A Self-adapting Turbulence Model for Flow Simulation at any Mesh Resolution.

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Abstract

A two-equation transport model is used to model turbulence at any mesh resolution, from RANS, to LES, to DNS. The two-equation model used is a slight modification of the standard k/ε model that allows the backscatter of energy to resolved scales. A mathematical explanation is provided for why RANS models (such as this two-equation model) are applicable to LES. The model automatically adapts to the mesh resolution provided and no interaction from the user is necessary. This approach is tested on the problem of moderately high Reynolds number isotropic decaying turbulence and gives good predictions at any mesh resolution and with different initial conditions. A detailed analysis shows that at LES resolutions the solution remains fully unsteady and three-dimensional and does not approach a RANS like solution.

I. Introduction

Turbulence models are frequently classified by the ratio of how much turbulent energy is represented by the model compared to how much turbulent energy is computed via first principles. RANS (Reynolds averaged Navier-Stokes) models represent the most turbulent energy in the model. LES (large eddy simulation) computes considerably more of the turbulent energy via first principles and DNS (direct numerical simulation) simulates all the turbulent energy correctly and models none. If a more detailed terminology is desired, in between RANS and LES lies URANS (unsteady RANS) and VLES (very large eddy simulation). It is sometimes even useful to use the term QR-LES (quasi-resolved LES) referring to an LES simulation that is very nearly DNS. This range of turbulence models is shown on Fig. 1 in relation to a turbulent energy spectrum. Each model, tries to represent the energy in the spectrum to the right of the model's name.

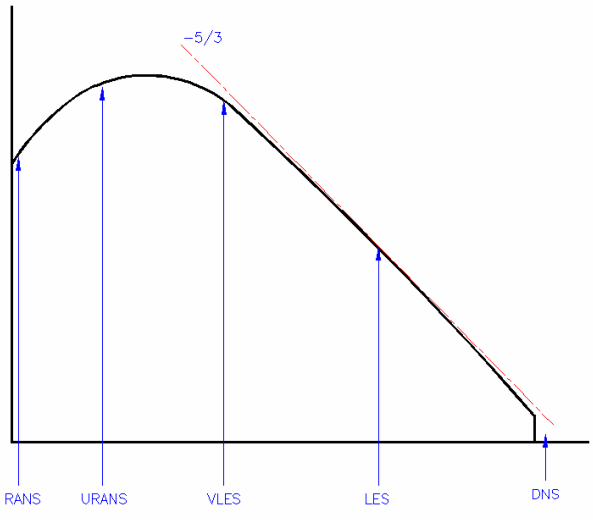

FIG. 1. Illustration of RANS – DNS energy spectra.

Very recently there have been a number of hybrid turbulence models developed that are designed to be able to perform over a broad range of the spectrum (i.e. URANS down through LES) depending on the situation. For example, the very popular DES (direct eddy simulation) model [1] behaves like a RANS or URANS model near walls, but away from walls the lengthscale is changed to the mesh size and the model has an LES character. Other hybrid models do not change their character based on location relative to a wall, but on whether the mesh is much smaller than the energy containing turbulence scales (leading to LES) or not (leading to RANS or URANS). The earliest implementation of such an approach Speziale[2] used classic LES (Smagorinsky) and RANS (k/ε) models to solve for both an LES and a RANS eddy viscosity and then blended these two viscosities together based on a function of the mesh size. Girimaji[3] has developed a hybrid model that can change its character based on input from the user (the user sets the desired ratio of modeled turbulent kinetic energy). The SAS model of Menter[4] is an attempt to fix the lengthscale deficiency of DES. The SAS idea has also been applied to k/ε models (and involves adding another term to the dissipation equation similar to the RNG correction).[5]

In this work, we demonstrate a self-adapting turbulence modeling approach that works at any mesh resolution and over the entire spectrum. It can therefore do, RANS, URANS, VLES, LES and even DNS. More importantly, the character of the model is not set by the user or the geometric location, but adapts to whatever level the mesh can support. The proposed approach therefore models only as much turbulent kinetic energy is necessary (for that mesh) and resolves as much of the energy using first principles as possible. It is not correct to consider the proposed approach to be a hybrid model in the classical sense (though it has many similarities to those

models) because it does not blend an LES and a RANS model together. The proposed model is closer to DES or SAS in philosophy in that it is a single set of transport equations that changes its character (RANS, LES or DNS) depending on the flow situation.

In Sec. II the mathematical background for a 'universal' model is presented following the ideas of Germano.[6] The model itself is presented in Sec.III and numerical tests of the model are analyzed in Sec. IV where tests of isotropic decaying turbulence are performed. Section V has a review of the key findings of this work and a brief discussion of the results.

II. Mathematical Background

The classic mathematical theory behind RANS and LES makes these two modeling approaches look fundamentally different. RANS is based on ensemble averages (although time averaging is very frequently substituted) and LES is based on filtering. At first glance, the possibility of a single model that does both (without some sort of switch or blending function) seems remote. However, a closer examination by Germano[6] revealed some very important insights. Most importantly, the exact but unclosed governing equations for RANS and LES (and URANS, VLES and DNS) are mathematically identical. So while the RANS equations can be derived from the assumption of ensemble averaging and the LES equations from filtering operations, these assumptions are overly restrictive and neither system *must* be derived with those assumptions. The only required assumption is that the velocity field can be split into two parts and that this splitting operation commutes with differentiation. With this assumption, the equations for turbulence evolution are (from Appendix A)

$$\overline{u}_{i,t} + (\overline{u}_i \overline{u}_j)_{,j} = -\overline{p}_{,i} + \nu \overline{u}_{i,jj} - R_{ij,j} \tag{1a}$$

where $\overline{u}_i$ and $\overline{p}$ and the computed velocity and pressure and $R_{ij} \equiv \overline{u_i u_j} - \overline{u}_i \overline{u}_j$ is the unknown turbulent stress tensor. The exact (but unclosed) evolution equation for this stress tensor is

$$\begin{aligned} R_{ij,t} + \overline{u}_k R_{ij,k} &= \nu R_{ij,kk} - (R_{jk}\overline{u}_{i,k} + R_{ik}\overline{u}_{j,k}) \\ &\quad - T_{ijk,k} - (< p_{,i}, u_j > + < p_{,j}, u_i >) - 2\nu < u_{i,k}, u_{j,k} > \end{aligned} \tag{1b}$$

where the bracket operation is given by $< a_i, b_j > \equiv \overline{a_i b_j} - \overline{a}_i \overline{b}_j$ and turbulent transport T_{ijk} is defined by $T_{ijk} \equiv \overline{u_i u_j u_k} - \overline{u}_i R_{jk} - \overline{u}_j R_{ik} - \overline{u}_k R_{ij} - \overline{u}_i \overline{u}_j \overline{u}_k$. The turbulent transport and the bracketed terms require a model if the system is to be solved. In RANS the overbar might denote an ensemble average. In LES the overbar might be an explicit filtering operation. However it can also be an implicit operation, because in practice (when these equations are modeled and then solved on a computer), the overbar operation is never actually performed. In this case, it is assumed that an overbar represents whatever the calculation computes. It is not possible to prove that an implicit filter commutes with differentiation, but it is a fairly reasonable assumption to make (at least to first order).

To accurately model some of the terms in Eqns (1a) and (1b) a third 'scale' equation is often postulated that captures the turbulent energetic length or timescale. The epsilon equation[7] is perhaps the most commonly used scale equation, but omega (inverse timescale)[8] and lengthscale equations[9-10] are also possible and can be advantageous. These scale equations can, in theory, be

derived from first principles, but the subsequent modeling assumptions, in practice, makes them empirical.

Starting from the exact equation (1b) numerous modeling approaches are possible. In order of increasing simplicity we briefly describe some of the more common approaches. Reynolds stress transport (RST) models use a modeled form of equation (1b). The more popular two equation RANS models (such as k/ε or k/ω) are a simplification of equation (1b) from a tensor equation to a scalar equation (by taking its trace). The primary unknown, R_{ij} must then be reconstructed from the scalar kinetic energy, k , using a hypothesized algebraic relation such as the eddy viscosity hypothesis. One equation models, such as the Spalart-Allmaras model[11] used in DES, solve a single transport equation directly for the eddy viscosity and use an algebraic expression for the lengthscale (such as the distance to the wall, or the mesh size). Classic LES models (such as Smagorinsky and its variants) are the simplest of all, as they solve only Eqn (1a) and algebraically obtain the necessary length and timescales from the mesh size and the resolved velocity gradients.

Traditionally, LES models have used the very simplest modeling approach because of the issue of cost. Early attempts at using more complex transport equations were not deemed worth the extra effort. Deardorff[12] used a RST model and Schumann[13] used a k/ε model. Models and computing power have changed considerably in the 30 years since those first simulations. Some more recent LES models now carry a single transport equation. This is certainly the case in DES, and a kinetic energy transport equation was used by Ghosal *et al*.[14] As the mathematical analysis of Germano[6] makes clear, there is no fundamental reason why the more complex modeling approaches (used currently only by RANS models) can not also be applied to LES. The apparent natural evolution of turbulence models to include more physics and therefore complexity, suggests that two-equation transport models for LES are, in fact, the next logical step.

