A model for the dissipation rate tensor in inhomogeneous and anisotropic turbulence

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A model for the dissipation rate tensor in anisotropic inhomogeneous turbulence is developed. By including terms that depend on gradients a dissipation model is developed that is exact in the limit of very strong inhomogeneity (such as near solid walls or free surfaces). Rapid distortion theory and equilibrium theory are used to motivate the anisotropic terms in the model. The resulting model has only one free constant (from the equilibrium theory) which is determined via comparisons with turbulent channel flow at Re=590. *A priori* tests of the model for two shear-free boundary layers, channel flow at lower Reynolds numbers, and a backward facing step are presented. Full simulations using the model in channel flow are also performed. Comparisons are made with a variety of existing tensor dissipation rate models. © *2004 American Institute of Physics*.
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I. INTRODUCTION

Reynolds stress transport (RST) equation closures for turbulence (also referred to as single-point second-moment closures) are theoretically capable of predicting a wide variety of complex industrial flows. However, after many years of development RST are still not widely used in industrial applications. This may be because in practice RST models often do not perform significantly better than two equation models in complex flows. Why has the potential of RST models not been achieved? One possible explanation is that the development of RST models is largely based on quasihomogeneous or quasi-isotropic assumptions.[1,2] These assumptions are frequently not applicable in engineering flows, particularly those involving walls.

In this work, the modeling of strongly inhomogeneous turbulence is explored. In particular, the focus of this paper is on the modeling of one of the unclosed terms in the RST equations, often referred to as the dissipation rate tensor. As pointed out by Bradshaw and Perot,[3] this tensor is not actually equal to the dissipation rate in inhomogeneous turbulent flows (the case of interest in this paper), so for brevity and historical reasons we simply refer to this tensor as the dissipation tensor in this paper. Our particular interest in the dissipation tensor is due to the fact that this term dominates in the region near a wall. Correct prediction of the dissipation tensor is therefore an important first step towards accurate RST models for complex wall bounded turbulent flows.

The Reynolds stress transport equation can be written as

$$\frac{d\mathbf{R}_{ij}}{dt} = -(\mathbf{R}_{ik}U_{j,k} + \mathbf{R}_{jk}U_{i,k}) + \nu\mathbf{R}_{ij,kk} - \varepsilon_{ij} + \mathbf{\Pi}_{ij} - \mathbf{T}_{ijk,k}. \tag{1}$$

The first term on the right-hand side is the production term. It does not need to be modeled. The next two terms are the viscous diffusion and dissipation (rate) terms. The diffusion term does not require a model, and the dissipation term is given by $\varepsilon_{ij}=2\nu u'_{i,k}u'_{j,k}$. This dissipation term is the focus of the paper. The final two terms, the pressure-gradient velocity correlation $\mathbf{\Pi}_{ij}=-(p'_{,j}u'_i+p'_{,i}u'_j)$, and the turbulent transport term $\mathbf{T}_{ijk}=u'_iu'_ju'_k$ also require models. Near a wall, the turbulent transport is small and is not critical. The pressure-gradient velocity correlation (closely related to the pressure-strain term) is important just away from the wall.

Early models for the dissipation tensor[4,5] assumed that the dissipation tensor was isotropic and given by the expression $\varepsilon_{ij}=\frac{2}{3}\varepsilon\delta_{ij}$. Note that the dissipation ε is a scalar equal to one-half of the trace of the dissipation tensor. The scalar dissipation is assumed to be a known quantity that is determined by its own transport equation. The assumption of isotropy is based on the argument that large velocity derivatives should primarily occur at the smallest turbulence scales and turbulence is thought to be isotropic at the smallest scales (Kolmogoroff[6]).

While small-scale isotropy of turbulence has support from some experiments,[7] it is contradicted by some others.[8–11] The recent theoretical analyses of Hallbäck, Groth, and Johansson[12] and Durbin and Speziale[13] suggest that under the action of mean velocity gradients, even the smallest scales and hence the dissipation tensor must become anisotropic. Brasseur[14] discusses the issue in detail.

Since it is now widely recognized that the dissipation tensor is not isotropic in practice, it is often argued that the dissipation anisotropy should be modeled along with the pressure-gradient velocity correlation following the practice of Lumly and Newman.[15] There is, indeed, significant evidence to suggest that the *slow* pressure-strain correlation and the dissipation tensor anisotropy are closely related. However, it should be observed that the dependence is one way. The pressure terms respond to and tend to counteract the production and dissipation terms. Fast pressure strain reduces the production anisotropy and slow pressure strain

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counteracts the dissipation anisotropy. In order to develop effective slow pressure-strain models it is important to be able to model the dissipation tensor anisotropy first.

Some insight into the dissipation tensor anisotropy in homogeneous turbulence can be obtained by using a Fourier decomposition of the fluctuating velocity field. The dissipation tensor can then be written as $\varepsilon_{ij}=\int 2\nu k^2 \hat{u}_i' \hat{u}_j'^* d\mathbf{k}$. If the turbulence exists almost entirely at one wave-number magnitude then k^2 can be removed from the integral and $\varepsilon_{ij}=2\nu k^2 \mathbf{R}_{ij}$, or solving in terms of the scalar dissipation $\varepsilon_{ij}=(\varepsilon/K)\mathbf{R}_{ij}$. This model was first proposed by Rotta.[16] It suggests that the dissipation anisotropy is equal to the Reynolds stress anisotropy, $e_{ij}=(\varepsilon_{ij}/\varepsilon)-\frac{2}{3}\delta_{ij}=(R_{ij}/K)-\frac{2}{3}\delta_{ij}=a_{ij}$. In decaying turbulence, at low turbulent Reynolds numbers only the large-scale structures (a single significant k magnitude) exists and this model for the dissipation tensor becomes exact. The Rotta model is therefore frequently referred to as the low Reynolds number limit. However, it should be noted that in many low turbulent Reynolds number situations (such as near walls) this critical hypothesis of a single wave number magnitude is not satisfied.

A number of dissipation tensor models[17,18] are based on the idea of blending the isotropic model and the Rotta model using a function that depends on the turbulent Reynolds number. These models have the form,

$$\varepsilon_{ij} = (1-f)\tfrac{2}{3}\varepsilon\delta_{ij} + f(\varepsilon/K)R_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij} + f\varepsilon a_{ij}, \tag{2}$$

where f is 1 at low turbulent Reynolds numbers and 0 at high Reynolds numbers. The model of Hanjalic and Launder[18] used $f=1/[1+0.1(k^2/\nu\varepsilon)]$. This model did not show very good agreement with direct numerical simulation (DNS) data of channel flow at Re=180 (Ref. 19) where the simpler expression $f=1$ (i.e., the Rotta model) was shown to perform better. Hallbäck, Johansson, and Burden[20] proposed $f=1/[1+(31/5\pi)(k^{1/2}L_f)/\nu]$ where L_f is the integral length scale. Note that the turbulent Reynolds number approaches zero near a wall, so any formulation that uses a Reynolds number dependent blending function (such as those described above) will go from approximately isotropic dissipation in the free stream to the Rotta model near the wall. An asymptotic expansion of the dissipation tensor near the wall (Sec. III) shows that the Rotta model captures the zeroth order terms correctly at a wall, so these models show improvement over pure isotropic dissipation for wall bounded flows.

Other researchers[21,22] have proposed using models other than the Rotta model for the near wall (or low Reynolds number) region. These models have the form, $\varepsilon_{ij}=\frac{2}{3}\varepsilon\delta_{ij}+fe_{ij}^{\mathrm{wall}}$, where the wall model e_{ij}^{wall} is trace free. Often, e_{ij}^{wall} is defined in terms of the wall normal vector, which is ill defined away from the wall or at corners. In addition, in these models the form of e_{ij}^{wall} is formulated specifically for walls and is incorrect at a free surface or at any other boundary other than a wall.

While Reynolds number dependent models capture the near wall region better, they all revert to the isotropic model at high Re numbers and evidence suggests that even in the high Re limit the dissipation tensor is not isotropic. In the rapid distortion limit Hallbäck, Groth, and Johansson[12] have shown that the dissipation tensor anisotropy is not zero, but half of the Reynolds stress anisotropy. The work of Speziale and Gatski[23] suggests that in equilibrium the dissipation tensor anisotropy should depend on the shear stress. Finally, Perot[24] has shown that these Reynolds number dependent models are not correct for boundaries other than walls, such as slip walls or free surfaces.

