Is the Keller Box Scheme Mimetic?

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SIAM Conference on Computational Science and Engineering

March 1, 2013

Driving Question

- Are all mimetic methods related to a discrete exterior calculus?
- Keller Box is ‘mimetic’.
- But not related to any (classic) discrete exterior calculus.

Keller Box is Mimetic

- **Multi-Symplectic:** (Reich, JCP, 2000)
- **Wave Propagation:** (Frank, J. Phys. A 2006)
- **Exact Discretization** (Perot and Subramanian, 2007)
- **Long History**
  Priessman Box Scheme (1961)
  Boundary Layer Eqns. (Cebeci and Bradshaw, 1977)
  Navier-Stokes (Chattot, 1999)
  Advection-Diffusion (Croisille and Greff, 2005)

Discrete Calculus Philosophy

Separate Discretization		**PDE -> LA**
from	**Approximation**	**LA -> square LA**

- **Do ALL discretization exactly.**
- **This means that the calculus and the physics remain exact.**

- **All approximations = interpolation problems.**
- **Numerical approximation only in material laws**

(which are engineering approximations already)

Example: Heat Eqn

$\frac{\partial i}{\partial t} = -\nabla \cdot \vec{\mathbf{q}}$	Conservation of Energy (Physics)
$i \approx \rho C T$	Perfectly Caloric Material (Mat.)
$\vec{\mathbf{q}} \approx -k\vec{\mathbf{g}}$	Fourier's Heat Flux Law (Mat.)
$\vec{\mathbf{g}} = \nabla T$	Def. of Gradient (Math)

- **Discretize Physics and Math - exactly.**
- **Approximate Material laws - using interpolation**

Exact Discretization

$$I_c^{n+1} - I_c^n = -\sum_{faces} \bar{Q}_f^{out}$$

where

$$I_c^{n+1} = \int_{\Omega_c} i dV$$

$$\bar{Q}_f = \int dt \int \mathbf{q} \cdot \mathbf{n} dA$$

Energy: FV approach.
Not very interesting.

$$\bar{\mathbf{g}}_c^{n+1} = \int_{\Omega_c} \nabla T dV \,|^{n+1} = \sum_{faces} \mathbf{n}_f^{out} \bar{T}_f^{n+1}$$

$$\bar{\mathbf{g}}_c^{n+1} = \int_{\Omega_c} \vec{\mathbf{g}} dV \,|^{n+1}$$

$$\bar{T}_f^{n+1} = \tfrac{1}{A} \int T dA$$

Math: Still Exact via Gauss' theorem.
Different from all other mimetic methods.

Exact Discretization of Heat Eqn.

$$\begin{bmatrix} \mathbf{I} & \mathbf{D} & 0 & 0 \\ 0_d & 0_d & \textbf{-I}_d & \mathbf{N}_d \end{bmatrix} \begin{pmatrix} I_c^{n+1} \\ \bar{Q}_f \\ \bar{\mathbf{g}}_c^{n+1} \\ \bar{T}_f^{n+1} \end{pmatrix} = \begin{pmatrix} I_c^n \\ 0_d \end{pmatrix}$$

But not CLOSED. Too many unknowns.

$$I_c^{n+1} \approx (\rho C)_c \sum_{faces} V_{cf} \bar{T}_f$$

$$-k_c \tfrac{\Delta t}{2} (\bar{\mathbf{g}}_c^{n+1} + \bar{\mathbf{g}}_c^{n}) \approx \sum_{faces} \mathbf{r}_f \bar{Q}_f$$

-Material Constitutive Eqns
-Physical Approximation.
-Numerical Approximation.
- Interpolation Problem.

Summary of Keller Box

- **Linear T in elements**
- **Constant q in elements**

$$\begin{bmatrix} (\rho C)_c A_{f\,2c} & \mathbf{D} \\ \mathbf{N}_d & \frac{2}{\Delta t k_c}\mathbf{R}_d \end{bmatrix} \begin{pmatrix} \bar{T}_f^{\,n+1} \\ \bar{Q}_f \end{pmatrix} = \begin{pmatrix} (\rho C)_c A_{f\,2c} \bar{T}_f^{\,n} \\ -\mathbf{N}_d \bar{T}_f^{\,n} \end{pmatrix}$$