III. A Two-Equation LES Model

The k/ε system is used in this work in order to reach the largest audience possible. There are very good reasons to prefer other two-equation model systems. However, since all two-equation transport models are closely related, the proposed modeling ideas can be easily generalized to these other two-equation frameworks.

The unclosed equations (1a) and (1b) can be modeled using the follow transport equations,

$$u_{i,t} + (u_i u_j)_{,j} = -(p + \tfrac{2}{3}k)_{,i} + [(\nu + \nu_T \alpha)(u_{i,j} + u_{j,i})]_{,j} \quad (2a)$$

$$k_{,t} + (k u_j)_{,j} = [(\nu + \nu_T / \sigma_k) k_{,j}]_{,j} + \alpha P - \varepsilon \quad (2b)$$

$$\varepsilon_{,t} + (\varepsilon u_j)_{,j} = [(\nu + \nu_T / \sigma_\varepsilon)\varepsilon_{,j}]_{,j} + \tfrac{\varepsilon}{k}[C_{\varepsilon 1} P - C_{\varepsilon 2}\varepsilon] \quad (2c)$$

where the overbar on the velocity and pressure have been dropped for convenience. The production is given by $P = \nu_T (u_{i,j} + u_{j,i}) u_{i,j}$ and eddy viscosity is given by $\nu_T = C_\mu \frac{k^2}{\varepsilon}(\frac{k}{k+k_r})$. The constants are fairly standard k/ε constants, $C_{\varepsilon 1} = 1.55$, $\sigma_\varepsilon = 1.2$, $\sigma_k = 1.0$, $C_\mu = 0.18$. The parameter $C_{\varepsilon 2} = \frac{11}{6} f + \frac{25}{\mathrm{Re}_T} f^2$ is sensitive to the local turbulent Reynolds number $\mathrm{Re}_T = \frac{k^2}{\nu\varepsilon}$ of the modeled turbulence via the function $f = \frac{\mathrm{Re}_T}{30}\left[\sqrt{1 + \frac{60}{\mathrm{Re}_T}} - 1\right]$ as per the analysis of Perot and de Bruyn Kops.[9] This varies $C_{\varepsilon 2}$ from its theoretical limits of 11/6 at high Reynolds numbers to 3/2 at low

Reynolds numbers. Any Reynolds number dependent $C_{\varepsilon 2}$ would probably be sufficient. Reynolds number dependence is important in LES because the effective turbulent Reynolds number $\mathrm{Re}_T = \frac{k^2}{\nu\varepsilon}$ get smaller as the mesh is refined (and goes towards zero in DNS). For incompressible flow, the pressure in Eqn (2a) is determined from the constraint $u_{j,j} = 0$. In this system k is the modeled turbulent kinetic energy and $k_r = \frac{1}{2}(u'^2_1 + u'^2_2 + u'^2_3)$ is the resolved turbulent kinetic energy. Similarly ε is the modeled dissipation and $\varepsilon_r = \nu\left(u'_{i,j}\, u'_{i,j}\right)$ is the resolved dissipation of turbulent kinetic energy.

A number of eddy viscosity formulations were tested but the formulation presented above performed the best and is consistent with standard LES. In the LES limit the turbulent length scale $L_T = \frac{k^{3/2}}{\varepsilon}$ should be proportional to the mesh size (in a later section we show that this is indeed the case). In this limit this formula for the eddy viscosity then becomes $\nu_T \propto C_\mu (\Delta x)^2 (\frac{\varepsilon}{k+k_r})$, which scales the same way with the mesh size as most LES models (including the Smagorinsky model and its derivatives). In the RANS limit $(\frac{k}{k+k_r}) \to 1$, and the classic RANS eddy viscosity is recovered.

The other deviation from the classic k/ε model is the presence of the additional parameter α. The proposed formulation assumes that the turbulent stress tensor is reconstructed using the modified eddy viscosity hypothesis, $R_{ij} = \frac{2}{3}k\delta_{ij} - \nu_T \alpha (u_{i,j} + u_{j,i})$. It will be shown that this modified hypothesis can backscatter energy (while the classic formulation cannot). While we stick with the simplest and most common reconstruction hypothesis for simplicity, the proposed ideas can easily be generalized to nonlinear eddy viscosity and algebraic Reynolds stress models as well. The simplest model possible is used in this work (along with the simplest test case possible) in order to focus as directly as possible on the key idea of this paper - that it is possible to develop models that automatically work at any mesh resolution.

One key component of a self-adaptive turbulence model is that it must be able to backscatter energy from the unresolved (modeled) turbulence to the resolved (calculated) turbulence. For example, if a simulation of isotropic turbulence is performed on a 64^3 grid, but the simulation is initialized with almost all the energy in the model and very little in the resolved field (i.e. it is initialized from a RANS simulation), the model should be able to take energy from the modeled turbulent energy and energize the resolved turbulent kinetic energy field. The idea of allowing backscatter in a turbulence model is not a new one. It has been shown by Chasnov[15] and Carati *et al.*[16] that the -5/3 power law in isotropic decay is better predicted by (dynamic) LES models that account for backscatter. Along similar lines, classical DES can not backscatter energy but it has been recently shown by Piomelli *et al.*[17] that adding noise, a crude form of energy backscatter, to the DES model improved results for channel flow.

RST models based on Eqn (1b) can, and do, backscatter energy. However, any model based on a positive definite eddy viscosity (such as the classic k/ε model or any two-equation model) is too simplified and can not backscatter energy. The eddy viscosity is always positive and therefore always removes energy from the mean flow (and puts it into the modeled turbulent kinetic energy k where it is eventually dissipated by ε to heat). This is the correct average behavior for a turbulence model, but not necessarily locally correct (in space or time).

The additional parameter α has been added to the classic k/ε model to correct this important flaw and control the energy flow. Usually α is positive (and order 1), but it can become small or even negative (soon to be shown). When $\alpha < 0$ the model is backscattering energy. The parameter α is not a model constant, instead it is a field that varies in space and time, so backscatter can happen in different regions at different times. The transfer variable α also appears in Eqn (2b), the k-equation, so that the total turbulent kinetic energy (resolved plus modeled) is a conserved quantity and can only disappear via dissipation. Its presence is not necessary in the scale equation, Eqn (2c).

When $\alpha < 0$, the eddy viscosity in Eqn (2a) is essentially negative. Negative viscosity is anti-diffusive, it amplifies (rather than damps) existing resolved velocity fluctuations. It amplifies small wavelength modes (those closest to the mesh resolution) the most rapidly. This is a very reasonable model for backscatter. It is not injecting energy via some random forcing of the resolved flow, rather it works to enhance the existing instabilities and modes. Moreover, the energy transfer is local in spectral space. It tends to take energy from the model (which has most of its energy at scales just below the mesh resolution) and preferentially delivers it to the resolved flow at almost the same length scale (but just above the mesh resolution).

The model for α, the energy flow parameter, uses ideas from error estimation for mesh adaptation. If the variation across one mesh size in the computed solution is small, then the mesh is deemed to be sufficient for first principles simulation and the model should go away (the modeled turbulent kinetic energy should become very small). This is akin to the situation described earlier, where a 64^3 grid is initialized with a RANS solution and some small resolved flow fluctuations. The model will detect this situation as very highly resolved (lots of mesh resolution for the given fluctuations). Due to energy conservation inherent in equations (2a) and (2b), the modeled turbulent kinetic energy can not just disappear. In order to become small the modeled turbulent energy must be transferred somewhere – the only possibility is to the resolved flow. Small estimated variations in the resolved quantities implies there should be energy backscatter (the mesh can handle more fluctuations via direct simulation). At some point later in time, the resolved velocity on the 64^3 mesh will be fully energized. If the Reynolds number is high, then a 64^3 mesh is still not sufficient to perform DNS (a model is still necessary). In this case the variation estimate will become larger and larger until α becomes positive and a normal forward energy scatter down the energy cascade will be imposed. The larger the variation the more energy is fed to the modeled turbulent kinetic energy and the more the model influences the resolved flow evolution. If the Reynolds number is low enough, then a 64^3 mesh may actually be sufficient resolution for DNS and the variation estimate stays small. In this case the model continues to backscatter until it has almost no energy. At this point the model has no effect on the resolved modes and DNS is achieved.

