Reformulation of the unstructured staggered mesh method as a classic finite volume method

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ABSTRACT *A generalization of the Harlow & Welch (1965) staggered mesh method to two-dimensional unstructured meshes is presented. With certain choices of the interpolation operators, it is shown that this method can be recast as a classic finite volume method using a single set of non-overlapping control volumes and collocated variables. When the divergence form of the Navier-Stokes equations are discretized using the unstructured staggered mesh method the resulting equations are equivalent to a classic finite volume method for the velocity vector. When the rotational form of the Navier-Stokes equations are discretized using the unstructured staggered mesh method the resulting equations are equivalent to a classic finite volume method for the vorticity vector.*

Key Words: unstructured, staggered mesh, reformulation, Navier-Stokes equations.

1. Introduction

The Cartesian staggered mesh method has a number of mathematical properties that make it a popular choice for simulations of incompressible fluids. In particular, the method does not have spurious 'pressure modes' and does not require stabilization or damping terms to control unphysical small-scale pressure fluctuations. In addition, the method is known to conserve mass, momentum, total energy, kinetic energy and vorticity. The latter two conservation properties are not found in generic control volume approaches and are particularly important in direct and large eddy simulations of turbulence where the cascade of turbulent kinetic energy (or enstrophy) from large to small scales (or vice versa) is critical to the overall predictions of the turbulence behavior.

The success of the Cartesian staggered mesh method originally developed by Harlow & Welch [HAR 65] has motivated the search for generalizations of the method to unstructured meshes. While such a generalization is a non-trivial exercise, the unstructured staggered mesh methods of Porsching [AMI 81], and Nicolaides [NIC 93] have demonstrated many of the attractive properties of the Cartesian staggered mesh method by taking advantage of the fact that every unstructured mesh has a locally orthogonal dual mesh – the Voronoi tesselation. Chou [CHO 97] has shown the connection of the unstructured staggered mesh methods to nonconforming finite element methods. In this paper we discuss the direct connection with classic finite volume methods. It is via this connection with classic finite volume methods that the hitherto uninvestigated conservation properties of unstructured staggered mesh methods can be evaluated.

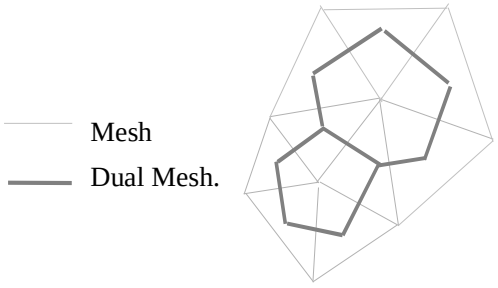

Figure 1. *Two dimensional unstructured mesh and the dual Voronoi mesh.*

2. Analysis of the Divergence Form

The unstructured staggered mesh discretization is simply a way of forming discrete difference operators. It is actually independent of the equations to which it is applied. Hence, different discretizations of the Navier-Stokes equations are possible depending on which form of the equations are discretized. In this section, we will look at unstructured staggered mesh discretizations of the divergence form of the Navier-Stokes equations.

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{uu}) = -\nabla p + \nabla \cdot \nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \qquad [1]$$

Discretizations based on the divergence form of the equations are of interest because they are able to discretely conserve momentum. While momentum conservation is a trivial consequence of a classical finite volume method, it is not an obvious trait of staggered mesh methods. This is because the staggered mesh methods only updates the normal velocity components at cell faces, tangential

velocity components are interpolated not evolved. It will be shown that with certain choices of the interpolation operators, the staggered mesh update of face normal velocities is directly equivalent to a classic finite volume method which updates the velocity vector at cell centers.

2.1 *Discretization of the Divergence Form*

The normal vector at each face is assumed to point from cell C1 to cell C2. At boundary faces the normal vector is assumed to point out of the domain and cell C2 is a virtual cell located at the domain boundary. The discrete equation for the evolution of the normal velocity component is then given by,

$$\begin{aligned} & W_f A_f \frac{u^{n+1} - u^n}{\Delta t} + \mathbf{n}_f \cdot (W_{C1}^f \mathbf{c}_{c1} + W_{C2}^f \mathbf{c}_{c2}) A_f \\ & \quad = -(p_{c2} - p_{c1}) A_f + \mathbf{n}_f \cdot (W_{C1}^f \mathbf{d}_{c1} + W_{C2}^f \mathbf{d}_{c2}) A_f \end{aligned} \qquad [2]$$

where $\mathbf{c}_c = \frac{1}{V_c} \sum_{\text{faces}}^{\text{cell}} \mathbf{u}_f \hat{u} A_f$ is a conservative discretization of the convection term evaluated in each cell, $\mathbf{d}_c = \frac{1}{V_c} \sum_{\text{faces}}^{\text{cell}} \nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \hat{\mathbf{n}}_f A_f$ is a conservative discretization of the diffusion term evaluated in each cell, V_c is the volume of each cell, A_f is the face area, W_f is the distance between neighboring cell circumcenters, and W_C^f is the distance between the face circumcenter and the cell circumcenter. Note that $u = \mathbf{u} \cdot \mathbf{n}_f$ is the normal velocity component at each cell face and $\hat{u}$ is the normal velocity component that points out of a particular cell. Similarly $\hat{\mathbf{n}}_f$ is the normal vector pointing out of a particular cell.

