Results on Finite Wireless Sensor Networks: Connectivity and Coverage

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Many analytic results for the connectivity, coverage, and capacity of wireless networks have been reported for the case where the number of nodes, n, tends to infinity (large-scale networks). The majority of these results have not been extended for small or moderate values of n; whereas in many practical networks, n is not very large. In this article, we consider finite (small-scale) wireless sensor networks. We first show that previous asymptotic results provide poor approximations for such networks. We provide a set of differences between small-scale and large-scale analysis and propose a methodology for analysis of finite sensor networks. Furthermore, we consider two models for such networks: unreliable sensor grids and sensor networks with random node deployment. We provide easily computable expressions for bounds on the coverage and connectivity of these networks. With validation from simulations, we show that the derived analytic expressions give very good estimates of such quantities for finite sensor networks. Our investigation confirms the fact that small-scale networks possess unique characteristics different from their large-scale counterparts, necessitating the development of a new framework for their analysis and design.

1. INTRODUCTION

In the past, many analytic results on the connectivity, coverage, and capacity of wireless ad-hoc and sensor networks have been obtained. In almost all of the results, it is assumed that the number of nodes, n, in the network tends to infinity (large-scale networks). In other words, these results are asymptotic. Asymptotic results are very important for two reasons. First, they give us good estimates for large-scale networks. Second, they show some fundamental trade-offs in the network. However, in many practical wireless networks, the number of nodes may be limited to a few hundred

(small-scale/finite networks). As it is shown in this article, the asymptotic results cease to be valid for these networks. Thus, it is very crucial from the practical point of view to analyze finite networks. These analytic results will essentially help us to understand, design, and analyze practical wireless networks, and also to design more suitable communication protocols.

For example, consider the capacity analysis of wireless networks, which has been studied extensively (e.g., [Balakrishan et al. 2004; Gupta and Kumar 2000, 2003; Grossglauser and Tse 2001; Li et al. 2001; Perevalov and Blum 2003; Liu et al. 2003]). Today we have a good understanding of scaling laws for the capacity of wireless networks. However, suppose we need to design a wireless sensor network consisting of an arbitrary deployment of a hundred sensor nodes. Some fundamental questions are as follows. What is the transport capacity? What are the connectivity and coverage probabilities of such networks? How do network parameters, such as the communication radius of nodes, number of nodes, and so on, affect these properties? Unfortunately, the available asymptotic results fail to give answers to these questions.

To address the aforementioned issues in small-scale networks, we need to address some inherent problems. First, in large-scale networks, we use asymptotic estimates that make the analysis much simpler. These estimates are not available in small-scale analysis. Thus, small-scale analysis is usually more difficult. Second, even if we can perform the small-scale analysis, we usually obtain very complicated formulas that are not very useful practically. In this article, we want to circumvent these problems and provide bounds for small-scale-analysis. In particular, we are looking for easily computable but acceptable estimates for fundamental network quantities. The main goal of this article is to initiate small-scale analysis of wireless sensor networks. To the best of our knowledge, this is the first work to analytically and systematically study this issue.

The main idea is the following. The first key point is to aim at simple and very good approximations instead of trying to find complicated exact formulas. To do so, we first consider asymptotic analysis. Some of the estimates in asymptotic analysis are still good for small-scale networks, while others are not. We identify those which are not valid and replace them with better estimates. However, this must be done carefully in order to obtain simple and easily computable formulas at the end. Specifically, in this article we list a few important differences between small-scale and large-scale analysis.

As a special case of finite sensor networks, we first study unreliable sensor grids in which the sensors are deployed in a grid and each sensor is active with a probability p. This probability is used to account for both sensor failures and sleeping sensors. A fundamental question is that given an area to be protected, how many sensors should be deployed so that every point in the region be covered by at least one sensor (more generally, we may require that every point in the region be covered by at least k sensors). Equivalently, one can ask, if n sensor nodes are deployed in an area, what should be the sensing radius of nodes to ensure coverage (or k-coverage)? The same question can be repeated for other network properties, such as connectivity and diameter. In this article, we study the behavior of the different parameters in finite sensor grids. We prove that all graph-theoretic properties of these networks, such as connectivity, network diameter, and capacity, follow a piecewise constant behavior, and this is even true for the coverage which is not a graph-theoretic property. This result shows a key difference between the behavior of sensor grids and that of randomly deployed sensor networks and has some important implications from the practical point of view. (1) It shows that increasing the communication and sensing radii does not necessarily improve coverage, connectivity, or any other graph-theoretic property. (2) It suggests that we could completely determine the behavior of a vast class of network properties