The proposed equation for the energy transfer variable α is,

$$\alpha = 1.5\left(1 - C^*\left(\frac{k}{k+k_r}\right)^2\left[\left(\frac{\Delta x_i}{\sqrt{k_r}}\frac{\partial\sqrt{k_r}}{\partial x_i}\right)^2 + 0.11\right]^{-1}\right) \qquad (3a)$$

where k_r is the resolved turbulent kinetic energy (at a certain location and time), k is the modeled turbulent kinetic energy, and the empirically determined constant $C^* = 0.28$. The quantity

$$\left(\Delta x_i \frac{\partial\sqrt{k_r}}{\partial x_i}\right)^2 / k_r = \left\{\left(\Delta x \frac{\partial\sqrt{k_r}}{\partial x}\right)^2 + \left(\Delta y \frac{\partial\sqrt{k_r}}{\partial y}\right)^2 + \left(\Delta z \frac{\partial\sqrt{k_r}}{\partial z}\right)^2\right\} / k_r \qquad (3b)$$

is a dimensionless measure of the gradients (similar to what is sometimes used in mesh adaptation) that is well defined even on anisotropic meshes. In this formulation, the resolved turbulent kinetic energy $k_r = \frac{1}{2}(u'^2_1 + u'^2_2 + u'^2_3)$ is the indicator function that is being used to estimate the mesh resolution. If the flow is DNS or over-resolved (like a RANS initial condition on an LES mesh) then this quantity is small, its inverse is large (but limited away from infinity by the fairly arbitrary 0.11 term), and the model tends to backscatter energy. In contrast, normal energy transfer (from resolved scales to the modeled scales) occurs in the regions of the flow where the gradient length scales are comparable to the mesh size. This is shown in the results section of the paper.

On very coarse meshes, RANS like behavior is the expected limit. In this case k_r is expected to be very small, but note that $(\Delta x_i \frac{\partial \sqrt{k_r}}{\partial x_i})^2 / k_r$ will remain finite and independent of the magnitude of the resolved fluctuations. It just senses the resolution. In our simulations of isotropic turbulence this term obtains an average value of 0.8 in the RANS limit. This means that $\alpha \to 1.5 - 0.42/(0.8 + 0.11) \approx 1.04$ and the standard RANS model is very closely recovered in the RANS limit.

The particular form of the energy transfer function was developed and tuned solely to obtain the correct limits. Many other functional expressions and/or indicator quantities are certainly possible. The goal of this paper is not to advocate for this particular function but to demonstrate that self-adaptive turbulence models are possible, and this particularly function serves this purpose adequately.

IV. Model Results

The governing equations were solved using a Cartesian staggered mesh method[18] with exact projection[19] for the pressure solution. Isotropic decaying turbulence was calculated using periodic boundary conditions on a box of size 18π x 18π x 36π. The extra length in the third direction was to compare directly with DNS results calculated elsewhere.[20] This third direction also has double the resolution so the mesh size is the same in all three directions. The numerical method is second order accurate in space and time and locally conserves mass and momentum. In addition, it locally conserves circulation and kinetic energy in the absence of viscosity.[21] The code is fully parallel (using MPI) and optimized for execution on PC clusters.[22]

A. High Reynolds Number Isotropic Decay

In isotropic turbulence, the turbulent kinetic energy decays with time. This process can be simulated with a simple RANS model using only one spatial grid cell, with LES using many cells, or with DNS using enough cells to resolve the smallest length scales of motion. Results from such a test are shown in Fig. 2. The initial turbulent Reynolds number $\mathrm{Re} = \frac{(k+k_r)^2}{\nu(\varepsilon+\varepsilon_r)}$ for this test case is 640 (comparable to the Compte-Bellot and Corrsin 1971 experiment).[23] The DNS result was performed independently by de Bruyn Kops[20] on a 768x768x1536 mesh using a Fourier spectral method and is given by the large circles. Many aspects of the DNS (including spectra) have been closely compared with experimental data and shown to agree well.[24] The mesh resolution is identical in each direction, and the box size in the z-direction is twice as large.

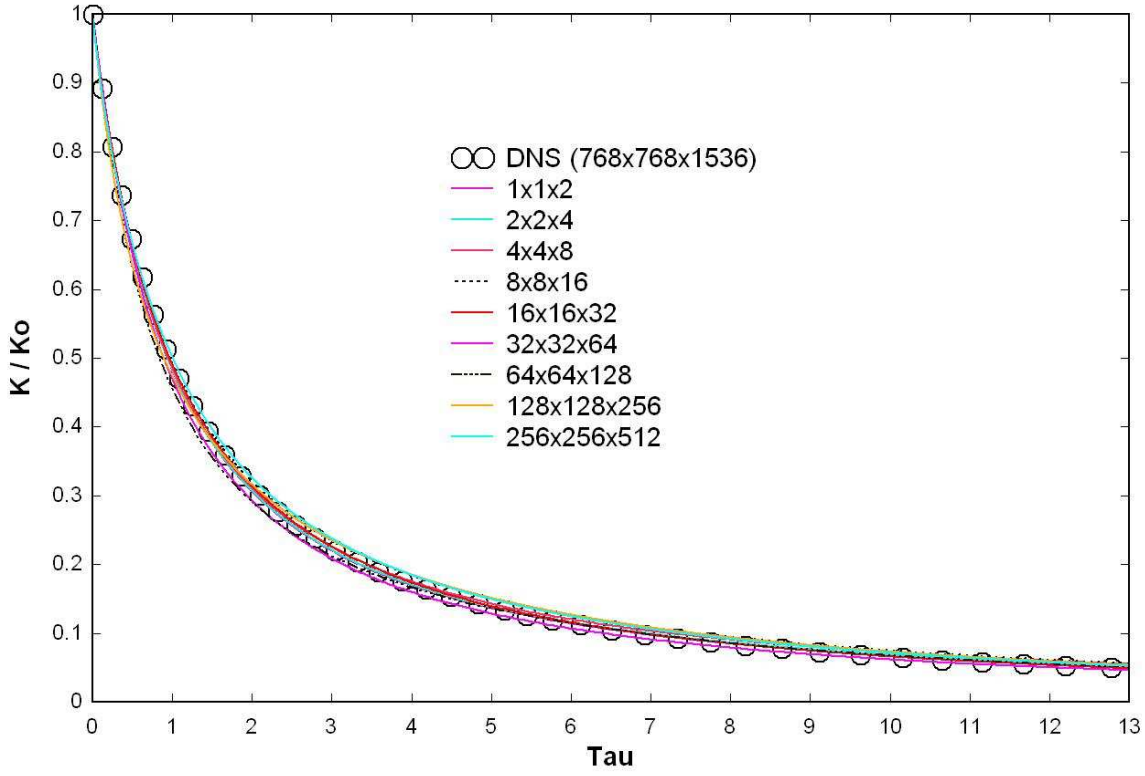

FIG. 2. Total turbulent kinetic energy predictions of isotropic decay at initial Re=640. Circles are DNS data and lines are model predictions for a variety of mesh resolutions from RANS through LES.

A number of different simulations were performed using mesh resolutions from 1x1x2 to 256x256x512. In each case the model is identical and only the mesh and initial conditions are changed. The initial conditions for the full DNS were obtained by running for a long time with as little forcing of the large scales as possible to maintain the kinetic energy. The initial condition is therefore not random Fourier modes – but Navier-Stokes turbulence. Each model initial condition was formed by averaging the same initial 768x768x1536 velocity field to the appropriate mesh size using a simple box average. The initial modeled turbulent kinetic energy at each mesh location was then determined by comparing the exact 768x768x1536 velocity field to the smoother coarse mesh field and calculating the sum of its difference. The initial modeled dissipation was calculated similarly by using a box average of the exact DNS dissipation field.

The simulations in Fig. 2 show that at any mesh resolution, the model predicts the decay of the turbulence well. The ordinate is total turbulent kinetic energy normalized by the initial total turbulent kinetic energy at t=0.0s, the abscissa (Tau) is time, non-dimensionalized by the inverse time scale, $\left(\frac{\varepsilon+\varepsilon_r}{k+k_r}\right)$ at t=0.0s. Tau can appropriately be interpreted as "eddy" turn over times.

The very lowest mesh resolution is clearly a RANS (or URANS) simulation and the largest mesh, 256x256x512, is a QR-LES simulation (with the mesh size roughly in the inertial range). The intermediate resolutions might be considered URANS, VLES, or LES. The spectra for the initial conditions are shown in Fig. 3. These spectra represent VLES (32x32x64), LES (64x64x128,

128x128x256), QR-LES (256x256x512), and DNS (768x768x1536) simulations. The spectra show that the box averaging procedure of the initial data does remove some energy from the low wavenumbers, especially at VLES (32x32x64) and LES (64x64x128). However, as resolution increases, most of the energy is taken from the highest wavenmbers and the resolved spectra does approach the DNS limit. All energy that is not resolved is modeled by the variable *k*.