2.2 *Reformulation as a Classic Control Volume Scheme*

The reformulation of the divergence form is accomplished by multiplying each evolution equation for the face normal velocity component (Eqn. [2]), by the face normal vector, and summing over all the faces in the computational domain. This results in the following equation,

$$\begin{aligned} & \sum^{\text{faces}} \mathbf{n}_f W_f A_f \frac{u^{n+1} - u^n}{\Delta t} + \sum^{\text{faces}} \mathbf{n}_f \mathbf{n}_f \cdot (W_{C1}^f \mathbf{c}_{c1} + W_{C2}^f \mathbf{c}_{c2}) A_f \\ & \quad = -\sum^{\text{faces}} \mathbf{n}_f A_f (p_{c2} - p_{c1}) + \sum^{\text{faces}} \mathbf{n}_f \mathbf{n}_f \cdot (W_{C1}^f \mathbf{d}_{c1} + W_{C2}^f \mathbf{d}_{c2}) A_f \end{aligned} \qquad [3]$$

The goal is then to recast this into a form that looks like a summation over control volume cells. Recognizing that $W_f = W_{C1}^f + W_{C2}^f$ and that $W_{C2}^f = 0$ at

boundary faces, and also noting that at boundary faces $p_{c2} = p_f$, [3] can be rewritten as a summation over cells.

$$\sum^{\text{cells}} \sum^{\text{cell faces}} \mathbf{n}_f W_f A_f \frac{u^{n+1} - u^n}{\Delta t} + \sum^{\text{cells}} \mathbf{c}_c \cdot \sum^{\text{cell faces}} \mathbf{n}_f \mathbf{n}_f W_C^f A_f = \sum^{\text{cells}} p_c \sum^{\text{cell faces}} \hat{\mathbf{n}}_f A_f - \sum^{\text{boundary faces}} p_f \hat{\mathbf{n}}_f A_f + \sum^{\text{cells}} \mathbf{d}_c \cdot \sum^{\text{cell faces}} \mathbf{n}_f \mathbf{n}_f W_C^f A_f \quad [4]$$

This can be further simplified using three identities derived from Gauss' divergence theorem. Gauss' Divergence Theorem for an arbitrary bounded volume and a vector quantity **f** is,

$$\int_\Omega \nabla \cdot \mathbf{f} dV = \int_{\partial\Omega} \mathbf{f} \cdot \hat{\mathbf{n}} dA \quad [5]$$

where Ω is the volume and $\partial\Omega$ is the boundary of the volume with unit normal vector $\hat{\mathbf{n}}$ oriented outwards from the volume. We are actually interested in convex polygonal volumes where Gauss' Theorem simplifies to,

$$\int_\Omega \nabla \cdot \mathbf{f} dV = \sum^{\text{cell faces}} \hat{\mathbf{n}}_f \cdot \int_{\partial\Omega_f} \mathbf{f} dA \quad [6]$$

If **f** is a nonzero constant vector then $\sum^{\text{cell faces}} \hat{\mathbf{n}}_f A_f = 0$. If $\mathbf{f} = (\mathbf{x} \cdot \mathbf{a})\mathbf{b}$ where **a** and **b** are nonzero constant vectors and **x** is the position vector with an origin located at the cell circumcenter then it follows from [6] that in two-dimensions $\sum^{\text{cell faces}} \mathbf{n}_f \mathbf{n}_f W_C^f A = V_c \mathbf{I}$ where **I** is the identity matrix. Finally, if $\mathbf{f} = (\mathbf{a} \cdot \mathbf{x})\mathbf{u}$ where **u** is the velocity vector and **a** is an arbitrary nonzero constant vector, then Gauss' Theorem gives,

$$\int_\Omega \nabla \cdot [(\mathbf{a} \cdot \mathbf{x})\mathbf{u}] dV = \sum^{\text{cell faces}} \hat{\mathbf{n}}_f \cdot \int_{\partial\Omega_f} (\mathbf{a} \cdot \mathbf{x})\mathbf{u} dA \quad [7]$$