by knowing their values for only a finite number of points. We then find simple lower and upper bounds for the k-coverage probability of sensor grids and show that these bounds are adequately close to the real value, as an estimate of the coverage probability.

Next, we consider finite sensor networks in which nodes are randomly distributed in the unit square. We study k-connectivity and coverage of these networks. We give several results pertaining to these properties. We first show that the previous asymptotic results on coverage and k-connectivity are not accurate for the finite case. We then provide a very simple formula for the k-connectivity probability of finite sensor networks and show that the formula is very precise. We also study the coverage probability of random networks where we prove simple lower and upper bounds for the coverage probability.

The remainder of the article is organized as follows: the next section provides an overview of the related work. In Section 2, we study connectivity and coverage of finite sensor grids. Section 3 investigates the fundamental properties of random sensor networks, such as connectivity and coverage. Finally, Section 4 concludes the article.

1.1. Related Work

Related problems have been studied in the context of random graph theory [Bollobás 2001], continuum percolation, and geometric probability [Meester and Roy 1996; Penrose 2003], as well as the study of wireless network graphs [Gupta and Kumar 1998, 2000; Xue and Kumar 2004, Booth et al. 2003, 2005; Shakkottai et al. 2003; Dubhashi et al. 2003; Li et al. 2003; Wan and Yi 2004; Kumar et al. 2004]. In random graph theory, the model $G(n, p)$ is extensively studied, in which edges appear in a graph of n vertices with probability p independent of each other. In continuum percolation theory, usually infinite graphs on $\mathbb{R}^d$ are studied. Finally, in geometric probability and the study of graphs of wireless networks, large-scale graphs over the plane are usually studied.

In Franceschetti and Meester [2008, 2006] studied connectivity and critical node lifetime for a model of random networks in which the density of nodes is kept constant while the area of interest tends to infinity. Furthermore, the throughput scaling of wireless relay networks is studied in Dousse et al. [2006] for this model. However, the results in these papers are all based on asymptotic analyses, thus their method cannot be applied to the case of finite networks, that is, networks with a finite number of nodes (e.g., less than 1,000) on a finite plane. In the analysis of these networks, boundary effects and constant factors (see Section 3.2) cannot be neglected, as can be for the case of asymptotic analysis.

The connectivity and k-connectivity of large-scale wireless networks have been investigated [Gupta and Kumar 1998; Li et al. 2003; Wan and Yi 2004; Pishro-Nik et al. 2004; Dousse et al. 2002]. Dousse and Thiran [2004] examined the trade-off between connectivity and capacity of dense networks. The transport, information-theoretic, and MAC-layer capacities have been investigated extensively (e.g., [Balakrishan et al. 2004; Gupta and Kumar 2000; Grossglauser and Tse 2001; Gupta and Kumar 2003; Li et al. 2001; Perevalov and Blum 2003; Liu et al. 2003]. The grid model for sensor networks has also been investigated. In particular, connectivity, coverage, and diameter of sensor grids were studied [Shakkottai et al. 2003]. Kumar et al. [2008] and Janson [1986], considered the k-coverage problem for sensor grids and other deployment methods. Balister et al. [2009] and Balister and Kumar [2009] also studied coverage for sensor networks in the presence of failures and placement errors. However, almost all previous analytical results are asymptotic, since they consider large-scale networks.