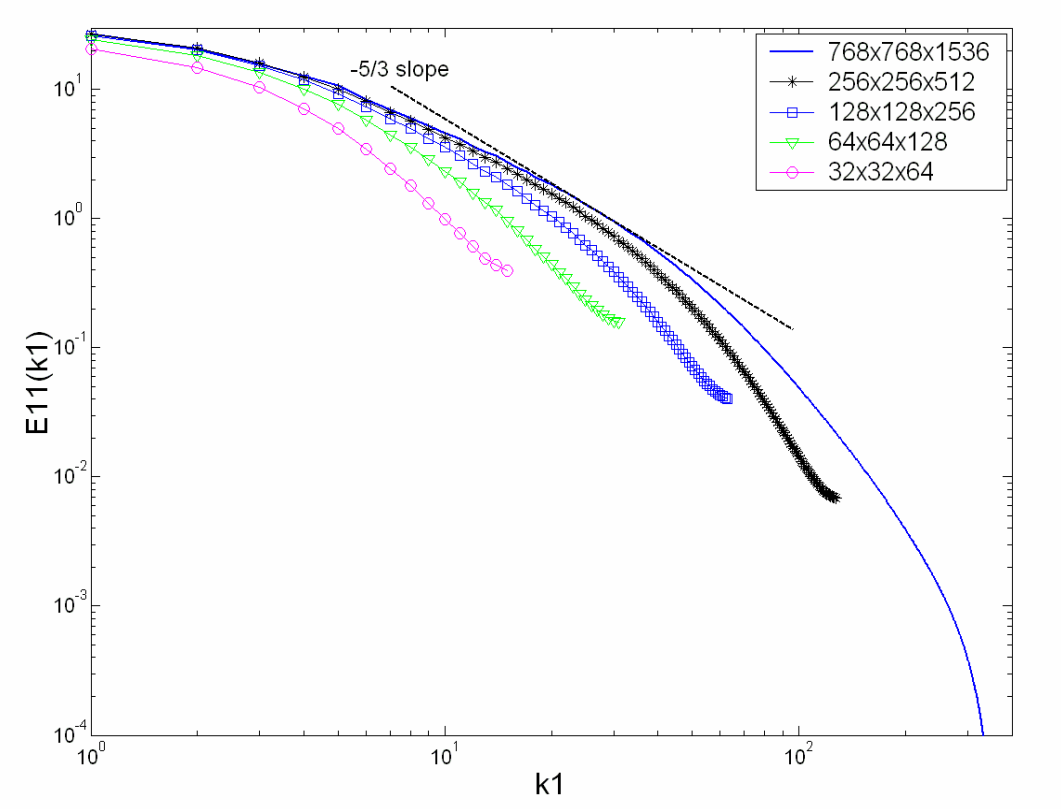

FIG. 3. Spectrum of the initial conditions for the Re = 640 isotropic decaying turbulence test case.

It might be expected that RANS equations should give RANS (essentially the 1x1x2) results irrespective of the mesh resolution used to solve those equations. This is not the case, for two reasons. First the variations will be different for different meshes which will affect the backscatter term. More importantly, the equations are nonlinear, and like the Navier-Stokes equations, have multiple possible solutions (some of which are unsteady and three-dimensional). The ratio of the modeled turbulent kinetic energy to the total turbulent kinetic energy is shown in Fig. 4 with one curve for each of the mesh resolutions. The 1x1x2 solution is the top curve, with all its turbulent kinetic energy contained in the model (giving a ratio of 1.0) and the 256x256x512 simulation is the bottom curve, with the smallest ratio of modeled turbulent kinetic energy (< 10%). Note that these curves are relatively constant but decrease slightly with time (as the simulation proceeds). Even though the equation system looks like a classic RANS model – it is not. This two-equation model is actually a 'universal' turbulence model applicable at any mesh resolution. There is no tendency for the solutions to move towards the RANS solution (a ratio of 1.0).

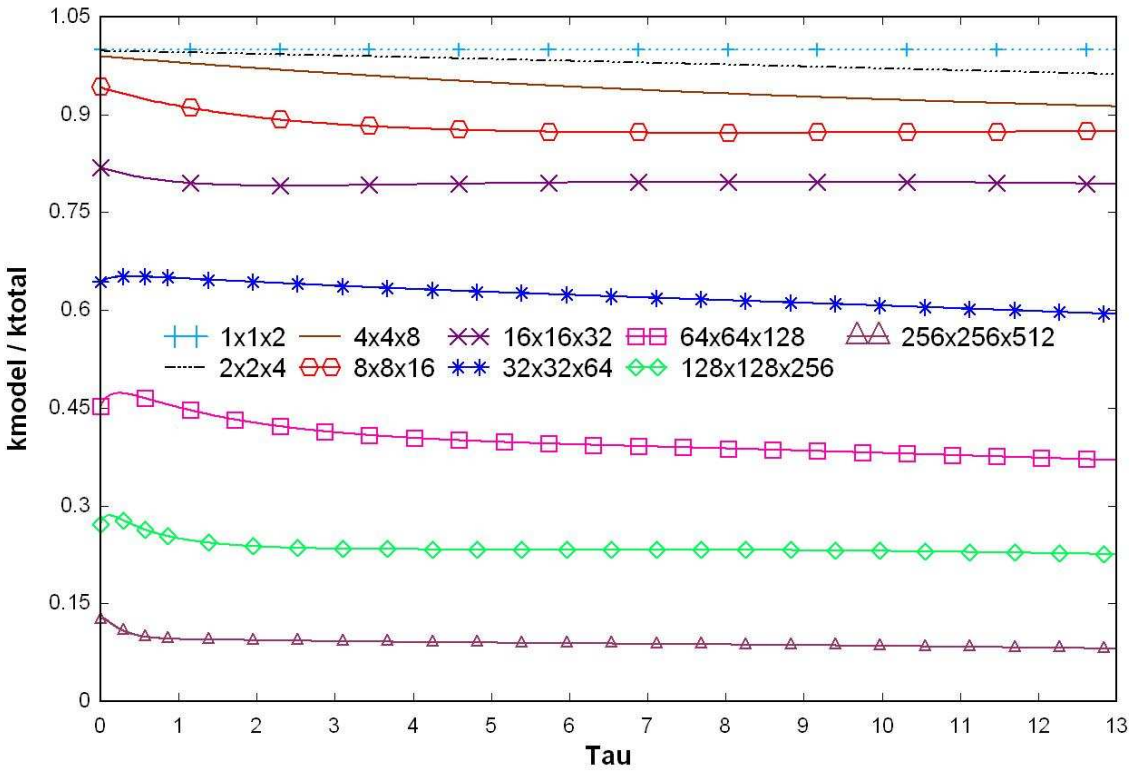

FIG. 4. Ratio of modeled turbulent kinetic energy (k) to total turbulent kinetic energy ($k+k_r$). Re=640

The LES solutions stay entirely unsteady and three-dimensional. The slight decrease in this ratio over time is the correct behavior. It is due to the fact that over time the Reynolds number of the flow is slowly decreasing and the mesh can, and therefore does, resolve a larger percentage of the turbulent fluctuations. If the simulation was run long enough, then the Reynolds number would drop sufficiently for the 256x256x512 mesh to perform DNS, and at this point the ratio should be essentially zero (see Fig. 11). The small variation at the beginning of each simulation shows the initial condition is close to the correct ratio supportable by that mesh, but not perfect. The long time behavior of this ratio is shown in Fig. 5. At these very long times, the turbulence scales grow large enough to feel the influence of the box size and the decay rate no longer follows the classical theory for homogeneous decay in an infinite domain. For this reason the total turbulent kinetic energy at these long times is not of interest. However, the long time behavior of the model ratio is still of interest. Figure 5 shows that the coarse mesh (4x4x8) appears to be approaching the RANS solution (a ratio of 1.0) at very long times. The 8x8x16 simulation initially tries to approach the RANS solution, however this ratio eventually starts to decrease. The finer meshes (16x16x32 – 64x64x128) have enough resolution to compute a solution that does not return to the RANS limit. It is hypothesized that steady (RANS-like) solutions occur when the resolution is not sufficient to maintain an energy cascade. For this test case, the initial large eddy lengthscale of the turbulence ($L = k^{3/2}/\varepsilon$) is 6, and the 8x8x16 simulation has a mesh size of close to 7. Nikitin *et al*. has noted that a similar effect occurs in under-resolved DES simulations.[25]