In order to account for the rapid distortion theory (RDT) limit Hällback, Groth, and Johansson[12] (HGJ) proposed a nonlinear dissipation tensor model. This model adds an additional term proportional to the square of the anisotropy and has the form

$$\varepsilon_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij} + f_1\varepsilon a_{ij} + f_2\varepsilon\left(a_{ik}a_{kj} - \tfrac{1}{3}II_a\delta_{ij}\right), \tag{3}$$

where $II_a=a_{ij}a_{ji}$ and the functions are given by $f_1=\frac{1}{2}+\frac{3}{8}II_a$ and $f_2=-\frac{3}{4}$. This model is realizable, meaning that the dissipation tensor in a certain direction is zero if the turbulence in that direction is zero. A similar model that depends on the two-componentality parameter, $F=\det[(3R_{ij})/2k]$ was suggested by Sjögren and Johansson[25] (SJ). The two-componentality factor F is 1 in isotropic turbulence and 0 in two-component (2C) turbulence such as near a wall or a free surface. Initial calibration of the SJ model suggests $f_1=1-0.67F$ and $f_2=-1.18F$ (and these are the values used in our tests). However, ultimately the SJ model uses $f_1=1-\frac{1}{2}F$ and $f_2=0$. This final SJ model goes to the Rotta model in the 2C limit, and satisfies the RDT condition that the dissipation anisotropy is half the Reynolds stress anisotropy under the action of large mean strains. These more complex models perform well (away from boundaries) and will be used for comparison in Sec. V where the model performance is evaluated.

Speziale and Gatski[23] have proposed an algebraic formulation for the dissipation tensor that is similar in construction to algebraic models for the Reynolds stress tensor. In the resulting model the dissipation tensor anisotropy is solely a function of the mean velocity gradients. Unfortunately, the resulting model reverts to the (incorrect) isotropic model in the absence of mean velocity gradients. This model is therefore incapable of representing the shear-free boundary layers studied in Sec. V. However, the basic premise of using mean velocity gradients to parametrize the dissipation anisotropy (particularly in the equilibrium limit) is a reasonable idea which is adopted later.

Transport equations for the dissipation tensor can also be formulated.[26–28] The Speziale and Gatski model mentioned above is a simplification of such a transport equation. However, this level of complexity may be unwarranted at this time given the level of model uncertainty in the other RST model terms (particularly the pressure-strain).

In Sec. II of this paper, near boundary terms for the dissipation tensor are developed that are accurate near walls and surfaces. These near wall terms are derived from first principles and introduce no model constants. Section III analyzes the near wall asymptotics of these models near both walls and free surfaces, and considers the limit of strong inhomogeneity. In Sec. IV the model development in regions away from boundaries is considered. *A priori* tests of the model are presented in Sec. V and compared with a variety of existing model formulations. A brief discussion and conclusions appear in Sec. VI.

II. MODELING STRONG INHOMOGENEITY

In strongly inhomogeneous flows, turbulent correlations such as the dissipation tensor change rapidly as a function of their position. Some of the change with position is due a change in the underlying structure of the turbulence. However, most of the change is simply due to the spatial change in the turbulence intensity. In the specific case of the dissipation tensor, $\varepsilon_{ij}=2\nu\overline{u'_{i,k}u'_{j,k}}$, the dissipation can change spatially for two reasons. Either the gradients correlate differently, or (more likely) the magnitude of the velocity fluctuations has simply changed. These different effects can be isolated by using the following decomposition. Let the fluctuating velocity be decomposed as $u'_i=Q_{ij}\tilde{u}_j$. The tensor Q_{ij} is assumed to be a known quantity (related to the velocity fluctuation magnitude). It is an average quantity and does not change in time for statistically steady flows or along homogeneous directions. The underlying temporal and spatial fluctuations of the velocity field are captured by the dimensionless quantity $\tilde{u}_j$. Changes in the dissipation due to changes in the turbulence magnitude will be captured by Q_{ij}. Changes in the underlying turbulent structure will be manifest in $\tilde{u}_j$.

Substituting this formula into the equation for the Reynolds stress tensor gives a relationship between the structure correlation and the Reynolds stress tensor:

$$R_{ij}=\overline{u'_iu'_j}=\overline{Q_{in}\tilde{u}_nQ_{jm}\tilde{u}_m}=Q_{in}\overline{\tilde{u}_n\tilde{u}_m}Q_{jm}. \tag{4}$$

The magnitude tensor is not a fluctuating quantity and therefore can come out of the average. Substituting this decomposition into the dissipation tensor formula gives

$$\begin{aligned}\varepsilon_{ij}&=2\nu\overline{u'_{i,k}u'_{j,k}}=2\nu\overline{(Q_{in,k}\tilde{u}_n+Q_{in}\tilde{u}_{n,k})(Q_{jm,k}\tilde{u}_m+Q_{jm}\tilde{u}_{m,k})}\\&=2\nu\Bigg\{Q_{in,k}Q_{jm,k}\overline{\tilde{u}_n\tilde{u}_m}+\frac{1}{2}(Q_{in}Q_{jm})_{,k}(\overline{\tilde{u}_m\tilde{u}_n})_{,k}\\&\quad+Q_{in}Q_{jm}\overline{\tilde{u}_{n,k}\tilde{u}_{m,k}}\\&\quad+\frac{1}{2}(Q_{in,k}Q_{jm}-Q_{in}Q_{jm,k})(\overline{\tilde{u}_{m,k}\tilde{u}_n}-\overline{\tilde{u}_m\tilde{u}_{n,k}})\Bigg\}.\end{aligned} \tag{5}$$

If it is required that Q_{ij} be invertible then the first two terms in the expression can be found from Eq. (4) and are exact. The third term is the dissipation of the velocity structure. It requires a model. However, the velocity structure is quasihomogeneous (by design), and so standard dissipation models are expected to perform well in this context. The final term is the product of two differences. It is assumed to be small and evidence to that effect can be found in Ref. 29. In regions of strong inhomogeneity the first term dominates and Eq. (5) becomes exact irrespective of the model used for the third term in Eq. (5) or the size of the fourth term.

One possible definition for Q_{ij} is that it represents all the magnitude information (Perot and Moin[30]). In this case $\overline{\tilde{u}_n\tilde{u}_m}=\delta_{nm}$ and Eq. (4) becomes $R_{ij}=Q_{in}Q_{in}$ or $R=QQ^T$ and Q is the matrix square root of the Reynolds stress tensor. This definition of Q is actually not unique, Q can be symmetric or lower triangular, for example. The symmetric square root, however, seems to be the most natural. Like regular square roots, the sign of Q is also not well defined. Since Q always appears in pairs, this distinction is not important. With this definition of Q, the second term of Eq. (5) is identically zero, and the model is given by

$$\varepsilon_{ij}=2\nu Q_{in,k}Q_{jn,k}+Q_{in}\tilde{\varepsilon}_{nm}Q_{jm}, \tag{6}$$

where the fourth term of Eq. (5) is assumed to be negligible.

This near wall model is elegant, but inconvenient to implement. Finding Q requires determining the eigenvectors and eigenvalues of R. In this paper we consider a simpler implementation of Eq. (5). In order to gain implementation simplicity we therefore assume that Q is isotropic and is scaled by the turbulent kinetic energy, $Q_{ij}=K^{1/2}\delta_{ij}$. With this definition of Q, Eq. (4) gives the relation $R_{ij}/K=\overline{\tilde{u}_i\tilde{u}_j}$ and the fourth term in Eq. (5) is identically zero. Equation (5) then becomes

$$\varepsilon_{ij}=2\nu(K^{1/2})_{,n}(K^{1/2})_{,n}\frac{R_{ij}}{K}+\nu K_{,n}\left(\frac{R_{ij}}{K}\right)_{,n}+K\tilde{\varepsilon}_{ij}, \tag{7a}$$

which is an exact relation. This can alternatively be written as

$$\varepsilon_{ij}=2\nu(K^{1/2})_{,n}\left(\frac{R_{ij}}{K^{1/2}}\right)_{,n}+K\tilde{\varepsilon}_{ij}. \tag{7b}$$

Note that Eq. (7) only becomes a dissipation tensor model when a quasihomogeneous dissipation tensor model (for $K\tilde{\varepsilon}_{ij}$) is hypothesized. The quasihomogeneous dissipation tensor should be significantly easier to model than the dissipation tensor itself. The quasihomogeneous model is discussed in Sec. IV. In the following section, the near wall behavior of Eq. (7) is analyzed.

III. ASYMPTOTIC ANALYSIS NEAR BOUNDARIES

The behavior of turbulence quantities near a boundary can be determined by using Taylor series expansions in the coordinate direction normal to the boundary (Launder and Reynolds[31]). Using the convention that y is the direction normal to a wall the fluctuating velocity can be expanded as

$$u'_i=a_i(x,z,t)+yb_i(x,z,t)+y^2c_i(x,z,t)+\cdots. \tag{8}$$

At a solid wall the velocity goes to zero, so all the a_i are zero. Continuity applied very close to the wall implies $b_2=0$.