The gradient of the position vector is the identity matrix ($x_{s,j} = \delta_{sj}$), and since **a** is an arbitrary vector

$$\int_\Omega \mathbf{u} dV + \int_\Omega \mathbf{x}(\nabla \cdot \mathbf{u}) dV = \sum^{\text{cell faces}} \int_{\partial\Omega_f} \hat{\mathbf{u}} \mathbf{x} dA \quad [8]$$

where $\hat{u}$ is the outwards normal component of the velocity at the cell faces. This is an exact equation for polygonal volumes. If we assume that the velocity field **u** is a constant function (a first order approximation), then the second term will be zero and the integrals can be evaluated. In two dimensions, this gives the relation,

$$\mathbf{u}_c V_c = \sum_{\text{faces}}^{\text{cell}} \hat{u}\hat{\mathbf{n}}_f W_C^f A_f = \sum_{\text{faces}}^{\text{cell}} u\mathbf{n}_f W_C^f A_f \qquad [9]$$

The first two identities are geometric and they are exact. The last expression (Eqn. [9]) is really not an identity, it is a first order approximation for the cell velocity vector given the normal velocity components at the cell faces. With these geometric identities and this definition for the cell velocity vector, [4] becomes,

$$\sum_{\text{cells}} V_c \frac{\mathbf{u}_c^{n+1} - \mathbf{u}_c^{n}}{\Delta t} + \sum_{\text{cells}} \mathbf{c}_c V_c = -\sum_{\text{faces}}^{\text{boundary}} p_f \hat{\mathbf{n}}_f A_f + \sum_{\text{cells}} \mathbf{d}_c V_c \qquad [10]$$

This equation is true for a collection of cells, but it is also true for a single mesh cell. The preceding analysis makes no distinction as to the number of cells. Applying the previous definitions for $\mathbf{c}_c$ and $\mathbf{d}_c$ we can therefore write that

$$V_c \frac{\mathbf{u}_c^{n+1} - \mathbf{u}_c^{n}}{\Delta t} + \sum_{\text{faces}}^{\text{cell}} \mathbf{u}_f \hat{u} A_f = -\sum_{\text{faces}}^{\text{cell}} p_f \hat{\mathbf{n}}_f A_f + \sum_{\text{faces}}^{\text{cell}} \nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \hat{\mathbf{n}}_f A_f \qquad [11]$$

This is true for each mesh cell, and has the form of a classic control volume scheme for the velocity vector in the mesh cells. It is important to note however, that despite the apparent similarity there remains a subtle distinction from classic control volume schemes. In the staggered mesh scheme, the normal velocity component u is the primary unknown and $\mathbf{u}_c$ is a derived quantity. In classic control volume schemes, $\mathbf{u}_c$ is the primary velocity unknown and the normal velocity component at faces is derived.

3. Analysis of the Rotational Form

In this section, we will look at unstructured staggered mesh discretizations of the rotational form of the Navier-Stokes equations.

$$\frac{\partial \mathbf{u}}{\partial t} - (\mathbf{u} \times \omega) = -\nabla p^d - \nabla \times (\nu \omega) \qquad [12]$$

where **u** is the velocity vector, ω is the vorticity, $p^d = p + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$ is the specific dynamic pressure, and ν is the kinematic viscosity. This equation assumes that viscosity is constant, but it is otherwise equivalent to other forms of the incompressible Navier-Stokes equations. Variable viscosity diffusion can be still be

represented in rotational form but the extra term (involving second derivatives of viscosity) complicates the analysis unnecessarily.

This particular form of the Navier-Stokes equations is of interest because it appears to be inherently suited to the staggered mesh discretization. The classic staggered mesh method can be rearranged to look like a discretization of [12]. It will be shown that in two dimensions the staggered mesh update of face normal velocities is directly equivalent to a classic finite volume method which updates the vorticity at nodes and where the control volumes are the dual mesh Veronoi polyhedra.

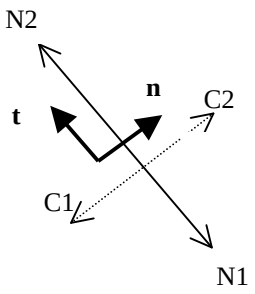

Figure 2. *Notation for a cell face in relation to neighboring cells and nodes.*

3.1 *Discretization of the Rotational Form*

Using the rotational form of the Navier-Stokes equations, the normal component of the face velocity is discretized as,

$$W_f A_f \frac{u^{n+1} - u^n}{\Delta t} - \tfrac{1}{2}(\omega_{n1} v_{n1} + \omega_{n2} v_{n2}) W_f A_f = -(p^d_{c2} - p^d_{c1}) A_f - (v_{n2}\omega_{n2} - v_{n1}\omega_{n1}) W_f \qquad [13]$$

where ω_n is the vorticity at a node in the direction out of the two-dimensional plane. The face tangential points from node N1 to node N2 and is oriented 90 degrees counterclockwise to the face normal vector. The tangential velocity at the nodes in the convection term is given by $v_n = \mathbf{u}_n \cdot \mathbf{t}_f$.