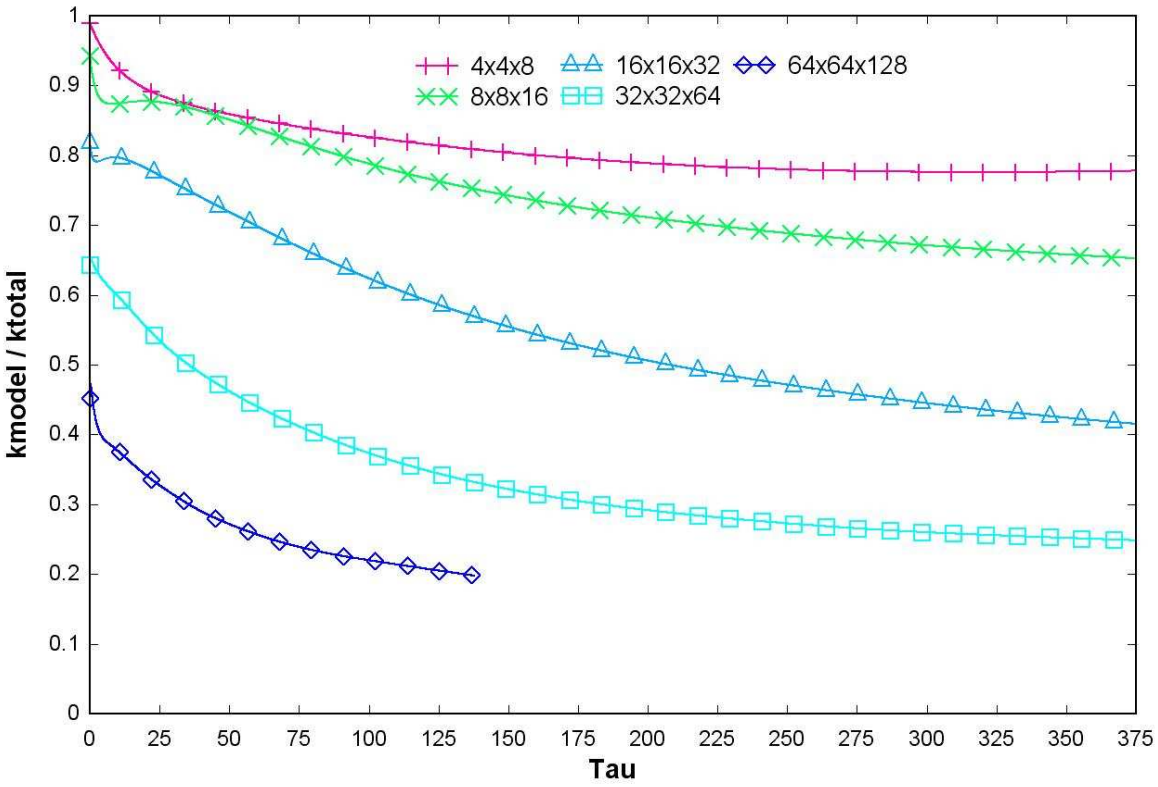

FIG. 5. Ratio of modeled turbulent kinetic energy (k) to total turbulent kinetic energy ($k+k_r$) for extended time. Re=640

B. Effect of Initial Conditions

A truly adaptive model should be able to obtain the correct behavior from incorrect initial conditions. For example, it is of considerable interest to see if a 64x64x128 mesh initialized with a RANS solution can, over time, develop into a full LES simulation. In order to test the model in this way, the initial conditions were either smoothed or sharpened using a filtering operation. The filter used to alter the initial conditions was a nearest neighbor averaging procedure,

$$u_{ijk}^{filtered} = \beta u_{ijk} + (1-\beta)(u_{i+1\,jk} + u_{i-1\,jk} + u_{ij+1k} + u_{ij-1k} + u_{ijk+1} + u_{ijk-1})\tfrac{1}{6} \qquad (4)$$

For smoothing $\beta = 0$ was used. This replaces the value at a mesh point by the average of its nearest neighbors. This type of filter removes energy primarily from the highly oscillatory modes with wavelengths close to the mesh size. In spectral terms it damps the spectra in the region just above the cutoff wavenumber. The effect is shown in Fig. 6, which shows the original initial spectra for the 64x64x128 simulation, and the spectra for the smoothed and sharpened initial conditions. Sharpening is performed by using $\beta = 1.5$, this adds energy to the existing high frequency modes.

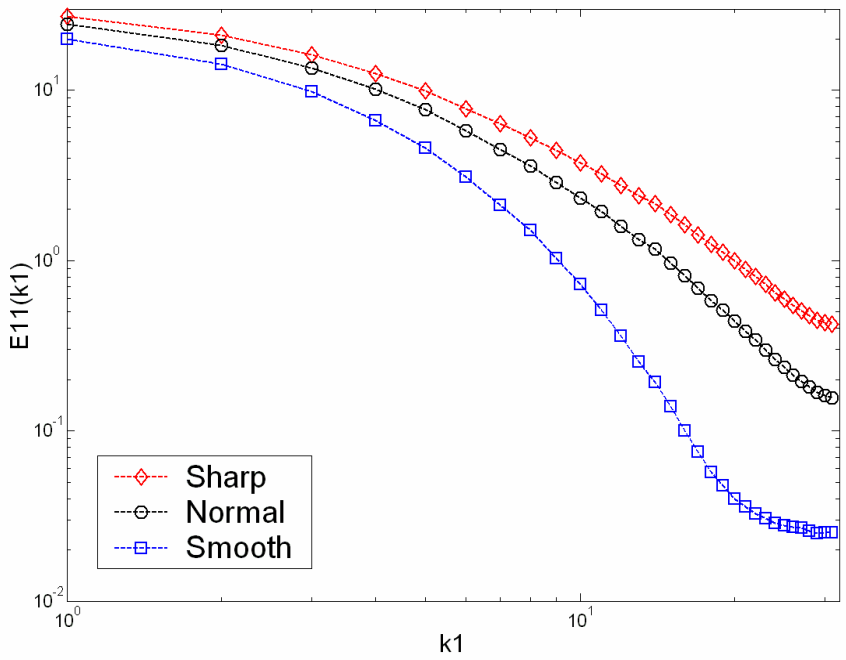

FIG. 6. 1D energy spectra for 64x64x128 simulation. (Sharp, Normal, Smooth)

Figure 7 shows the affect of smoothing and sharpening the initial conditions on the turbulent kinetic energy ratio. When smoothing is used, energy is removed from the resolved modes. In order to keep the total turbulent kinetic energy the same, the model now must start with more energy. Therefore the ratios for the smoothed case start higher than before.

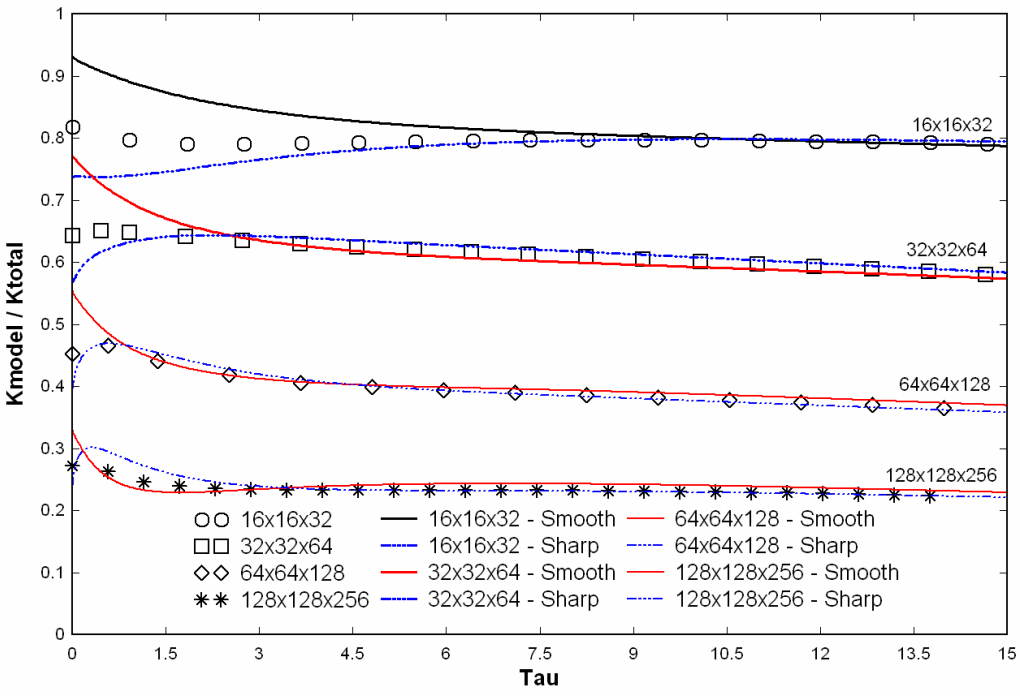

FIG. 7. Ratio of modeled turbulent kinetic energy (k) to total turbulent kinetic energy ($k+k_r$). Re=640

As time proceeds the model achieves the same ratio irrespective of the initial conditions. At early times, the smoothed solution has less variation and therefore backscatters somewhat more than the unperturbed initial condition. This removes energy from the model and makes the ratio decrease faster, so that it approaches its original state. A similar (but opposite) process happens when the spectra is sharpened. In this case, the model senses that the mesh can not support the input resolved fluctuations, the eddy viscosity is increased by α, and damping of the resolved modes occurs with the resulting energy transfer to the model. Note that the rate at which the model adjusts to incorrect initial conditions depends on the mesh resolution. The higher mesh resolutions adjust much more quickly. It is hypothesized that the time it takes to transfer the energy scales on the timescale of the turbulence at the cutoff (transfer) lengthscale (k/ε).