Substituting Eq. (8) into the definition for the dissipation tells us that near a wall,

$$\varepsilon_{11} = 2\nu\{(\overline{b_1^2}) + y(4\overline{b_1c_1}) + y^2(4\overline{c_1^2} + 6\overline{b_1d_1} + \overline{b_{1,1}^2} + \overline{b_{1,3}^2}) + \cdots\},$$

$$\varepsilon_{12} = 2\nu\{y(2\overline{b_1c_2}) + y^2(3\overline{b_1d_2} + 4\overline{c_1c_2}) + y^3(4\overline{b_1e_2} + 6\overline{c_1d_2} + 6\overline{d_1c_2} + \overline{b_{1,1}c_{2,1}} + \overline{b_{1,3}c_{2,3}}) + \cdots\}, \quad (9)$$

$$\varepsilon_{22} = 2\nu\{y^2(4\overline{c_2^2}) + y^3(12\overline{c_2d_2}) + y^4(9\overline{d_2^2} + 16\overline{c_2e_2} + \overline{c_{2,1}^2} + \overline{c_{2,3}^2}) + \cdots\}.$$

The ε_{33} component behaves just like ε_{11}. A similar expansion for the Reynolds stress tensor can also be performed:

$$R_{11} = y^2(\overline{b_1^2}) + y^3(2\overline{b_1c_1}) + \cdots,$$

$$R_{12} = y^3(\overline{b_1c_2}) + y^4(\overline{b_1d_2} + \overline{c_1c_2}) + \cdots, \quad (10)$$

$$R_{22} = y^4(\overline{c_2^2}) + y^5(2\overline{c_2d_2}) + \cdots.$$

The leading order terms in the dissipation tensor and the Reynolds stress tensor are very similar. However, the coefficient is different in each case. Rotta's model gets the 0(1) terms of the dissipation tensor correctly (i.e., the leading term of the ε_{11} and ε_{33} expansion), but it will underpredict the leading order terms of the other two dissipation components. Although the wall is at a low turbulent Reynolds number, Rotta's model does not work entirely correctly. The amplitude of fluctuations normal to the wall and those parallel to the wall are very different, and the basic assumptions used to derive the Rotta model are violated.

Even if leading order terms of ε_{12} and ε_{22} are wrong, does it matter? They still go to zero at the wall. Interestingly, if wall functions are not used it matters a great deal (using wall functions with a RST model largely defeats the purpose of having a RST model, see Speziale[32]). Near the wall, the dissipation and pressure-gradient velocity correlation exactly balance the diffusion term. If the leading order behavior of the dissipation is incorrect, the Reynolds stresses are too large near the wall and as a result they are also too large away from the wall. Trying to reduce these Reynolds stresses via terms in the model (rather than fixing the root cause) often leads to instability in the wall bounded RST equation system. Note that one reason elliptic relaxation models work well has nothing to do with ellipticity. These models allow an extra boundary condition to be imposed (because they hypothesize an extra equation). This additional boundary condition forces the correct near wall behavior of the Reynolds stresses. In essence, the elliptic relation forces the near wall behavior of the dissipation tensor to be correct via additional boundary conditions. In standard RST models (where six additional equations and their boundary conditions are not available), correct leading order behavior of each dissipation term is highly desirable.

As mentioned earlier, it is also possible to formulate models with the correct near wall asymptotics by using the wall normal vector or distance to the wall along with a blending function. This works, and is standard practice, but these models have serious deficiencies in their generality. Typically they work *only* at walls.

The boundary conditions at a slip wall (or stationary free surface), impose different constraints on the expansion. We now find that $b_1 = a_2 = b_3 = 0$, and continuity implies

$$a_{1,x} + a_{3,z} + b_2 = 0.$$

At a stationary free surface the dissipation behaves as

$$\varepsilon_{11} = 2\nu\{(\overline{a_{1,1}^2} + \overline{a_{1,3}^2}) + y^2(\overline{a_{1,1}c_{1,1}} + \overline{a_{1,3}c_{1,3}} + 4\overline{c_1^2}) + \cdots\},$$

$$\varepsilon_{12} = 2\nu\{y(\overline{a_{1,1}b_{2,1}} + \overline{a_{1,3}b_{2,3}} + 2\overline{c_1b_2}) + y^2(\overline{a_{1,1}c_{2,1}} + \overline{a_{1,3}c_{2,3}} + 3\overline{d_1b_2} + 4\overline{c_1c_2}) + \cdots\}, \quad (11)$$

$$\varepsilon_{22} = 2\nu\{(\overline{b_2^2}) + y(4\overline{c_2b_2}) + \cdots\},$$

and the Reynolds stress tensor is

$$R_{11} = (\overline{a_1^2}) + y^2(2\overline{a_1c_1}) + \cdots,$$

$$R_{12} = y(\overline{a_1b_2}) + y^2(\overline{a_1c_2}) + \cdots, \quad (12)$$

$$R_{22} = y^2(\overline{b_2^2}) + y^3(2\overline{b_2c_2}) + \cdots.$$

At a free surface there is no longer a clear relationship between the dissipation tensor and the Reynolds stress tensor. Rotta's model will cause ε_{22} to be zero at the surface when it should be finite. Also note that a free surface is no longer a low turbulent Reynolds number situation, so blending models [Eq. (2)] will produce the isotropic limit near the surface. The isotropic model does give a finite value for ε_{22} but it will be shown in Sec. V that it is far too large, and that the dissipation near a free surface is not close to isotropic.

The near boundary behavior of the proposed model can be determined from the behavior of the Reynolds stresses. For a no-slip wall we find that

$$K = y^2\tfrac{1}{2}(\overline{b_1^2} + \overline{b_3^2}) + y^3\tfrac{1}{2}(2\overline{b_1c_1} + 2\overline{b_3c_3}) + \cdots \quad (13)$$

and

$$(K^{1/2})_{,k}(K^{1/2})_{,k}\frac{1}{K} = \left(\frac{K_{,k}}{2K}\right)^2 = \frac{1}{y^2}\left\{1 + y\frac{(2\overline{b_1c_1} + 2\overline{b_3c_3})}{(\overline{b_1^2} + \overline{b_3^2})} + \cdots\right\} \quad (14)$$

plugging into the model equation [Eq. (7)] gives

$$\varepsilon_{11} = 2\nu\{(\overline{b_1^2}) + y(4\overline{b_1c_1}) + O(y^2)\} + K\tilde{\varepsilon}_{11},$$

$$\varepsilon_{12} = 2\nu\left\{y(2\overline{b_1c_2}) + y^2\left(3\overline{b_1d_2} + 3\overline{c_1c_2} + \overline{b_1c_2}\frac{(\overline{b_1c_1} + \overline{b_3c_3})}{(\overline{b_1^2} + \overline{b_3^2})}\right) + O(y^3)\right\} + K\tilde{\varepsilon}_{12}, \quad (15)$$

$$\varepsilon_{22} = 2\nu\left\{ y^2(3\overline{c_2^2}) + y^3\left(8\overline{c_2 d_2} + 2\overline{c_2^2}\frac{(\overline{b_1 c_1} + \overline{b_3 c_3})}{(\overline{b_1^2} + \overline{b_3^2})}\right) + O(y^4)\right\} + K\tilde{\varepsilon}_{22}.$$

So the proposed expression for the dissipation tensor [Eq. (7)] captures the $O(1)$ and $O(y)$ terms exactly and at least 75% of the $O(y^2)$ terms, when implemented near a wall. Since K is $O(y^2)$ this analysis shows that the quasihomogeneous dissipation model can be as high as $O(1)$ at the walls, without affecting the near wall asymptotics described above. Before considering the behavior of the quasihomogeneous dissipation tensor in any more detail, let us consider the behavior of the proposed decomposition [Eq. (7)] near a free surface.

Near a free surface the kinetic energy is given by

$$K = \tfrac{1}{2}(\overline{a_1^2} + \overline{a_3^2}) + y^2\tfrac{1}{2}(\overline{b_2 c_2} + 2\overline{a_1 c_1} + 2\overline{a_3 c_3}) + \cdots \tag{16}$$

and

$$\left(\frac{K_{,k}}{2K}\right)^2 = \left\{\left[\frac{1}{2}(\overline{a_1^2} + \overline{a_3^2})_{,1}\right]^2 + \left[\frac{1}{2}(\overline{a_1^2} + \overline{a_3^2})_{,3}\right]^2\right\} \Big/ [(\overline{a_1^2} + \overline{a_3^2})]^2 + O(y^2).$$

We can also show that

$$K_{,k}\left(\frac{R_{11}}{K}\right)_{,k} = O(1),$$

$$K_{,k}\left(\frac{R_{12}}{K}\right)_{,k} = O(y), \tag{17b}$$

$$K_{,k}\left(\frac{R_{22}}{K}\right)_{,k} = O(y^2),$$

so the near boundary terms in Eq. (7) have the same type of behavior. This requires the $\tilde{\varepsilon}_{12}$ model to go like $O(y)$ near the surface and $\tilde{\varepsilon}_{22}$ to be $O(1)$. Looking at the exact expressions for the dissipation tensor near a free surface it is clear that capturing the leading order ε_{11} and ε_{12} terms exactly is not possible. Derivative information is not available to a RST model. However, the leading two terms of the ε_{22} expression can, in theory, be obtained exactly from Reynolds stress information. Also note that the error in both the wall and free-surface expressions for ε_{22} can be represented by $(2\nu/y^2)R_{22}$. The error is the same irrespective of the boundary type. At both boundaries, the flow becomes two component, so we will use the 2C parameter F to model this missing contribution for ε_{22}. This extra term is $2\nu(F^{1/2})_{,n}(F^{1/2})_{,m}R_{nm}\delta_{ij}/F$. Technically we are now modeling the quasihomogeneous dissipation $K\tilde{\varepsilon}_{ij}$. This is the near boundary contribution of the quasihomogeneous dissipation due to the 2C nature of the turbulence near these boundaries. This term is higher order for ε_{11} and ε_{12} terms near both walls and free surfaces, and so it only affects the ε_{22} dissipation component. At a solid wall, this enhancement has only a very small affect on the model. However, at a free surface the 2C affects can be seen very clearly. The importance of this 2C correction is demonstrated in Sec. V.