3.2 *Reformulation as a Control Volume for Vorticity*

The reformulation of the rotational form is accomplished by dividing each normal velocity evolution equation (Eqn. [13]) by the face area and then multiplying by –1 if the face normal points clockwise with respect to the node in question, and finally summing over all the faces touching a specific node. The result will be shown to be a control volume equation for the vorticity. In mathematical notation we start with the following equation,

$$\sum_{\text{faces}}^{\text{node}} W_f \frac{u^{n+1} - u^n}{\Delta t} - \sum_{\text{faces}}^{\text{node}} \tfrac{1}{2}(\omega_{n0} v_{n0} + \omega_{ni} v_{ni}) W_f = -\sum_{\text{faces}}^{\text{node}} (p^d_{c2} - p^d_{c1}) - \sum_{\text{faces}}^{\text{node}} (\nu_{n0}\omega_{n0} - \nu_{ni}\omega_{ni}) W_f / A_f \qquad [14]$$

where the normal vector at each face has been chosen to point in a direction counterclockwise with respect to the node in question. In addition, node n0 is the node around which the summation is occurring and node ni is the other node adjoining that face.

In this case, we use a discrete version of Stokes Curl Theorem to simplify the equations. Stokes theorem says that for an arbitrary bounded surface and vector quantity **f**,

$$\int_S (\nabla \times \mathbf{f}) \cdot \mathbf{z} dA = \int_{\partial S} \mathbf{f} \cdot d\mathbf{L} \qquad [15]$$

where S is the surface with normal **z**, ∂S is the boundary of the surface, and the integration takes place in a counterclockwise direction around the boundary with respect to the face normal. We are actually interested in the planar polygonal Veronio regions surrounding each node, it which case Stokes Theorem simplifies to

$$\mathbf{z} \cdot \int_S (\nabla \times \mathbf{f}) dA = \sum_{\text{faces}}^{\text{node}} \mathbf{n}_f \cdot \int_{\partial S_e} \mathbf{f} dL \qquad [16]$$

If we let **f** equal the velocity vector and make the first order assumption that the velocity is constant in the Veronio cell then we obtain,

$$\omega_n A_n = \sum_{\text{faces}}^{\text{node}} u W_f \qquad [17]$$

where A_n is the area of the Veronio cell surrounding the node, and the face normal vectors are assumed to point in a counterclockwise direction around the node.

In conjunction with [17] it is clear that for interior nodes, the pressure term is identically zero. The net result is that [14] can be written as,

$$A_n \frac{\omega_n^{n+1} - \omega_n^n}{\Delta t} + \sum_{\text{faces}}^{\text{node}} (v\omega)|_f W_f = \sum_{\text{faces}}^{\text{node}} (\nu_{ni}\omega_{ni} - \nu_{n0}\omega_{n0}) W_f / A_f \qquad [18]$$

where the face vorticity flux is given by $(\nu\omega)|_f = \frac{1}{2}(\omega_{n0}\mathbf{u}_{n0} + \omega_{ni}\mathbf{u}_{ni}) \cdot \hat{\mathbf{t}}_f$ and $\hat{\mathbf{t}}_f$ points out of the Veronio cell. This is a discrete version of the continuous two-dimensional vorticity evolution equation,

$$\frac{\partial \omega_3}{\partial t} + \nabla \cdot (\omega_3 \mathbf{u}) = \nabla^2 (\nu \omega_3) \qquad [19]$$

Again, it is important to note that despite the apparent similarity there remains a subtle distinction from classic control volume schemes. In the staggered mesh scheme, the normal velocity component u is the primary unknown and ω_n is the derived quantity. In a classic control volume schemes, ω_n would be the primary unknown. So the staggered mesh scheme differs from a standard vorticity-streamfunction or vorticity-velocity formulation in the fact that (often complex) boundary conditions on the vorticity are not required.

4. Discussion

The primary result of the current work is that staggered mesh methods are not just control volume methods applied on staggered control volumes, but are directly equivalent to classic collocated control volume methods. It was shown that unstructured staggered mesh discretizations of the divergence form of the Navier-Stokes equations are equivalent to classic control volume method for the velocity vector in mesh cells. Likewise, unstructured staggered mesh discretizations of the rotational form of the Navier-Stokes equations are equivalent to classic control volume method for the vorticity vector at mesh nodes (in Veronio cells). These equivalencies imply that the method possesses certain conservation properties.

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