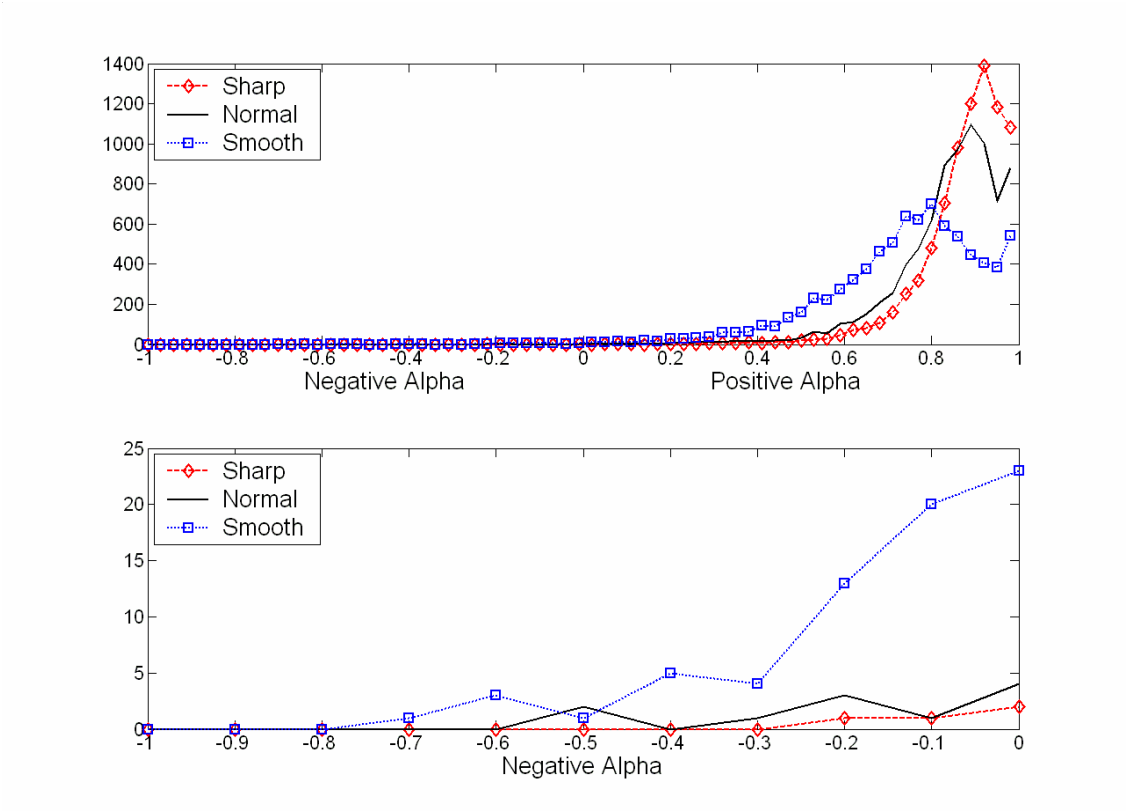

FIG. 8. Histogram of the energy transfer function alpha (top), enlarged view of the negative alpha values (bottom).

It had been stated earlier that the energy transfer function α was usually positive and order 1. In order to visualize exactly what happens during a simulation with perturbed initial conditions, a histogram of α is provided in Fig. 8. This figure (top view) shows that with the correct initial conditions (solid line), α obtains an average value of 0.87 for this particular mesh (64x64x128). The bottom view of Fig. 8 is an enlarged section of the negative values α obtains. The histograms include values of α for the smooth (squares) and sharpened (diamonds) initial conditions whose spectra were shown in Fig. 6. As mentioned previously, smoothing the velocity field removes energy from the resolved flow and places too much energy into the model. To compensate for this error Fig. 8 shows that α takes on a smaller average value around 0.80 and has a drastic increase in negative values (more backscatter). Because the model now has more backscatter, α is correctly displacing energy from the model to the resolved field to correct for the wrong initial condition.

The opposite effect is shown for the sharpened case. The simulation has been started incorrectly with too much energy in the resolved field. Here it is seen that α obtains a slightly larger value than 0.87 to ensure more normal forward transfer of energy. Because of this shift to the right, there is considerably less (almost non-existent) negative α values. This guarantees almost no backscatter and an increase in forward scatter to correct the initial conditions.

The histograms of Fig. 8 are very similar for all of the mesh resolutions. In all cases, the stability of Eqn (2b) is verified by α having an average positive value. This supports the energy cascade theory that overall the transfer of energy will be from large to small scales. However, it is shown that locally the energy transfer can go the other way and small amounts of backscatter can indeed provide a method for a self-adapting model.

C. Low Reynolds Number Isotropic Decay

In order to verify that the model works, even in the DNS regime, a lower Reynolds number simulation was performed which is well resolved (DNS) at the largest resolution of 256x256x512. The initial field for this lower Reynolds number case was generated by box averaging the high Re initial 768x768x1536 field to 256x256x512. The simulation was then run with a higher viscosity and with a small amount of modeled turbulent kinetic energy. Using the higher viscosity reduces the Reynolds number and also quickly damps the smallest scales of the turbulence.

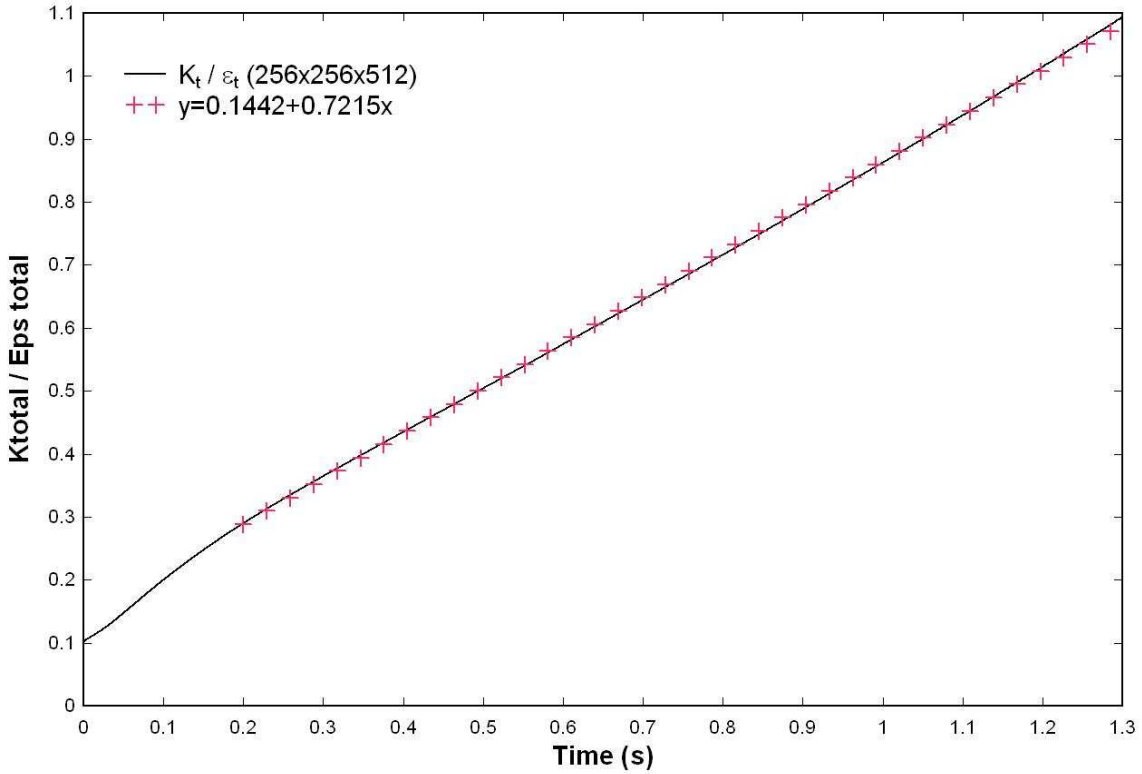

FIG. 9. Turbulent time scale $(k+k_r) / (\varepsilon+\varepsilon_r)$ vs. simulation time for a 256x256x512 simulation with very little modeled turbulent kinetic energy (DNS). Slope of curve fit (plus signs) give the inverse of decay exponent (n).

The turbulent time scale $\left(\frac{k+k_r}{\varepsilon+\varepsilon_r}\right)$, is plotted vs. simulation time (solid line) in Fig. 9 for the low Re case. At early times, the small scales are adjusting to the low Re. Eventually, a power law decay for the turbulent kinetic energy is obtained and the simulation is then considered to be fully adjusted. On Fig. 9, a power law decay should appear as a straight line. A linear curve fit is given by the plus symbols. The inverse of the slope given by the curve fit is the kinetic energy decay exponent, $n \approx 1.38$. The simulation was considered to be well developed at a time of 0.14s. This field (at t=0.14s) was used in all the subsequent simulations, as the initial well developed low Reynolds number (Re=211) initial condition. The same box averaging procedure as the high Reynolds number case was also used to create initial conditions for the smaller meshes in the low Reynolds case.

Figure 10 shows the total turbulent kinetic energy predictions for the low Re test case. The circles represent the essentially DNS simulation (though the model is on), the lines represent the various simulations from 1x1x2 through 128x128x256. These are in fairly good agreement with the DNS data, as was observed for the high Reynolds number case. The slight discrepancy at long times is probably due to errors in the RANS model at lower Re.