IV. QUASIHOMOGENEOUS DISSIPATION

In homogeneous turbulence, the boundary (or gradient) terms drop out entirely and the quasihomogeneous dissipation remains to be modeled. Hallbäck, Johansson, and Burden[20] show that in initially isotropic homogeneous turbulence the dissipation anisotropy should be half of the Reynolds stress anisotropy under the action of rapid irrotational strain or shear. This will be referred to as the RDT limit. The experiments of Crow[33] and Lee and Reynolds[34] show that this ratio does not remain $\frac{1}{2}$ when the turbulence is anisotropic, and in the extreme limit of axisymmetric 2C turbulence it is seen to be close to 1 (which is the Rotta model).

The practice of expanding model parameters in polynomial expansions of the potential unknowns is a rational way to proceed, and is certainly viable when the unknowns are known to be small. However, when the objective is to capture an entire functional range the use of polynomial expansions can be detrimental. Rational polynomials have a greater fitting capability. In this work, we propose a simple tensorally linear model for the quasihomogeneous dissipation, in which the blending parameter f is a function of F. This is similar to the models of Johansson, however, we hypothesize a rational polynomial expansion, $f=1/(1+F)$, rather than a simple polynomial series. This results in the quasihomogeneous model $\tilde{\varepsilon}_{ij}=\tilde{\varepsilon}[F/(1+F)\frac{2}{3}\delta_{ij}+1/(1+F)\times(R_{ij}/K)]=\tilde{\varepsilon}[\frac{2}{3}\delta_{ij}+1/(1+F)a_{ij}]$ and $a_{ij}=(R_{ij}/K)-\frac{2}{3}\delta_{ij}$. In isotropic turbulence, this model gives the correct RDT anisotropy ratio of $\frac{1}{2}$, and in 2C turbulence it gives the correct anisotropy ratio of 1. In theory, a slightly more complex blending might be desired in which $f=1/[1+g(\mathrm{Re}_T)F]$, where the function g goes to zero as the turbulent Reynolds number becomes small and approaches 1 at high turbulent Reynolds numbers. We have not pursued this added level of complexity at this time.

Finally, we note that the dissipation anisotropy could be a function of the mean flow gradients, not just the Reynolds stress anisotropy. Typically, dissipation anisotropy is not modeled in this way because one does not expect sudden changes in the mean flow to have an instantaneous affect on the dissipation. However, in equilibrium situations, there could be a good correlation between the two tensors. Reynolds stresses are frequently modeled using this type of hypothesis (eddy viscosity hypothesis of Bousinesq). In fact, models which only depend on the Reynolds stress anisotropy will have the dissipation anisotropy aligned along the same principal directions as the Reynolds stress anisotropy. We know that these anisotropy directions are not always aligned (in channel flow they disagree by 8° at $y+=30$). In this work we therefore hypothesize that the quasihomogeneous dissipation tensor can also be a linear function of the shear-stress tensor, $S_{ij}=\frac{1}{2}(U_{i,j}+U_{j,i})$.

The model for the dissipation tensor then becomes

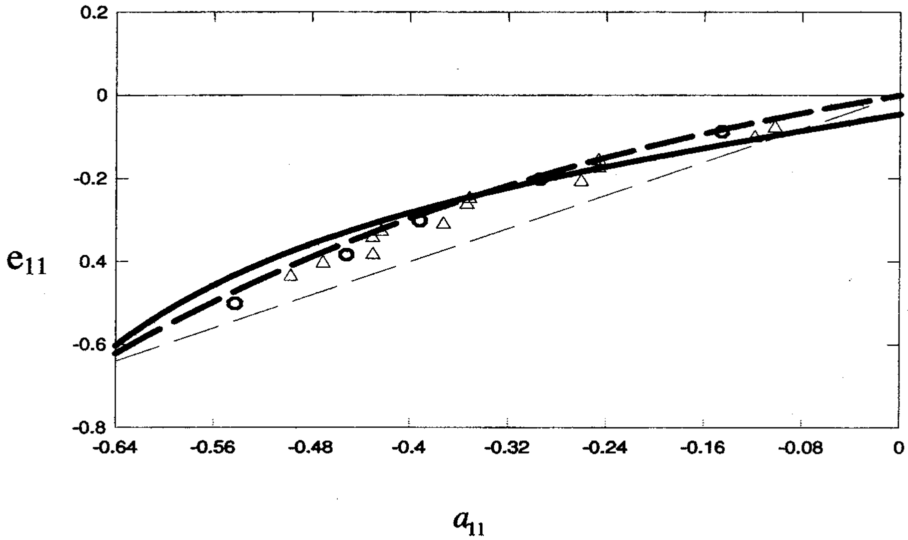

FIG. 1. Dissipation anisotropy in axisymmetric contraction. Open triangles denote the experimental data of Crow (Ref. 33) $[S_2^*(t=0)\approx 0.5-2.0, \mathrm{Re}_\lambda \approx 15-100]$, open circles denote the experimental data of Lee and Reynolds (Ref. 34) $[S_2^*(t=0)\approx 0.97$ to $0.71, \mathrm{Re}_\lambda \approx 50]$, thick dashed line denotes HGJ model, thin line denotes isotropic model, thin dashed line denotes Rotta model, and thick line denotes the proposed model.

$$\varepsilon_{ij} = 2\nu(K^{1/2}{}_{,n})^2 R_{ij} + \nu K_{,k}\left(\frac{R_{ij}}{K}\right)_{,k} + \tilde{\varepsilon}\frac{F}{1+F}\frac{2}{3}\delta_{ij}K + \tilde{\varepsilon}\frac{1}{1+F}R_{ij} + 2\nu\frac{F^{1/2}{}_{,m}R_{mn}F^{1/2}{}_{,n}}{F}\delta_{ij} + C^*KS_{ij}. \quad (18)$$

The single parameter $C^*=0.18F/(1+F)^2$ is set by comparing the ε_{12} component of turbulent channel flow at Re=590. In theory, the constant C^* should be a function of (ε/SK), such that C^* goes to zero when (ε/SK) is zero (the RDT limit). We have not explored this level of detail in this work.

The scalar $\tilde{\varepsilon}=\frac{1}{2}\tilde{\varepsilon}_{ii}$ is the trace of the quasihomogeneous dissipation. It has units of inverse time or frequency and can be obtained by taking one-half the trace of Eq. (18), $K\tilde{\varepsilon}=\varepsilon-2\nu(K^{1/2}{}_{,n})^2-3\nu(F^{1/2}{}_{,m}R_{mn}F^{1/2}{}_{,n})/F$. The quasihomogeneous dissipation $\tilde{\varepsilon}$ (or its closely related form $\hat{\varepsilon}=K\tilde{\varepsilon}$) is an interesting inverse time scale that has been used previously in some near wall turbulence models (e.g., Launder and Sharma[35]). It is attractive because at a wall it is finite, whereas the standard inverse timescale (ε/K) is singular and goes like y^{-2} at a wall. Note that from its definition, $\tilde{\varepsilon}$ is a positive quantity. However, due to numerical inaccuracy in the calculation of gradients, calculating $\tilde{\varepsilon}$ from the formula above can lead to large errors or negative values when implemented on a computer. In practical implementation either a transport equation is solved directly for $\tilde{\varepsilon}$ rather than the more common ε transport equation (as in many low Re number k/ε models), or we sometimes use $\tilde{\varepsilon}=(\varepsilon/K)1/[1+10\nu|\nabla(K^{1/2})|/K]$ to guarantee a positive inverse timescale with finite near wall behavior.

While the proposed model [Eq. (18)] looks somewhat complex, it is relatively easy to implement. Many of the terms combine with similar looking terms in the pressure-strain model, and if the Reynolds stress anisotropy equation is solved rather than the RST equation, then some of the terms drop out or simplify even further.

It is important when implementing this model to have a numerical method that is capable of accurately calculating gradients. At the wall, certain terms should exactly balance. Numerically they will only approximately balance and if the disagreement is large enough, the numerical implementation (not the model) becomes unstable. Quantities with high power law behavior $[R_{22}=O(y^4)]$ can be quite hard to differentiate accurately with low order numerical methods. For this reason, the anisotropy equations (rather than Reynolds stress equations) are somewhat easier to solve with low order numerical methods.

V. MODEL RESULTS

In this section, the proposed dissipation model [Eq. (18)] is compared against experimental and DNS data. The performance of the model is compared to a number of other dissipation tensor models that have been mentioned in the text. The majority of the tests are *a priori* tests using data for the Reynolds stresses and dissipation plugged directly into Eq. (18).