Analysis of wireless networks with a modest number of nodes recently has generated a lot of interest [Bai et al. 2006; Desai and Manjunath 2002; Gore 2006; Karmachandani et al. 2006; Yen and Yu 2004; Ghasemi and Nader-Esfahani 2006]. Desai and
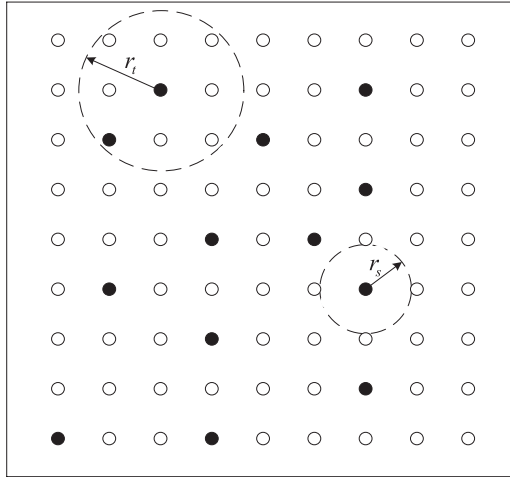
Fig. 1. A sensor grid is shown where the active sensors are shown by black nodes. Communication and sensing radii, r_t and r_s, are also shown.

Manjunath [2002] investigated the problem of connectivity for one-dimensional networks (i.e, line networks). Using probabilistic methods, they obtained the exact formulation for the probability of connectivity. Gore [2006] presented corrections and extensions to Desai and Manjunath [2002]. It is noted that both papers connsidered a line network, and the extension to two-dimensional networks was achieved by obtaining a loose bound using the results from the former case. Ghasemi and Nader-Esfahani [2006] also consider the line network and obtain connectivity results for one-dimensional networks. The threshold phenomena for finite wireless networks on a line is studied in Eslami et al. [2010]. The authors also find lower and upper bounds on the MAC-layer capacity for such networks. It should be noted that the main challenges in finite analysis arise in the two-dimensional case. Karmachandani et al. [2006] examined mobility and more realistic models. The authors obtained results on the connectivity for both finite and asymptotic cases in one-dimensional networks. Yen and Yu [2004] some simple local network characteristics, such as the link probability (occurrence of a link) and average node degrees. Their paper also obtained formulas for the average covered area. Balister et al. [2007] studied connectivity and coverage for networks on a thin strip of finite length. The authors provided reliable density estimates for achieving coverage and connectivity, assuming a Poisson distribution for the nodes.

2. FUNDAMENTAL PROPERTIES OF FINITE UNRELIABLE SENSOR GRIDS

In this section, we present properties of finite unreliable sensor grids. In particular, we prove that a large class of network properties, such as connectivity, coverage, and capacity, can be represented as a piecewise constant function of the communication and sensing radii, r_t and r_s, respectively. We also discuss the implications of this result and show the importance of boundary effects in finite networks. We then find an upper bound for coverage which can be used to approximate the exact value of the coverage.

Here, we consider the sensor network model introduced in Shakkottai et al. [2003]. In particular, it is assumed that n sensor nodes are arranged in a grid over a square region of unit area. This region is called the *deployment region* and it is assumed to be the unit square centered at the origin. Such a grid is depicted in Figure 1. We show the deployment region by S_0. The separation between adjacent nodes is assumed to be $\frac{1}{\sqrt{n}}$ units. Each sensor node can detect events within some distance from it, called the

sensing radius r_s. Each sensor is active with probability p independently from other nodes. The transmission radius of each node is assumed to be r_t. In other words, if the distance between two sensor nodes u and v is less than r_t, then they can communicate with each other, thus the edge $\{u, v\}$ belongs to edges of the graph. It is worth noting that our results apply to any deterministic placement of finite sensor networks and also any finite deployment region with smooth boundaries. However, for simplicity, we consider the preceding grid model in this article. We are interested in connectivity and coverage, in particular, we assume $p_{disc}(n, p, r_t)$ is the probability that the sensor grid with parameters n, p, and r_t constructs a disconnected graph. We also assume that $p_{cov}(n, p, r_s, k)$ is the probability that each point of the unit square (i.e., the deployment region) is covered by at least k sensors in the sensor grid with parameters n, p, and r_s. Thus, $p_{cov}(n, p, r_s, 1)$ is the probability that the whole unit quare is covered by the sensor nodes.