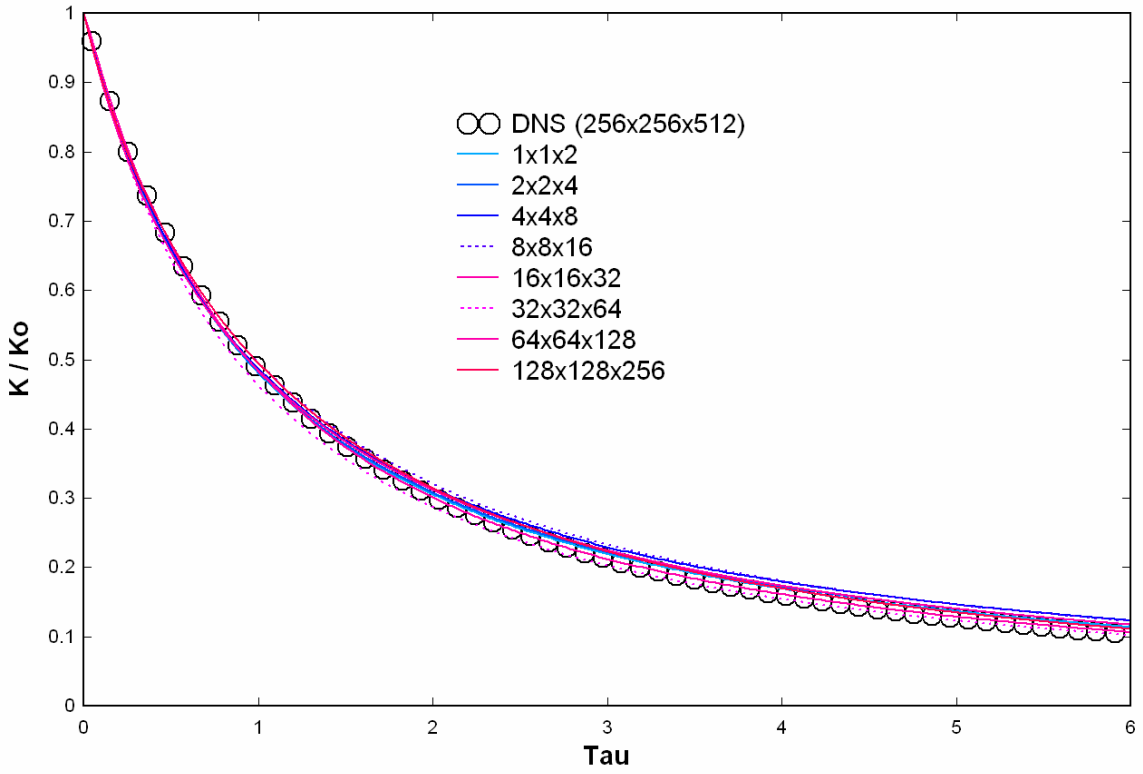

FIG. 10. Total turbulent kinetic energy predictions of isotropic decay at initial Re=211. Circles are DNS data and lines are model predictions for a variety of mesh resolutions from RANS through LES.

The ratio of modeled turbulent kinetic energy is plotted in Fig. 11. The full RANS simulation (1x1x2) is at the top of the figure (ratio of 1.0) and the DNS is shown at the bottom of the figure with a ratio of roughly 10^{-3}. Once again it is observed that there is no tendency for these solutions to move

towards the RANS solution (ratio of 1.0). The LES solutions stay entirely unsteady and three-dimensional, and the correct decrease in this ratio over time is shown.

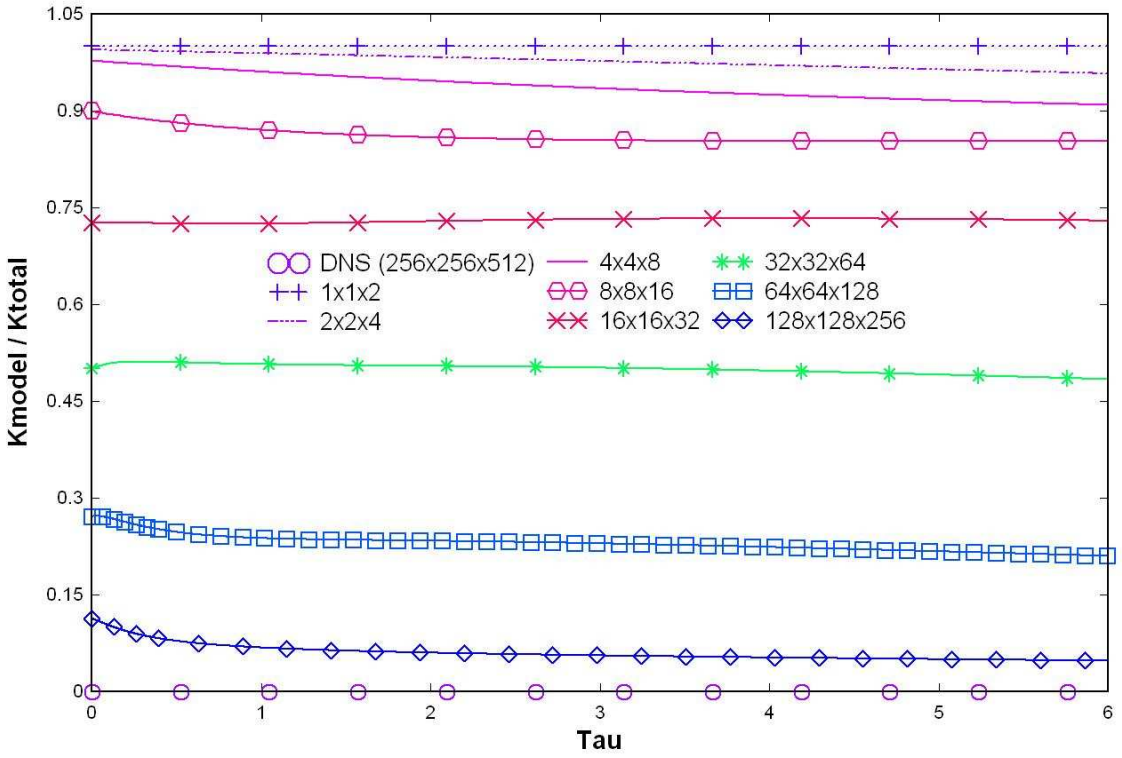

FIG. 11. Ratio of modeled turbulent kinetic energy (k) to total turbulent kinetic energy ($k+k_r$). Re=211

D. Scaling

In classic LES models the lengthscale is assumed to be proportional to the mesh size, Δ, and the gradients scale like $u_{i,j} \sim \varepsilon^{1/3}\Delta^{-2/3}$. The eddy viscosity is then constructed from this scaling and has the form, $\nu_T \sim u_{i,j}\Delta^2 \sim \varepsilon^{1/3}\Delta^{4/3}$. The use of the mesh size to infer the lengthscale is why classical LES fails outside of the inertial range (close to the DNS or RANS regimes), and one reason why the current model (which predicts the lengthscale and does not infer it) can operate at with any mesh size. However, the current model should still obtain the classical LES behavior in its range of applicability. The lengthscale predicted by the model, $L_m = k^{3/2}/\varepsilon$, should be proportional to the mesh size, when the model is operating in the LES regime.

Figure 12 looks at predicted lengthscale (in log scale) at a fixed time t=0.5s for the various meshes. At small values of Δx (large numbers of mesh points) it is obvious that the lengthscale is indeed a function of the grid spacing as would be expected with classic LES. A reference line has been added (dotted line) with a slope of 1.0 for comparison. If a well defined -5/3 slope were to be observed in the energy spectra, one would assume that classic LES would closely match the reference line in that -5/3 region. Because the spectrum shown (Fig. 3 and Fig. 6) show only a weakly observable -5/3 slope, it is not surprising that the lengthscale shown in Fig. 12 does not match the reference line exactly. One would expect viscous effects (the influence of a viscous lengthscale) to reduce the

scaling below linear and this is indeed observed. The figure clearly shows how the turbulent lengthscale is not related to the mesh spacing outside the LES regime.

Mesh resolutions smaller than LES (64x64x128) do not exhibit the linear behavior because now the relevant turbulent lengthscale is no longer related to the mesh size. There is a transition region that one might call VLES (32x32x64), but not full LES. The 8x8x16 mesh might be considered the start of URANS (as was shown in Fig. 5), where the largest scales can just be resolved, and unsteady 3D solutions maintained. Mesh resolutions smaller than 8x8x16 are where the solutions can not sustain a cascade because all the largest turbulence has scales that are smaller than the mesh size. Interestingly, Carati *et al.*[16] conducted LES simulations for isotropic decaying turbulence and determined that the smallest mesh size possible for a classic (dynamic model) LES simulation was 48^3. This is in very good agreement with the results of Fig. 12, and is just where the elbow in the curve occurs.