These tests are a useful way to directly isolate if the model can represent the dissipation tensor accurately. However, it is possible to construct models, which perform well in *a priori* tests but do not perform well in practice. These models are unstable and move away from the desired solution rather than towards it. To demonstrate stability we will also present at the end of this section some solutions of turbulent channel flow that use this dissipation tensor model in a full RST prediction.

Our first test case does not involve the inhomogeneous terms at all. It is a test of the quasihomogeneous part of the model. Figure 1 shows the model performance in axisymmetric rapid contraction of homogeneous turbulence. In this flow the turbulence is initially isotropic and becomes increasingly 2C as time proceeds. Because the turbulence is axisymmetric only one component of the dissipation needs to be analyzed. The figure shows the e_{11} component as a function of a_{11}, at various times during the simulation (experiment). The isotropic model gives a flat line, and the dashed line with a slope of 1 is the Rotta model. This is a relatively high Reynolds number test case, so models that switch between the isotropic model and the Rotta model based on a blending parameter, which is a function of the turbulent Rey-

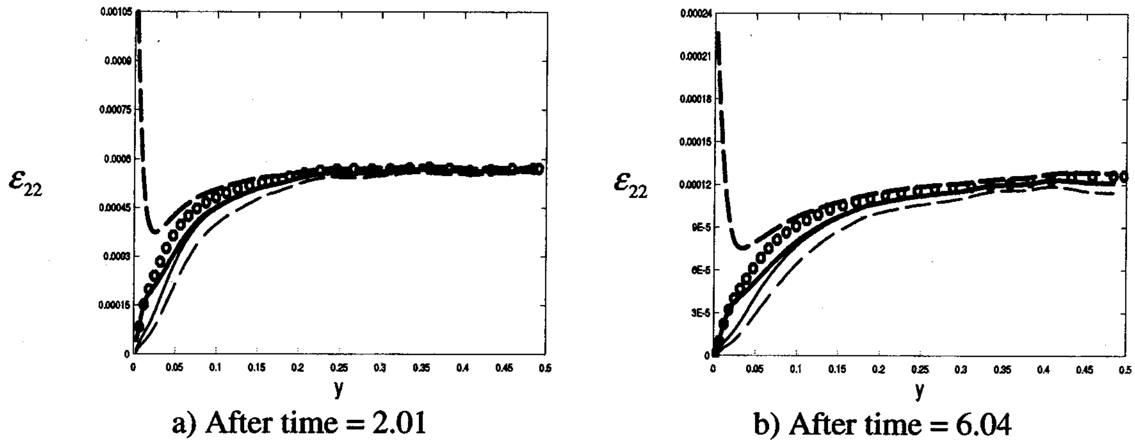

FIG. 2. Shear-free turbulent boundary layer next to a solid wall. Circles denote DNS data of Perot and Moin (Ref. 36), thick dashed line denotes isotropic model, thin dashed line denotes Rotta model, thin line denotes HGJ model, and thick line denotes proposed model.

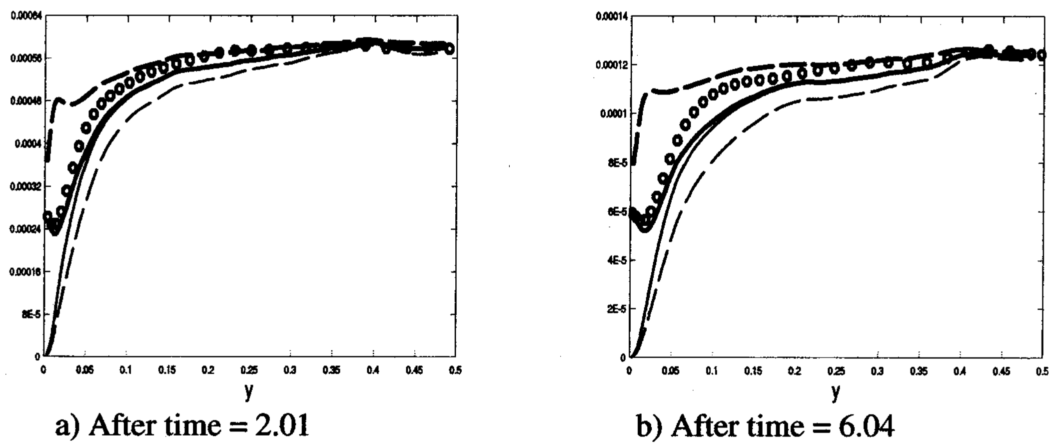

FIG. 3. Shear-free turbulent boundary layer next to a free surface. (See Fig. 2 for caption.)

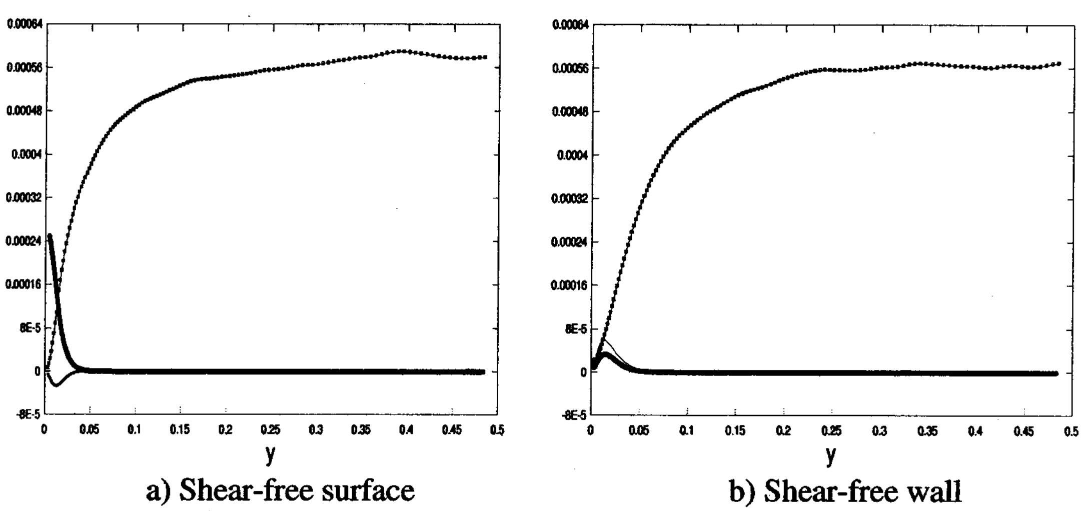

FIG. 4. Each term in Eq. (18) for shear-free turbulent boundary layers at time 2.0. The chain dotted line denotes $2\nu(K^{1/2}{}_{,n})^2R_{ij}+\tilde{\varepsilon}(F/1+F)\frac{2}{3}\delta_{ij}K+\tilde{\varepsilon}(1/1+F)R_{ij}$, thin line denotes $\nu K_{,k}(R_{ij}/K)_{,k}$, and thick line denotes $2\nu(F^{1/2}{}_{,m}R_{mn}F^{1/2}{}_{,n}/F)K\delta_{ij}$. The shear-stress term is zero for both these flows.

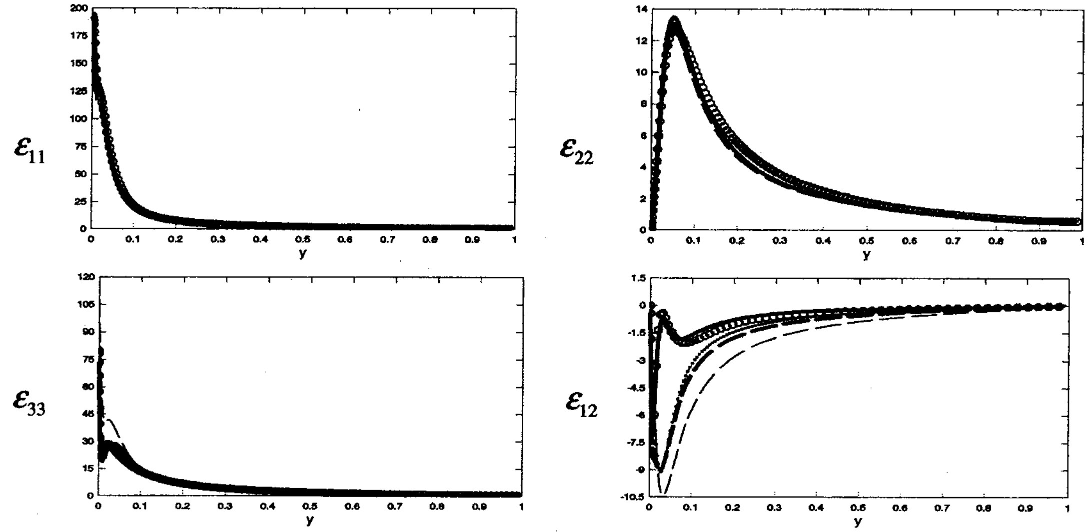

FIG. 5. Dissipation tensor in channel flow (Re=590). Circles denote DNS data from Ref. 37, long dashed line denotes SJ model, chained dashed line denotes HGJ model, and thick line denotes proposed model. (In the figure of ε_{12}, thick dashed line represents the proposed model without the shear-stress term.)