2.1. Sensor Grids: Asymptotic versus Finite Analysis

We now present some evidence to show that previous asymptotic results diverge significantly from actual values for finite grids. To show this, we consider connectivity and coverage. Let us first consider coverage. The asymptotic coverage probability $p_{cov}(n, p, r_s, k)$ has been found in Kumar et al. [2008]. In particular the following fundamental result has been obtained.

THEOREM 1. (Kumar et al. 2008). *Let ϵ be an arbitrary constant positive real number and k be a constant positive integer. Then, for n chosen large enough, we have the following two cases.*

—*If $r_s(n) \geq \sqrt{\frac{(1+\epsilon)\log(np)}{\pi np}}$, then the unit square is almost always k-covered completely, that is, $p_{cov}(n, p, r_s, k) = 1 - o(1)$.*
—*If $r_s(n) \leq \sqrt{\frac{(1-\epsilon)\log(np)}{\pi np}}$, then $p_{cov}(n, p, r_s, k) = o(1)$.*

Using simulations, Kumar et al. [2008] have shown that this theorem results in accurate estimation of $p_{cov}(n, p, r_s, k)$ when n is large (say $n > 10,000$). Thus, the theorem is very useful in the design of large-scale sensor networks. Let us now consider a sensor grid consisting of 100 unreliable sensor nodes with $p = 0.2$. If we want to use the asymptotic result for this network, choosing $\epsilon = 0.1$, we conclude that if $r \geq 0.229$, then $p_{cov}(n, p, r_s, k) \approx 1$ and if $r \leq 0.207$, then $p_{cov}(n, p, r_s, k) \approx 0$. We have used exhaustive simulations to obtain an accurate estimate of $p_{cov}(n, p, r_s, k)$. In Figure 2, we compare the results obtained by exhaustive simulations and Theorem 1. It is observed that the two results differ considerably. For example, at $r_s = 0.25$, the asymptotic result predicts that the unit square is covered with probability close to one. However, simulations show that this probability is only $p_{cov}(n = 100, p = 0.2, r = 0.25, k = 1) = 0.018$. It is clear that for this network, the asymptotic analysis cannot provide results that are sufficiently accurate. Figure 3 shows that the same situation exists when we consider k-coverage for $k > 1$. Thus, it is very important to provide finite-size analysis. We also observe that the coverage probability obtained by simulations shows several discontinuities. We prove this phenomenon in Section 2.2. We performed many simulations for different values of n, p, and k to further validate the insufficiency of asymptotic results. However, we omit them for brevity.

2.2. Discontinuity in Properties of Sensor Grids

Here we prove that a vast class of network properties can be represented by piecewise constant functions of r_t and r_s. We stress that the piecewise property is one of the key differences between sensor grids (i.e., deterministic deployment) and randomly

