ij} = \overline{u_i u_j}$.

Considerable headway can be made towards this end by rewriting the dissipation equation. The starting point for this transformation, which is purely mathematical in nature, is a decomposition of the fluctuating velocity given by,

$$u_i = \overline{Q}_{ik}\tilde{u}_k. \tag{2}$$

An overbar has been placed on the tensor function $\overline{Q}_{ik}$ to emphasize that this function is a statistical average. Eventually it will be specified in terms of known quantities. The quantity $\tilde{u}_k$ will be referred to as the *rescaled fluctuating velocity.* It contains all the random statistical variations of the original velocity field, and its properties will become clearer when the scaling function $\overline{Q}_{ik}$ is precisely defined.

Substituting this decomposition into the original dissipation definition, and using the fact that $\overline{Q}_{ij}$ is a statistic and can therefore be extracted from any further averaging, results in an expression of the form,

$$\begin{aligned}\frac{\epsilon_{ij}}{2\nu} &= \overline{Q}_{im,k}\left(\overline{\tilde{u}_m\tilde{u}_n}\right)\overline{Q}_{jn,k} + \overline{Q}_{im}(\overline{\tilde{u}_{m,k}\tilde{u}_{n,k}})\overline{Q}_{jn}\\ &+ \frac{1}{2}\left(\overline{Q}_{im,k}(\overline{\tilde{u}_m\tilde{u}_n})_{,k}\overline{Q}_{jn} + \overline{Q}_{im}(\overline{\tilde{u}_m\tilde{u}_n})_{,k}\overline{Q}_{jn,k}\right)\\ &+ \frac{1}{2}\left(\overline{Q}_{im,k}W_{mnk}\overline{Q}_{jn} - \overline{Q}_{im}W_{mnk}\overline{Q}_{jn,k}\right)\end{aligned} \tag{3}$$

where the tensor $W_{mnk} = (\overline{\tilde{u}_m\tilde{u}_{n,k}} - \overline{\tilde{u}_{m,k}\tilde{u}_n})$ is antisymmetric in m and n. Although initially daunting, this equation for the dissipation is quite straightforward. The first term is the contribution to the dissipation due to the inhomogeneity of the turbulence. The second term involves the *renormalized dissipation*, and accounts for the dissipation due to statistical fluctuations. Finally, the last two terms are redistribution terms that will be discussed in more detail within the text.

The first two terms of equation (3) reveal the purpose of this mathematical manipulation of the dissipation equation. By isolating the inhomogeneity of the turbulence in the statistic $\overline{Q}_{ij}$, the inhomogeneous contributions to the dissipation can be explicitly determined. This transformation also leaves the renormalized velocity looking very much like isotropic, homogeneous turbulence. Classical quasi-homogeneous models can then be expected to perform reasonably well for the statistics of the renormalized velocity. In this manner, quasi-homogeneous models can be extended into strongly inhomogeneous regions in a mathematically sound way, and with surprisingly good results. The following section makes this clear.

BASIC DISSIPATION MODEL

The simplest renormalization choice is $\overline{Q}_{ij} = q\delta_{ij}$, where $q = (R_{kk})^{1/2}$. This means that $\overline{\tilde{u}_i \tilde{u}_j} = \frac{R_{ij}}{q^2}$, and the dissipation equation becomes,

$$\frac{\epsilon_{ij}}{2\nu} = \left(\frac{q_{,k}}{q}\right)^2 R_{ij} + \frac{1}{2}\left(q^2\right)_{,k}\left(\frac{R_{ij}}{q^2}\right)_{,k} + q^2 L_{ij}^{-2} \qquad (4)$$

where $L_{ij}^{-2} = \overline{\tilde{u}_{i,k}\tilde{u}_{j,k}}$. Fortuitously the third term of equation (3), involving the unknown antisymmetric tensor W_{mnk} has vanished, and only the symmetric tensor L_{ij}^{-2}, with the units of inverse length squared, must be modeled. In essence, equation (4) abstracts the problem of modeling the dissipation to the problem of modeling the renormalized dissipation, $\overline{\tilde{u}_{i,k}\tilde{u}_{j,k}}$. It is expected that the renormalized dissipation will be much more amenable to quasi-homogeneous dissipation models than the dissipation itself.