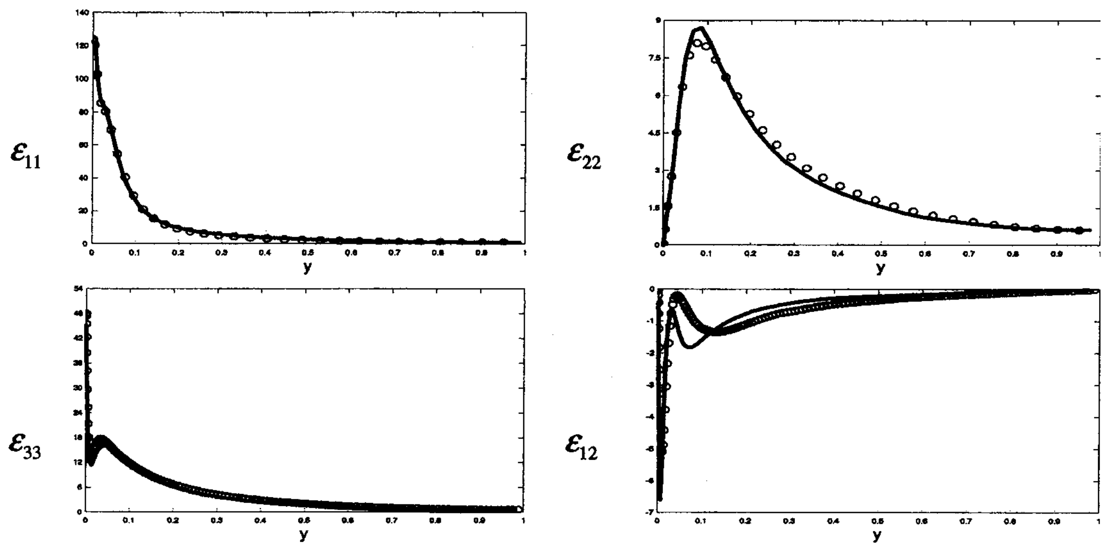

FIG. 6. Dissipation anisotropies in channel flow (Re=395). Circles denote DNS data from Ref. 37 and thick line denotes proposed model.

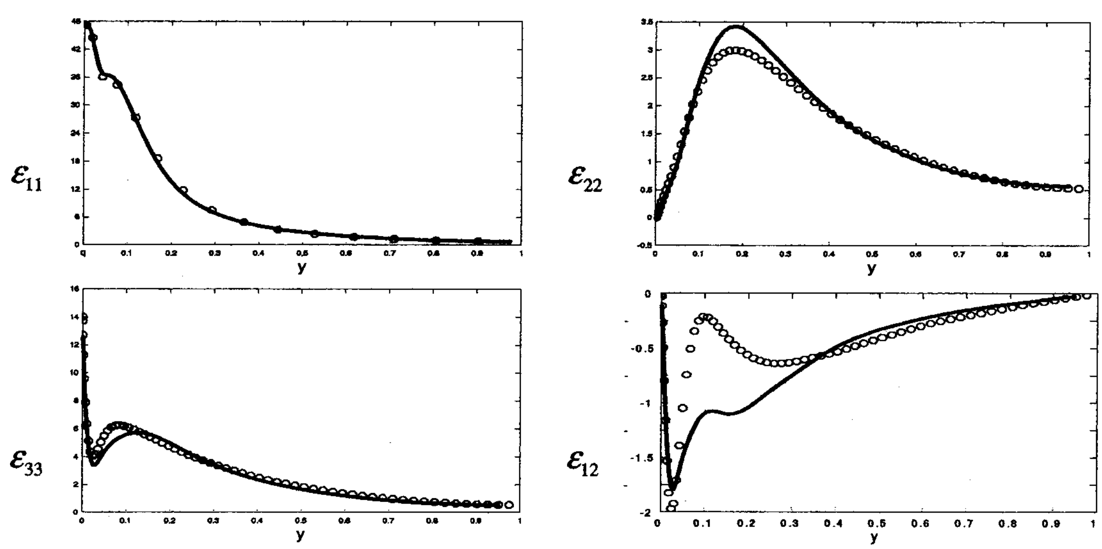

FIG. 7. Dissipation anisotropies in channel flow (Re=180). Circles denote DNS data from Ref. 37 and thick line denotes proposed model.

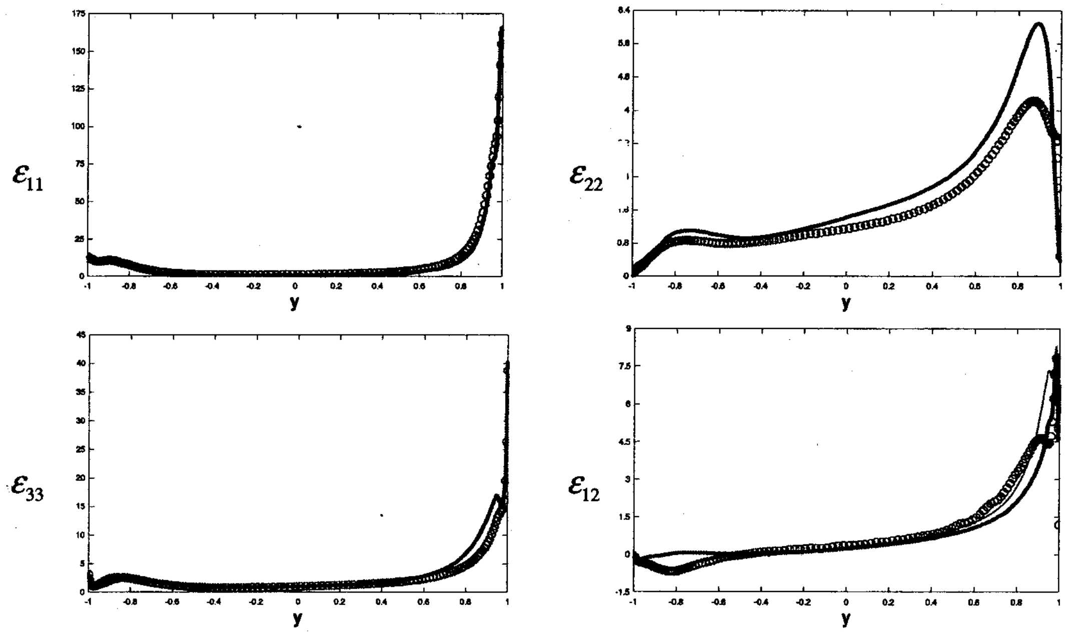

FIG. 8. Dissipation anisotropies in rotating channel flow (Ro=0.15, Re=194). Circles denote DNS data from Ref. 38 and thick line denotes proposed model. Thin line in the last figure is the model with $C^*=0$.

nolds number (most models), will be essentially isotropic (very close to a horizontal line through the origin). The HGJ model was designed for this flow and therefore performs well for this case. The SJ model (not shown) performs much like HGJ.

Next, we wish to examine the inhomogeneous terms. The quasihomogeneous term cannot be completely eliminated in any flow, but shear-free boundary layers provide an opportunity to evaluate the model in a strongly inhomogeneous situation with few other complicating effects. In Fig. 2, the model predictions of a shear-free boundary layer next to a solid wall are compared with DNS data.[36] Like the previous case, this flow is axisymmetric and time developing, but unlike the previous case, it also has strong gradients in the direction normal to the wall. The figure shows the ε_{22} dissipation at two different times after the wall appears. As predicted by the asymptotic analysis, the Rotta model is too small near the wall. The HGJ model transits from a combination of $\frac{1}{2}$ isotropic and $\frac{1}{2}$ Rotta well away from the wall to all Rotta near the wall where the turbulence is 2C. The SJ model (not shown) is almost identical to HGJ. Other models

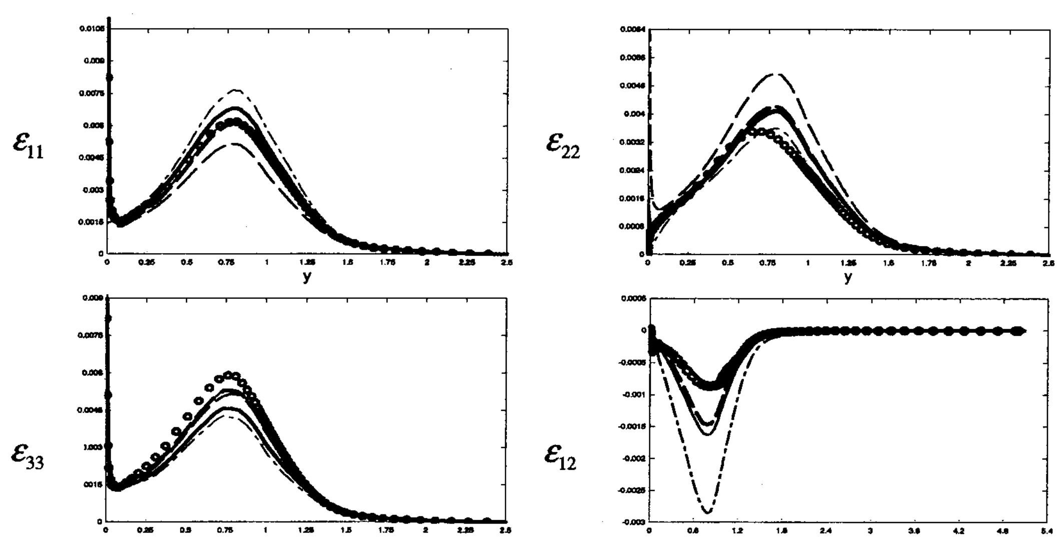

FIG. 9. Dissipation tensor in the flow over backward-facing step (at 4 h). Circles denote DNS data from Ref. 39, long dashed line denotes the isotropic model, chain-dotted line denotes the Rotta model, thin line denotes the SJ model, the small dashed line denotes the HGJ model, and the thick line denotes the proposed model.