Fig. 2. Comparison of asymptotic results and simulation results for the coverage probability of a sensor grid with $p = 0.20$. For simulations we picked $n = 100$, while the asymptotic result holds when n tends to infinity.
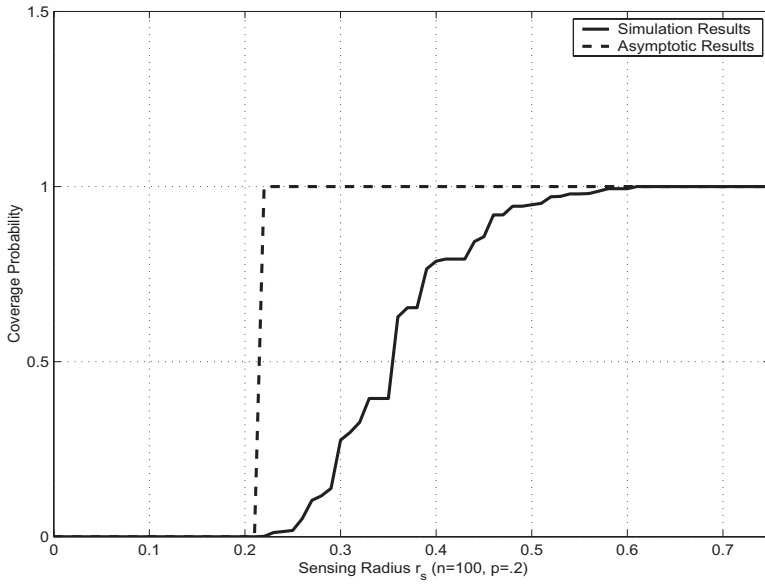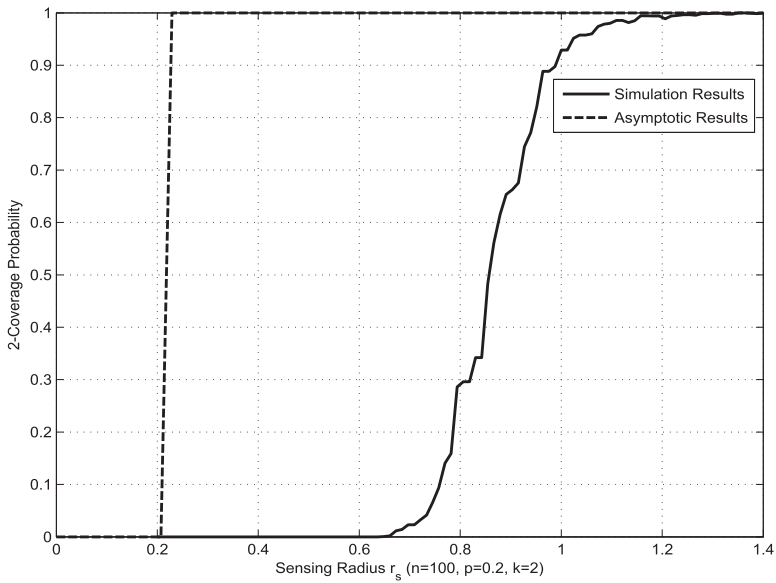

Fig. 3. Comparison of asymptotic results and simulation results for two-coverage probability ($k = 2$) of a sensor grid with $p = 0.20$. For simulations we picked $n = 100$, while the asymptotic result holds when n tends to infinity.

deployed sensor networks. Consider a right-continuous function $f(x) : [0, \infty] \to \mathbb{R}$. The function f is said to be piecewise constant if there exists a set of real numbers $0 = x_1 < x_2 < x_3 < \ldots$, and $c_1, c_2, \ldots$, such that $f(x) = c_i$ for all $x \in [x_i, x_{i+1})$. In this article we only deal with functions for which the number of x_i's is finite.

Let Q be a property for sensor grids, such as coverage, that is, we say that a grid has the property Q if it covers the deployment region. Coverage is an example of *geometric properties*. Another category of properties is *graph-theoretic properties*, such as connectivity. In particular, any sensor grid with parameters n, p, and r_t corresponds to a graph that can be shown by $g(V, E)$, where V and E are the set of vertices and edges in the graph, respectively. The sensor nodes construct V, that is, the set of vertices of the graph. There exists an edge between two vertices if their corresponding sensors are within the communication range of each other. Any property of $g(V, E)$ is a graph-theoretic property for the sensor grid. Thus, two different sensor grids will have the same graph-theoretic properties if they have isomorphic (identical) graphs. We note that coverage is not a graph-theoretic property.

Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ be a set of points on the plane. Define $g(\mathcal{X}, r_t)$ as the graph obtained by the following method. The vertices of g are the points in $\mathcal{X}$, and there is an edge between two vertices X_i and X_j, if their distance is less than or equal to r_t. We prove the following theorem.