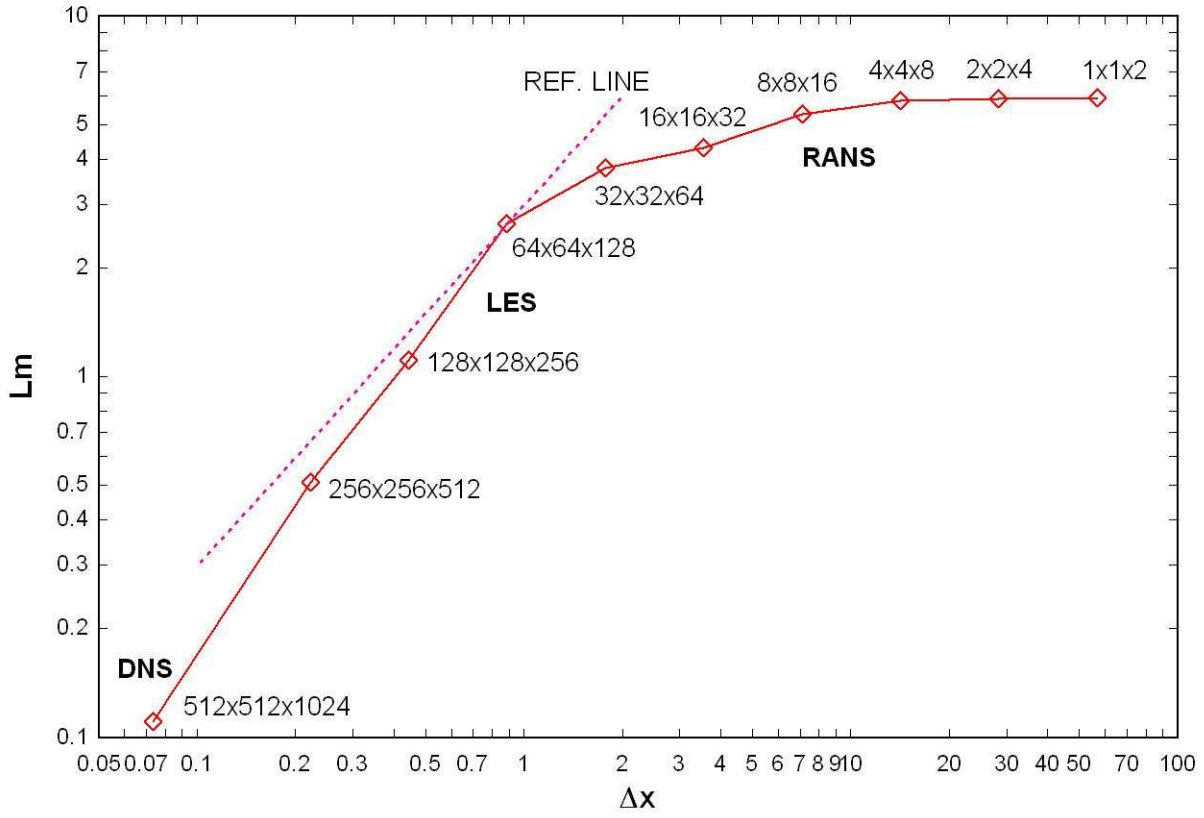

FIG. 12. Lengthscale behavior at the time = 0.5s.

V. Discussion

The primary goal of this paper is to demonstrate that RANS models can be used to model turbulence at any mesh resolution. The fundamental reason that this is possible is because the governing equations are the same for RANS, URANS, VLES, LES, and DNS. However, just because the exact equations are mathematically the same does not mean that the same models can be used in both the RANS and LES regimes. This work indicates that with slight modification, RANS models (which

can represent fairly complex turbulence physics) can also model subgrid scale turbulence equally well.

In order to reach as broad an audience as possible, this idea has been demonstrated with the well known k/ε model. There is every reason to suspect that other RANS models will work just as well. The paper shows that two-equation models are not limited to RANS modeling, they can be used to model subgrid scale turbulence for LES and all the regions in between. The 'universality' or automatic adaptivity of this approach is actually very important, because in many practical turbulent flows it is not possible to tell if a mesh is actually viable for LES ahead of performing the simulation.

The implication here is that subgrid scale turbulence is not fundamentally more complex than 'regular' turbulence. It can be modeled using the same RANS ideas. This paper shows that two-equation models produce chaotic unsteady three-dimensional solutions (just like the Navier-Stokes equations do) when there are enough degrees of freedom. RANS models do not necessarily result in RANS-like (steady) solutions.

The advantage of using a two-equation model for turbulence is that it is sufficiently powerful to model the turbulence at any mesh resolution. Classic LES models based on the Smagorinsky approach ($R_{ij} - \frac{2}{3} K \delta_{ij} = -C\Delta^2 \mid S \mid S_{ij}$) are limited to meshes that lie in the inertial range. This is because they all assume that the relevant length scale for the model is the mesh size. In the inertial range this assumption holds, but when the mesh is very coarse (in the energy containing range) or very fine (in the DNS range) this assumption is incorrect[26] and the classic LES approach and its variations like dynamic modeling are fundamentally flawed.

Turbulence models based on a single transport equation (like DES) also have some of the same problems of the classical approach. Single equation models must use an algebraically specified lengthscale to correctly represent dissipation processes. When this lengthscale is not correct the model fails. DES can therefore only be expected to perform well if the turbulence is near a wall, or the mesh is in the inertial range. The very simple test case presented in this paper is one in which DES will fail. There is no way to accurately specify the length scale for the coarse mesh cases. The relevant length scale is no longer the mesh size. It is the energy containing eddy size which can not be determined from a single model equation alone.

In our test case, the RANS model can predict the answer accurately. For real industrial problems we would expect the LES solutions to be better than the RANS solutions, since more of the turbulence is calculated via first principles. A single 'universal' model of the type demonstrated herein allows one to consider doing mesh refinement studies for turbulent flows that really do give a sense of whether the solution (and model) are converging.

The proposed model may be attractive for industrial situations since the same code can be used to perform coarse preliminary design simulations as well as highly refined 'performance' calculations of the final designs. The only input required by the user is the size of the mesh they can afford. The model produces the best solution possible for that mesh resolution, since it computes as much of the turbulence from first principles as the mesh will allow.

Acknowledgments

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Appendix A

The incompressible Navier-Stokes equations are:

$$u_{k,k} = 0 \tag{A.1}$$

$$u_{i,t} + (u_i u_k)_{,k} = -p_{,i} + \sigma_{ik,k} \tag{A.2}$$

Following the ideas of Germano[6], the moment of A.2 is taken with u_j, another moment of this same equation with indices interchanged yields equations A.3 and A.4.

$$u_j u_{i,t} + u_j (u_i u_k)_{,k} = -p_{,i} u_j + u_j \sigma_{ik,k} \tag{A.3}$$

$$u_i u_{j,t} + u_i (u_j u_k)_{,k} = -p_{,j} u_i + u_i \sigma_{jk,k} \tag{A.4}$$

Adding together A.3 & A.4 and using simple calculus it can be shown that the resulting equation can be written as:

$$\left(u_i u_j\right)_{,t} + (u_i u_j u_k)_{,k} = -\left[p u_i \delta_{jk} + p u_j \delta_{ik} - \nu \left(u_i u_j\right)_{,k} \right]_{,k} + 2 p s_{ij} - 2\nu u_{i,k} u_{j,k} \tag{A.5}$$

if a Newtonian fluid $\sigma_{ij} = \nu u_{i,j}$ is assumed.

Recall that the turbulent stress tensor was previously defined as $R_{ij} = \overline{u_i u_j} - \overline{u_i}\,\overline{u_j}$, the double bracket by $< a_i, b_j > \equiv \overline{a_i b_j} - \overline{a}_i \overline{b}_j$, and the turbulent transport term was defined by $T_{ijk} \equiv \overline{u_i u_j u_k} - \overline{u}_i R_{jk} - \overline{u}_j R_{ik} - \overline{u}_k R_{ij} - \overline{u}_i \overline{u}_j \overline{u}_k$. Simply averaging equations A.1, A.2, and A.5 and then substituting for the stress tensor, double bracket, and turbulent transport term will recover the equations below.

$$\overline{u}_{k,k} = 0 \tag{A.6}$$

$$\overline{u}_{i,t} + (\overline{u}_i \overline{u}_j)_{,j} = -\overline{p}_{,i} + \nu \overline{u}_{i,jj} - R_{ij,j} \tag{A.7}$$

$$\begin{aligned} R_{ij,t} + \overline{u}_k R_{ij,k} = {} & \nu R_{ij,kk} - (R_{jk} \overline{u}_{i,k} + R_{ik} \overline{u}_{j,k}) \\ & - T_{ijk,k} - (< p_{,i}, u_j > + < p_{,j}, u_i >) - 2\nu < u_{i,k}, u_{j,k} > \end{aligned} \tag{A.8}$$

Hence it is shown that the averaging invariance procedure of Germano does indeed provide evolutionary equations that are exactly the same as the Reynolds stress transport (RST) equations.

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