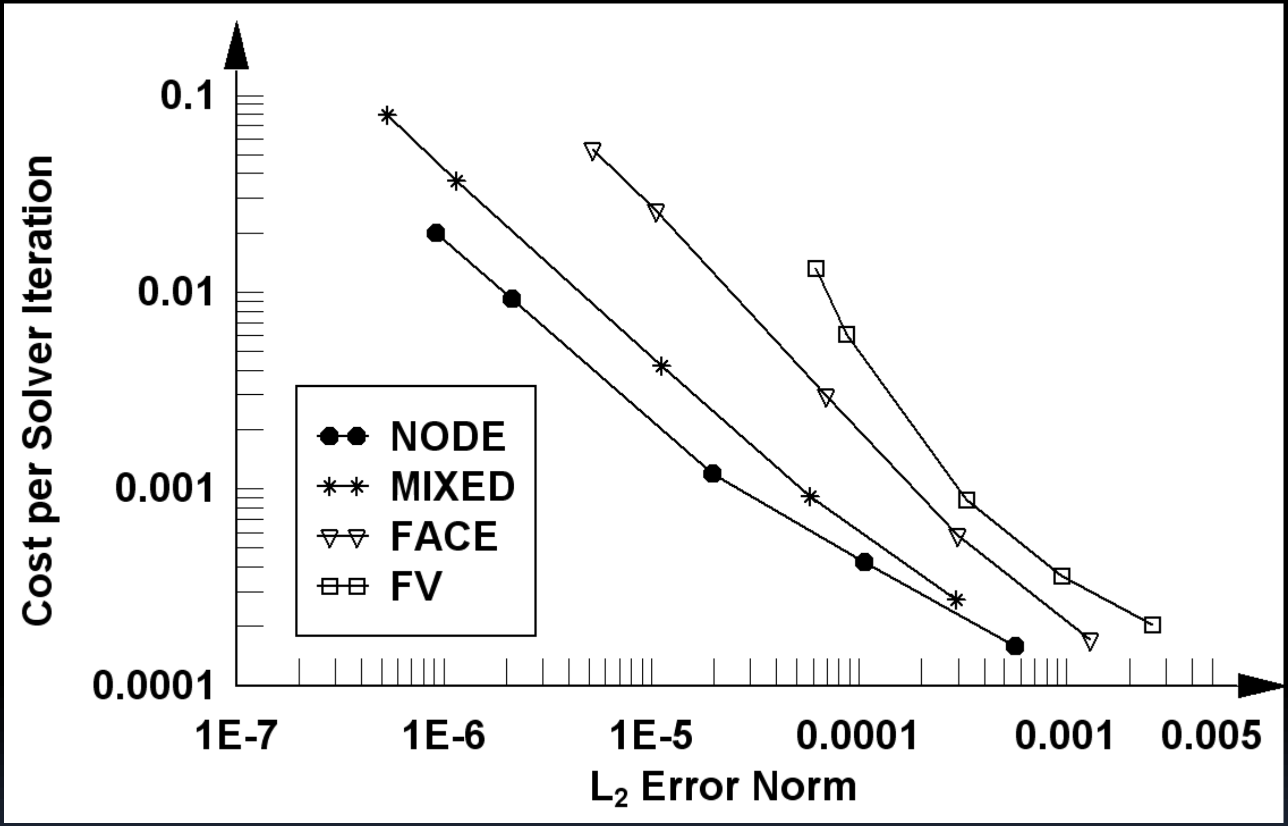
Not symmetric.

Square matrix (for simplices).

Invertible. (but not with CG)

Cannot easily eliminate Q_f .

Results



- KB is better than FV (due to higher accuracy).
- Other low-order DEC methods are better than KB.
- Log scale

Comparison

$$\overline{\mathbf{g}}_c^{n+1} = \int_{\Omega_c} \vec{\mathbf{g}} dV \,|^{n+1} = \sum_{faces} \mathbf{n}_f^{out} \overline{T}_f^{n+1} \qquad \text{KB}$$

$$\overline{\mathbf{g}}_e^{n+1} = \int \vec{\mathbf{g}} \cdot d\mathbf{l} \,|^{n+1} = T_2^{n+1} - T_1^{n+1} \qquad \text{Others}$$

- **KB has the gradient on elements.**
- **Others have it on primary or dual edges.**

- **KB has T as an integral unknown on faces.**
- **Others have T as a point value (0-form) at primary nodes or dual nodes (cell centers).**

Exterior Calculus ?

$$T_n \Rightarrow \nabla \Rightarrow g_e \Rightarrow \nabla\times \Rightarrow Q_f \Rightarrow \nabla\cdot \Rightarrow S_c$$ triangles

Node based

Cell based

$$S_{\tilde{c}} \Leftarrow \nabla\cdot \Leftarrow Q_{\tilde{f}} \Leftarrow \nabla\times \Leftarrow g_{\tilde{e}} \Leftarrow \nabla \Leftarrow T_{\tilde{n}}$$ polygons

$$T_n \Rightarrow \nabla \Rightarrow g_e \Rightarrow \nabla\times \Rightarrow Q_f \Rightarrow \nabla\cdot \Rightarrow S_c$$

KB

$$? \Leftarrow \vec{\mathbf{g}}_c \Leftarrow \nabla \Leftarrow T_f$$

Is the KB Scheme Related to other Mimetic methods?

Yes	No
• Captures Physics Well.	• Doesn't fit into DEC framework.
• Has an exact Discretization.	• Uses a fundamentally different exact gradient.

Summary

- KB has an Exact Discretization.
- KB is Mimetic.
- KB is an example where (classic) DEC may not work well to analyze the method.