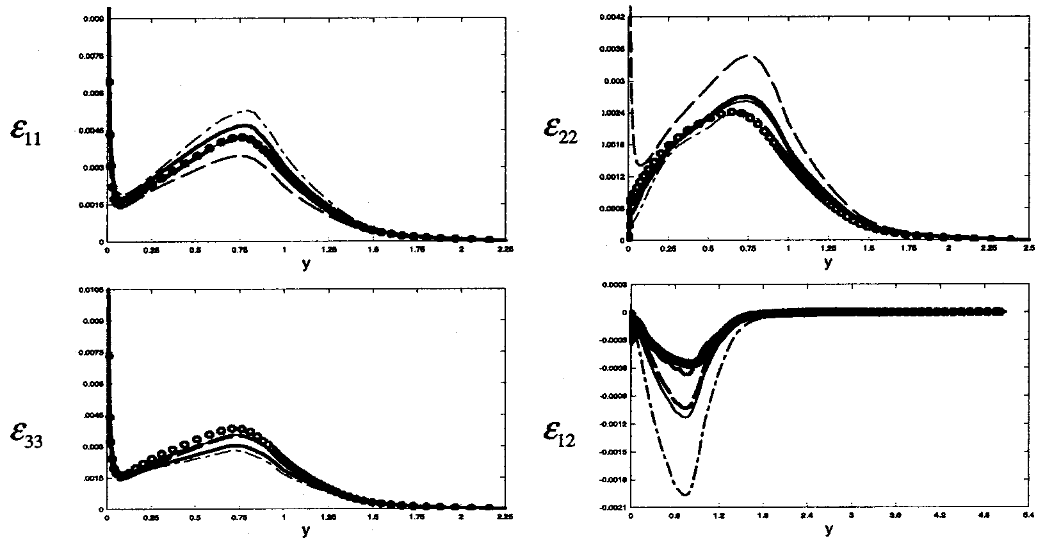

FIG. 10. Dissipation tensor in the flow over backward-facing step (at 6 h). Circles denote DNS data from Ref. 39, long dashed line denotes the isotropic model, chain-dotted line denotes the Rotta model, thin line denotes the SJ model, the small dashed line denotes the HGJ model, and the thick line denotes the proposed model.

which blend the isotropic and Rotta models based on the Reynolds number will behave like the Rotta model near the wall. All but the proposed model, underpredict the normal dissipation component near the wall.

The shear-free boundary layer next to stationary-free surface is shown in Fig. 3. Again two times are shown, and the definitions of the lines and symbols are the same as in Fig. 2. This flow is no longer low Reynolds number near the surface, and so the underprediction of the Rotta and HGJ models is even more obvious in this case. Most of the other models will behave like the isotropic model for this flow. The proposed model captures the near-surface dissipation correctly using no adjustable constants.

In Fig. 4, the various terms in the present model are split out so that the magnitude and location of each contribution can be ascertained. Figure 4(a) is the shear-free surface at time 2.0 and Fig. 4(b) is the shear-free wall at time 2.0. It is clear that the term involving $2\nu(F^{1/2}{}_{,m}R_{mn}F^{1/2}{}_{,n})/F$ is critical in the free-surface case.

In Fig. 5 the model is tested in turbulent channel flow at Re = 590.[37] The proposed model performs well for all the dissipation components. The other models have difficulty predicting the ε_{12} and ε_{22} components. The value of C^* was tuned for ε_{12} in this case. With $C^*=0$, the small dashed line is obtained. The model predictions for channel flow at Re = 395 and Re = 180 are shown in Figs. 6 and 7, respectively.

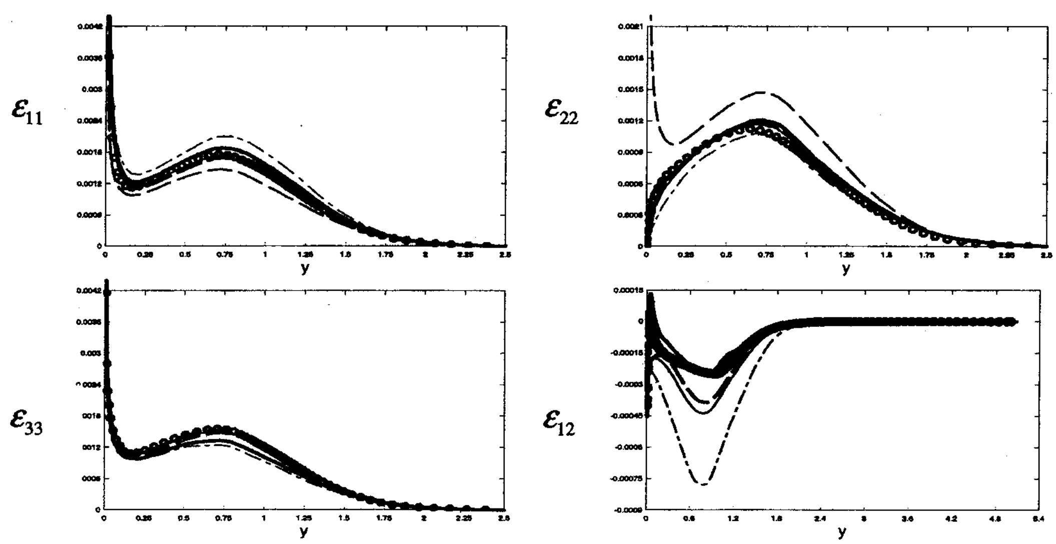

FIG. 11. Dissipation tensor in the flow over backward-facing step (at 10 h). Circles denote DNS data from Ref. 39, long dashed line denotes the isotropic model, chain-dotted line denotes the Rotta model, thin line denotes the SJ model, the small dashed line denotes the HGJ model, and the thick line denotes the proposed model.

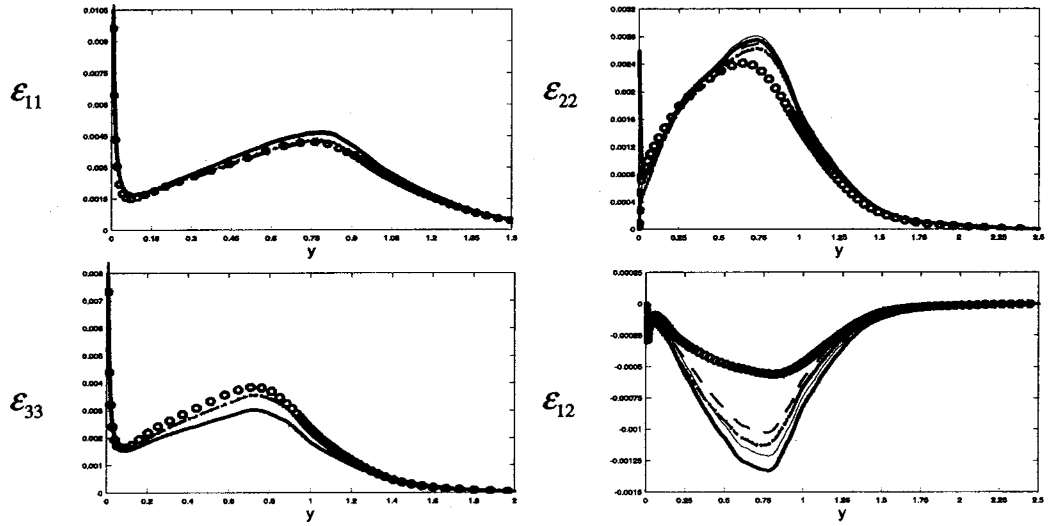

FIG. 12. Dissipation anisotropies in flow over backward-facing step (at 6 h). Circles denote DNS data from Ref. 39, chained dashed line denotes SJ model, dashed line denotes HGJ model, thin line denotes SJ model without nonlinear term, thick line denotes HGJ model without nonlinear term.

The lowest Reynolds number case shows some discrepancies for ε_{12}. This might be fixed by making C^* a function of ε/SK as suggested earlier.