Two classical quasi-homogeneous dissipation models are $\epsilon_{ij} = T^{-1}\frac{q^2}{3}\delta_{ij}$ and $\epsilon_{ij} = T^{-1}R_{ij}$ (Rotta, 1951). Usually the inverse time scale is given by $T^{-1} = \frac{\epsilon_{ii}}{q^2}$, which is exact in the homogeneous, isotropic limit. However, this choice for the inverse time scale is incorrect in the near wall region, because it becomes singular. Since the first two terms of equation (4) already account for the near wall region, a much better approximation (one used for the numerical tests of this model), is that the inverse time scale varies only slightly across the thin near wall region and can be approximated by its value at the far edge of that region. With this in mind, the corresponding models for the length scale tensor are $L_{ij}^{-2} = \frac{T^{-1}}{2\nu}\frac{\delta_{ij}}{3}$ and $L_{ij}^{-2} = \frac{T^{-1}}{2\nu}\frac{R_{ij}}{q^2}$, where T^{-1} is found from $\frac{\epsilon_{ii}}{q^2}$ far from the wall.

It is important that models have the correct asymptotic behavior as they approach the wall (Launder & Reynolds, 1983). For instance, at the wall the transverse components of the dissipation (ϵ_{11} and ϵ_{33}) must exactly balance the corresponding diffusion components or turbulence will spuriously be created by the wall. Fortunately, an asymptotic expansion of this model about a no-slip wall shows that the leading coefficient of every component of the model is exact (except the wall normal component ϵ_{22}, which has the right behavior, $O(y^2)$, and can be made exact with the right choice of L_{22}^{-2}). This remarkable behavior is obtained naturally, without resorting to *ad hoc* functions of the wall normal vector or wall normal coordinate. It is due to the fact that the inhomogeneous terms have been explicitly extracted by the model and do not involve any model parameters. These asymptotics will hold as long as the length scale tensor is non-singular.

The model given by equation (4) also satisfies a few basic mathematical constraints. These can best be seen by taking the trace of the model. Since the second term is solely a redistribution term it vanishes, and the following equation is obtained,

$$\frac{\epsilon_{ii}}{2\nu} = (q_{,k})^2 + q^2 L_{ii}^{-2}. \qquad (5)$$

If the length scale is a positive definite tensor, its trace will be positive, and equation (5) then implies that the dissipation will always be positive. In addition, equation (5) can be used in combination with the trace of the diffusion term to prove a mild form of the realizability constraint. That is, the trace of the viscous terms go to zero as the kinetic energy goes to zero. Therefore, the model can not produce negative kinetic energy.

Despite the many attractive mathematical, physical and numerical properties of this model its capabilities are limited. The modeled dissipation *tensor* can not be shown to be positive definite or to satisfy strict realizability. As a result, equation (4) is probably only sufficient for simple flows or simple models (such as k-ϵ). In the next section it is shown that these problems may be overcome by a better choice of the renormalization tensor.

IMPROVED DISSIPATION MODEL

The procedure used for the basic dissipation model can be generalized. Instead of using $Q_{ij} = q\delta_{ij}$ (i.e. the square root of the trace of R_{ij}), the square root of the entire Reynolds stress tensor is used, $Q_{ik}Q_{jk} = R_{ij}$. This decomposition is unique because R_{ij} is a positive definite tensor. With this definition of Q_{ij} it can be shown that $\overline{\tilde{u}_i\tilde{u}_j} = \delta_{ij}$ and the dissipation equation becomes,

$$\begin{aligned}\frac{\epsilon_{ij}}{2\nu} &= \overline{Q}_{im,k}\overline{Q}_{jm,k} + \overline{Q}_{im}L_{mn}^{-2}\overline{Q}_{jn} \\ &+ \frac{1}{2}\left(\overline{Q}_{im,k}W_{mnk}\overline{Q}_{jn} - \overline{Q}_{im}W_{mnk}\overline{Q}_{jn,k}\right).\end{aligned} \qquad (6)$$

Note that the third term of equation (3) vanishes, but the fourth term involving the unknown tensor W_{mnk} remains.