THEOREM 2. *Let Q be a graph-theoretic property of sensor grids with parameters n, p, r_t. Let n and p be fixed numbers and $p_Q(r_t)$ be the probability that the sensor grid with communication radius r_t has the property Q. Then p_Q is a piecewise constant function. In particular, there exist $0 = r_0 < r_1 < r_2 < \cdots < r_m \leq \frac{1}{\sqrt{2}}$, and $c_1, c_2, \ldots, c_m$ such that $p_Q(r_t) = c_i$ if $r_t \in [r_{i-1}, r_i)$.*

PROOF. Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ be the set of points in the sensor grid. Let also $\mathcal{X}_a \subset \mathcal{X}$ be the set of active sensors. Assume that $g(\mathcal{X}, r_t)$ is the corresponding graph. Let $p(\mathcal{X}_a)$ be the probability that $\mathcal{X}_a$ is the set of active sensors, then we have

$$p(\mathcal{X}_a) = p^{|p(\mathcal{X}_a)|}(1 - p)^{n - |p(\mathcal{X}_a)|}, \tag{1}$$

where $| \, p(\mathcal{X}_a) \, |$ is the number of active sensor nodes. Then

$$p_Q(r_t) = \sum_{g(\mathcal{X}_a, r_t) \text{ has } Q} p(\mathcal{X}_a). \tag{2}$$

It suffices to find $0 = r_0 < r_1 < r_2 < \cdots < r_m < \frac{1}{\sqrt{2}}$ such that the network graphs $g(\mathcal{X}_a, r_t)$ remain constant for $r_t \in [r_{i-1}, r_i)$ for any choice of $\mathcal{X}_a$ and any $i \in \{1, 2, \ldots, m\}$. Let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of distances between the points in $\mathcal{X}$, and assume that $0 < d_1 < d_2 < \cdots < d_l \leq \frac{1}{\sqrt{2}}$. In our grid model, we have $d_i = \sqrt{\frac{i}{n}}, i = 1, \ldots, l$. Then, the network graph remains the same when $r_t \in [d_i, d_{i+1})$ for any $i \in \{1, 2, \ldots, m\}$. This is because changing r_t within $[d_i, d_{i+1})$ will not add or remove any edges. This means that we can choose $r_i = d_i$. Thus p_Q in Equation(2) remains constant for $r_t \in [r_{i-1}, r_i)$. It is also easy to see that p_Q is right-continuous because the edges in the graphs are formed when the distance between two nodes is less than or equal to r_t. This completes the proof. □

Note that the preceding discussion shows that any graph-theoretic quantity is a piecewise constant function of r_t. This includes diameter of the network, MAC-layer capacity [Balakrishan et al. 2004], k-connectivity, etc. We now prove that coverage probabilities are piecewise constant functions of the sensing radius. Note that this cannot be concluded from Theorem 2, since coverage is not a graph-theoretic property.

THEOREM 3. *Consider a sensor grid with parameters n, p, r_s. Let n and p be fixed numbers. Then $p_{cov}(n, p, r_s, k)$ is a piecewise constant function of r_s. In particular, there exist $0 = r_0 < r_1 < r_2 < \cdots < r_m \leq \frac{1}{\sqrt{2}}$, and $c_1, c_2, \ldots, c_m$ such that $p_{cov}(n, p, r_s, k) = c_i$ if $r_s \in [r_{i-1}, r_i)$.*

PROOF. For simplicity, we prove the theorem for $k = 1$; the extension to $k > 1$ is straightforward. Let $p_{cov}(r_s) = p_{cov}(n, p, r_s, 1)$. We need to show $p_{cov}(r_s)$ ia a piecewise constant function of r_s. It is clear that $p_{cov}(r_s)$ is a nondecreasing function. In particular, we have $p_{cov}(0) = 0$ and $p_{cov}(r_s) = 1$ for $r_s \geq \frac{1}{\sqrt{2}}$. For a point X in the plane, let $circ(X, r)$ be the closed ball that is centered at X and has radius r. Define $cov(X, r_s)$ to be the area that is covered by a sensor node located at X with sensing radius r_s. In other words, $cov(X, r_s)$ is the portion of $circ(X, r_s)$ that lies within the deployment region. Again assume that $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ is the set of points in the sensor grid and that $\mathcal{X}_a \subset \mathcal{X}$ is the set of active sensors. Define