The behavior of the model in rotating channel flow is shown in Fig. 8. The test case is the Ro=0.15 DNS case of Andersson and Kristoffersen[38] at a Re of 194. The model does a good job of predicting the very different behaviors on each side of the channel (especially given the very low Re number of the simulation).

The proposed model is compared with DNS data for the flow past a backward facing step in Figs. 9–11. Several locations are shown, both before (4 h) and near (6 h) reattachment, and well downstream (10 h) during the boundary layer recovery. The figures show good agreement with the DNS data[39] in the turning mixing layer. The boundary layer near the wall is more difficult to see, but behaves similarly to the previous channel flow results. In the mixing layer, this tensoraly linear model performs similarly (or better in the ε_{12} case) to the more complex nonlinear model of HGJ and is much more accurate than the isotropic and Rotta models.

It is interesting to quantify the effect of the nonlinear term in the SJ and HGJ models. In Fig. 12, these models are shown with f_2 set to zero, for the 7 h downstream location on the backward-facing step. The contribution of the nonlinear term is not that large, given its added complexity we have chosen to follow the example of Sjögren and Johansson and

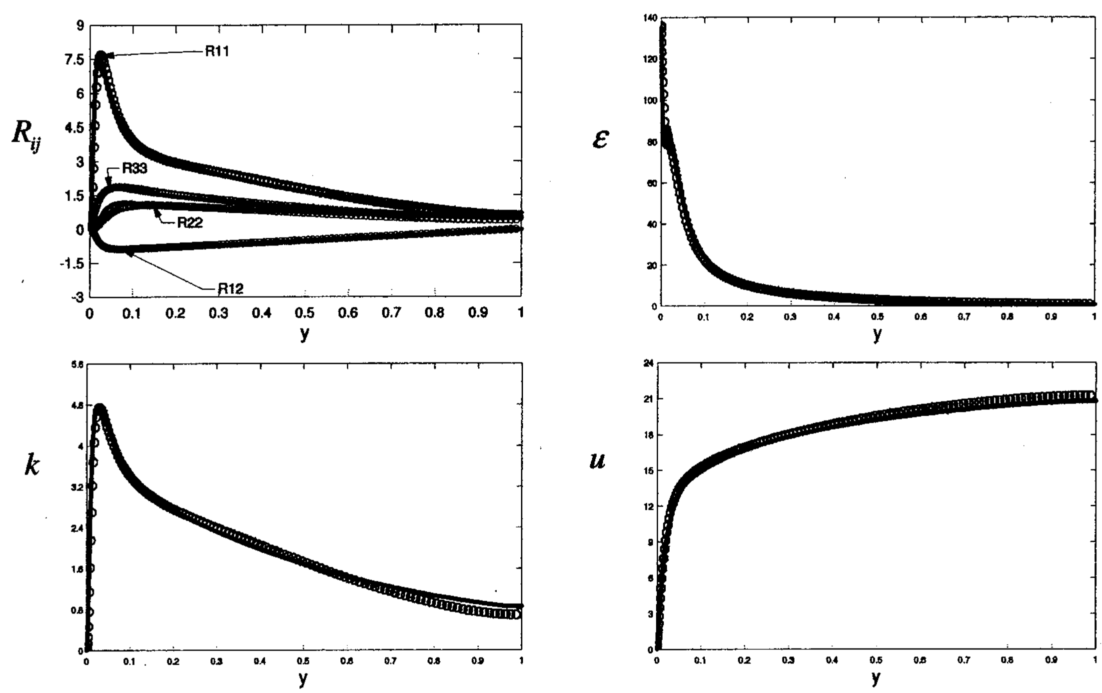

FIG. 13. Channel flow (Re=590). Circles denote DNS data from Ref. 37 and thick line denotes full RST model predictions.

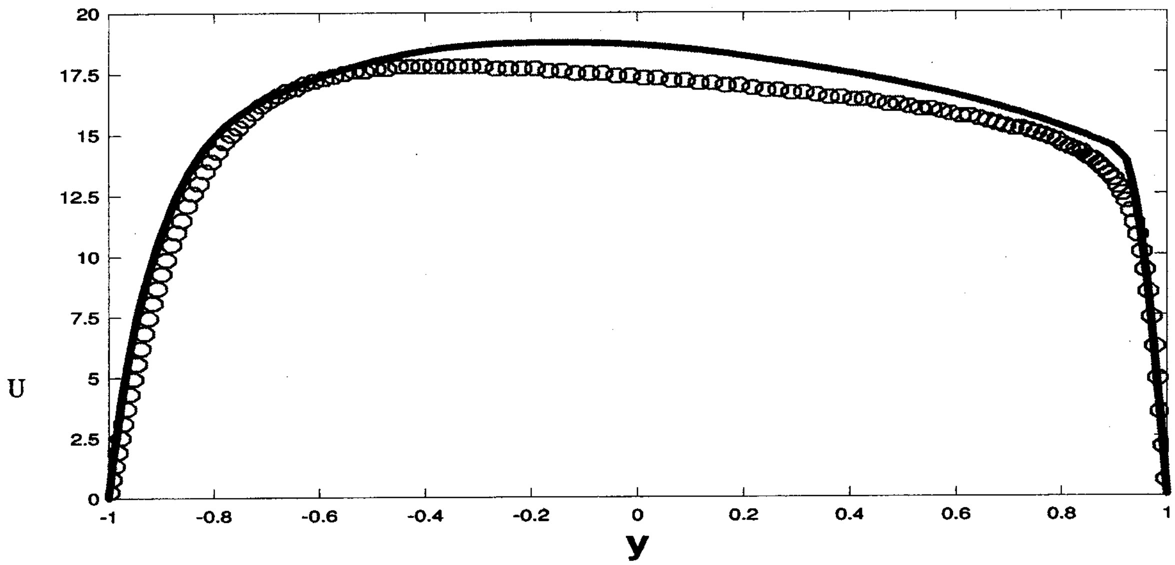

FIG. 14. Full RST prediction of the mean velocity in rotating channel flow $(\mathrm{Ro}=0.15,\mathrm{Re}=194)$. Circles denote DNS data from Ref. 38 and thick line denotes the model prediction.

not include such a nonlinear term in the proposed model. The choice of the function $f=1/(1+F)$ allows us to satisfy the RDT limit and realizability without the nonlinear term present.

A priori tests such as those described above can be very informative about the quality of a model. Nevertheless, it is possible to formulate models which work well in *a priori* test but which fail in practice due to the inherent instability of the proposed formulation. The proposed dissipation model has been incorporated into a full RST closure and solved for turbulent channel flow. The details of the full RST closure are given by Natu.[40] Results of these simulations for channel flow at $\mathrm{Re}=590$ are shown in Fig. 13, and for the rotating channel flow case in Fig. 14. These full model results are highly dependent on the chosen pressure-strain model. They are not, therefore, an indication of the accuracy of the dissipation tensor model. They are, however, an indication of the dissipation tensor model's stability and computability.

VI. CONCLUSION

The proposed model for the dissipation tensor deviates from all prior models in its use of terms which involve gradients of various turbulence quantities. The appearance of these gradient terms is not particularly surprising. There is no reason to believe that the source terms in the RST equations should not be functions of such gradients and, in fact, there is every reason to believe that gradients should dominate in regions of strong inhomogeneity, such as near walls.

When attempting to add gradient information to a model, the variety of choices is so large that the traditional approach of expanding a quantity (such as the dissipation) in terms of all the possible unknowns becomes intractable. This paper has demonstrated a rational approach to derive the gradient terms in the model. The resulting terms [Eq. (7)] do not have any additional model constants, but do tend to greatly improve the model performance in regions where gradients are large.

It is interesting to note that the resulting gradient terms do have associations in other contexts. The term $2\nu(K^{1/2})_{,n}(K^{1/2})_{,n}$ is a common modification of the dissipation near a wall such that time scales at the wall remain finite, and the term $\nu K_{,n}(R_{ij}/K)_{,n}$ appears in the Reynolds stress anisotropy transport equation.

The quasihomogeneous term in Eq. (7) is in many senses the harder part of the dissipation tensor to model. Our quasihomogeneous model introduces a parameter to represent the affects of mean strain on the dissipation. In theory, this term should only be present in equilibrium situations (when production is roughly equal to dissipation). The near-wall 2C term does not introduce any addition model constants but is, like the strain dependent term, motivated rather than derived. Fortunately, these two modifications are not large in most flow situations. The bulk of the dissipation model is carried by the term $\tilde{\varepsilon}(F/\{1+F\})\frac{2}{3}\delta_{ij}K+\tilde{\varepsilon}(1/\{1+F\})R_{ij}$. This model satisfies an RDT limit and is realizable in the 2C limit. It agrees reasonably well with data from axisymmetric expansion at both small and large anisotropies. The key innovation in this term of the model is the functional form of the blending parameter f. It is suggested that the common practice of expanding parameters in simple polynomial series can be detrimental. Such expansions do not perform well when the expansion variable (such as F) is $O(1)$.

ACKNOWLEDGMENT

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