The third term of equation (6), the redistribution term, is particularly interesting. It is zero if the Reynolds stress tensor is either isotropic or homogeneous. In fact it is zero if $Q_{im,k} = \lambda Q_{im}$ where λ is a scaler quantity. This turns out to be the case in spatially decaying turbulence if there is no return to isotropy in the sense of Lumley (1978). So in some sense this term can be thought of as a return to isotropy term.

Further insight can be gained by assuming that the Reynolds stresses are evaluated in their principal coordinates. Then Q_{ij} is a diagonal tensor with $Q_{\alpha\alpha} = R_{\alpha\alpha}^{1/2}$ (here, and throughout the text, no summation is implied for greek indices). In this arrangement the first term of equation (6) only contributes to the diagonal components of the dissipation tensor, and the redistribution term only contributes to the off-diagonal components of the dissipation tensor. So along with the off-diagonal components of the length scale tensor, the redistribution term is responsible for the fact that the dissipation tensor does not have the same principal coordinates as the Reynolds stress tensor. This is a useful property of the model but one which is also burdensome since the tensor W_{mnk} introduces nine new unknowns for which no model (even quasi-homogeneous) now exists.

The term involving W_{mnk} can be eliminated by assuming that the dissipation tensor and Reynolds stress tensors have the same principal directions, (an assumption that was hypothesised to be true at low Reynolds numbers by Rotta (1951), and which has some numerical support in the work of Mansour, et. al. (1988)). This assumption also necessitates that the length scale tensor has the same principal directions as the Reynolds stress tensor. Under these circumstances the model can be written in the principal coordinates as,

$$\frac{\epsilon_{\alpha\alpha}}{2\nu} = \frac{(R_{\alpha\alpha,k})^2}{4R_{\alpha\alpha}} + R_{\alpha\alpha}L_{\alpha\alpha}^{-2}. \tag{7}$$

It can be shown that for every component of this model, the leading (and sometimes higher order) coefficients of an asymptotic expansion about a no-slip wall are exact. This non-trivial result holds irrespective of the model for the length scale, as long as the length scale approaches a constant near the wall. It is a result of the fact that inhomogeneity dominates in the near wall region, and the inhomogeneous term of equation (7) is exact.

This model also satisfies certain strict mathematical constraints. By its construction the model is Galilean and tensor invarient. It is clear from equation (7) that if the length scale tensor is positive definite then the dissipation tensor can also be guaranteed to be positive definite, since each of its principal components must be positive. To prove strict realizability (Shumann, 1977) we again need to incorporate the diffusion tensor, $D_{ij} = \nu R_{ij,kk}$. Then, in principal coordinates,

$$-\epsilon_{\alpha\alpha} + D_{\alpha\alpha} = 2\nu R_{\alpha\alpha}^{1/2} R_{\alpha\alpha,kk}^{1/2} - 2\nu R_{\alpha\alpha}L_{\alpha\alpha} \tag{8}$$

Therefore, whenever a principal stress goes to zero, the corresponding viscous component will also go to zero, and the model will not cause the Reynolds stress tensor to become indefinite.

Numerically this model is not overly complicated. It requires finding the eigenvalues and eigenvectors of the Reynolds stress tensor, constructing the dissipation tensor in principal coordinates, and then using the eigenvectors to transform the dissipation back. The unknown length scale tensor is easily modeled with standard quasi-homogeneous dissipation models, as long as these models are prevented from becoming singular. The performance of the quasi-homogeneous model in strongly inhomogeneous regions is not particularly important since the exact inhomogeneous terms of the model will tend to dominate the solution in those regions.

RESULTS

The two models described above have been numerically tested against dissipation data from direct numerical simulations (Perot, 1992). The flow in question is a shear-free turbulent boundary layer, which is formed by instantaneously placing a wall in isotropic homogeneous turbulence. This temporally developing flow provides a unique environment for testing near wall turbulence models, providing many of the aspects of a turbulent boundary layer without the complications due to shear. It is a flow which is well suited to testing these models because it provides large isotropy and inhomogeneity, but the principal directions of the Reynolds stress tensor remain cartesian. In addition, in the flow far from the wall an exact expression for the inverse time scale is known. In the tests of the models this exact value will be used at all points in the flow including the boundary layer (where it is no longer exact, but is at least not singular). Since the model equations are exact except for the determination of the length scale tensor, these numerical comparisons are principally a test of the quasi-homogeneous model, and the constant inverse time scale hypothesis.

The basic dissipation model that was tested was;

$$\frac{\epsilon_{ij}}{2\nu} = \left(\frac{q_{,k}}{q}\right)^2 R_{ij} + \frac{1}{2}\left(q^2\right)_{,k}\left(\frac{R_{ij}}{q^2}\right)_{,k} + \frac{T^{-1}}{2\nu}R_{ij} \tag{9}$$

and the general dissipation model was;

$$\frac{\epsilon_{\alpha\alpha}}{2\nu} = \frac{R_{\alpha\alpha,k}R_{\alpha\alpha,k}}{4R_{\alpha\alpha}} + \frac{T^{-1}}{2\nu}R_{\alpha\alpha}. \tag{10}$$

In both cases $T^{-1} = \frac{\epsilon_{ii}}{R_{ii}}|_\infty$.

Figure 1. shows a plot of the calculated (symbols) and modeled (lines) dissipation tensor as a function of the distance from the wall. Only the ϵ_{11} and ϵ_{22} components are shown since all the off diagonal components are zero and $\epsilon_{33} = \epsilon_{11}$. This plot is at roughly one large eddy turn-over time after the wall has been inserted, and the microscale Reynolds number is approximately 15. It is evident that the models give very good agreement near the wall, and predict the asymptotic value at the wall correctly. Globally, the error in the improved dissipation model never exceeds 15%.

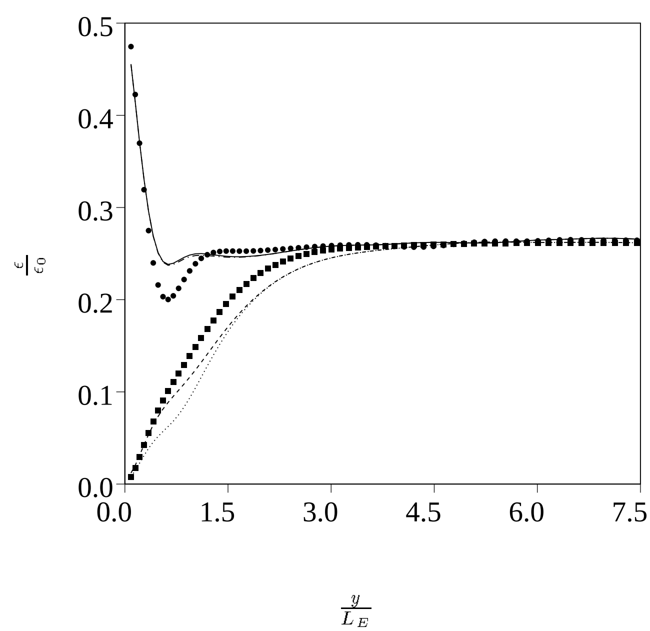

FIGURE 1. The ϵ_{11} and ϵ_{22} components of the dissipation tensor as a function of the distance from the wall. ● , ϵ_{11} calculated; — , ϵ_{11} from improved model; $-\cdot-\cdot$, ϵ_{11} from basic model; ■ , ϵ_{22} calculated; - - - , ϵ_{22} from the improved model; $\cdots$, ϵ_{22} from the basic model; ($T_E = 1$, $Re_\lambda = 15$, $L_E = \frac{q^{3/2}}{\epsilon}$)

The same curves are plotted in figure 2. at almost three large eddy turn-over times after the wall has been inserted, and at a much higher microscale Reynolds number of about 65. The models continue to show very good agreement with the simulation data under these conditions. Further agreement could be achieved by a more complicated expression for the length scale tensor which took into account the changes in length scale as the wall is approached. It is sufficient for the purposes of this paper, however, to demonstrate that even the simplest of length scale models can produce reasonable results.

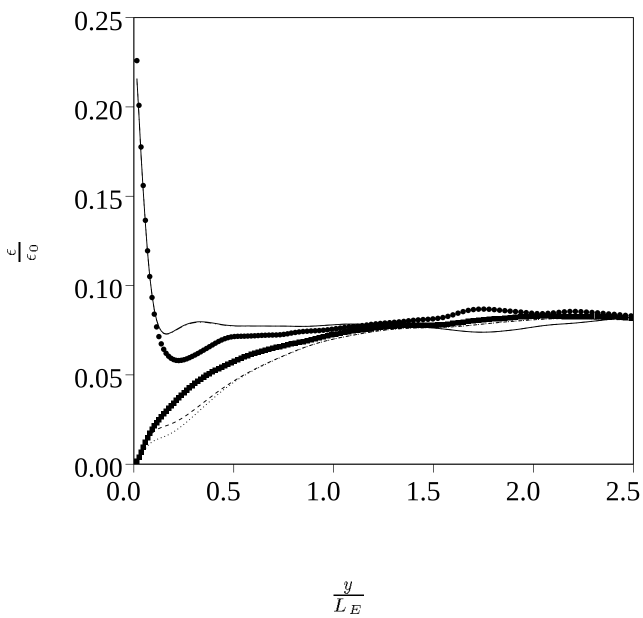

FIGURE 2. The ϵ_{11} and ϵ_{22} components of the dissipation tensor as a function of the distance from the wall. • , ϵ_{11} calculated; — , ϵ_{11} from improved model; $- \cdot - \cdot$, ϵ_{11} from basic model; ▪ , ϵ_{22} calculated; - - - , ϵ_{22} from the improved model; $\cdots$, ϵ_{22} from the basic model; ($T_E = 3$, $Re_\lambda = 65$, $L_E = \frac{q^{3/2}}{\epsilon}$)

Note that in both cases the model tends to underpredict ϵ_{22} and overpredict ϵ_{11}. This discrepancy can not be fixed by a more complicated expression for the inverse time scale, (which was assumed to be a constant). It must be a result of the model for the length scale tensor, which probably should include a return to isotropy term. However, it is hardly worth resolving these small details until equally accurate near wall models for the pressure strain and turbulent transport terms of the Reynolds stress equations are also developed.

SUMMARY

The dissipation model that has been presented satisfies a host of mathematical and physical constraints. These constraints were not prescribed *a priori*, or imposed with *ad hoc* modifications, their satisfaction is simply a consequence of the rigorous way in which the model was developed. The fundamental idea underlying this model is that the length scales associated with the inhomogeneity and the length scales associated with the dissipative fluctuations are vastly different and can therefore be separated. This is the basis for the decomposition $u_i = \overline{Q}_{ik}\tilde{u}_k$.

The model assumes only that a reasonable expression for the quasi-homogeneous renormalized dissipation exists, and that the dissipation and Reynolds stress tensors have the same principal directions (a restriction that could be removed). In addition, a simple form of the model suitable for $k - \epsilon$ type modeling exists (eqn. (5)). Finally, the non-linearity and simplicity of this modeling approach makes it attractive both numerically and aesthetically.

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