$$cov(\mathcal{X}_a, r_s) = \bigcup_{X \in \mathcal{X}_a} cov(X, r_s). \tag{3}$$

Thus, the unit square S_0 is completely covered whenever $cov(\mathcal{X}_a, r_s) = S_0$. Let $0 \leq r_s \leq \frac{1}{\sqrt{2}}$. If $cov(\mathcal{X}_a, r_s) = S_0$, then for all $r > r_s$, we have $cov(\mathcal{X}_a, r) = S_0$. On the other hand, we prove that if $cov(\mathcal{X}_a, r_s) \neq S_0$, there exists $\epsilon > 0$ such that for all $r \in [r_s, r_s + \epsilon)$, we have $cov(\mathcal{X}_a, r) \neq S_0$. To prove this, note that the covered area $cov(\mathcal{X}_a, r_s)$ is a closed set because it is the union of a finite number of closed sets. Thus, the uncovered area is an open set, and hence, to cover the uncovered area, the sensing radius r_s must increase by a strictly positive amount.

We now prove that for any r_s, there exists a strictly positive ϵ such that $p_{cov}(r)$ remains constant as the sensing radius r varies within $[r_s, r_s + \epsilon)$. Define

$$\mathbf{X}_a^{r_s} = \{\mathcal{X}_a : cov(\mathcal{X}_a, r_s) = S_0\}, \quad \text{and} \quad \overline{\mathbf{X}}_a^{r_s} = \{\mathcal{X}_a : cov(\mathcal{X}_a, r_s) \neq S_0\}. \tag{4}$$

Note that $\mathbf{X}_a^{r_s}$ and $\overline{\mathbf{X}}_a^{r_s}$ are finite sets. Using Equation (4) we have

$$p_{cov}(r_s) = \sum_{\mathcal{X}_a \in \mathbf{X}_a^{r_s}} p(\mathcal{X}_a). \tag{5}$$

For any $\mathcal{X}_a \in \overline{\mathbf{X}}_a^{r_s}$, define $\epsilon(\mathcal{X}_a) = \min\{r' \text{ s.th. } cov(\mathcal{X}_a, r_s + r') = S_0\}$ and let $\epsilon = \min\{\epsilon(\mathcal{X}_a) : \mathcal{X}_a \in \overline{\mathbf{X}}_a^{r_s}\}$. Then $\epsilon > 0$. Further, for all $r \in [r_s, r_s + \epsilon)$, we have $\overline{\mathbf{X}}_a^{r} = \overline{\mathbf{X}}_a^{r_s}$. Thus we conclude that for all $r \in [r_s, r_s + \epsilon)$, we have $\mathbf{X}_a^{r} = \mathbf{X}_a^{r_s}$. Using Equation (5) we conclude that $p_{cov}(r_s)$ does not change as r varies in $[r_s, r_s + \epsilon)$. This proves that $p_{cov}(r_s)$ is a right-continuous piecewise constant function.

It remains to show that the number of discontinuities is finite. This follows easily from the fact that the number of $\mathcal{X}_a$'s is finite. Note that by Equation (5), any discontinuity occurs when the set $\mathbf{X}_a^{r_s}$ changes due to an increase in r_s. However, $\mathbf{X}_a^{r_s}$ can have at most 2^n elements. Further, at each discontinuity, at least one element is added to $\mathbf{X}_a^{r_s}$. This implies that the number of discontinuities is upper-bounded by 2^n. It is worth noting that in practice, the number of discontinuities is much smaller than 2^n. This completes the proof. □

Theorems 2 and 3 determine the behavior of a vast class of network quantities when they are considered as functions of communication and sensing radii. In particular, these are important from the viewpoint of finite sensor grids. We note that for very large network sizes, the piecewise constant functions tend to continuous functions. Thus, we do not observe the discontinuities. However, in such networks as finite sensor

grids, this property is noticeable, as in Figures 2 and 3. We clarify that because the simulation results are approximations for the actual values, the figures are not completely piecewise constant. In fact, one of the implications of Theorems 2 and 3 is to simplify simulations, since the piecewise constant functions can be completely determined by knowing their values for only a finite number of points. Furthermore, the preceding results suggest that increasing the communication and sensing radii does not necessarily improve coverage, connectivity, or any other graph-theoretic properties. This is an important observation for designing the network and choosing its parameters optimally.

2.3. Bounds on the Coverage Probability

We now consider coverage probability for finite sensor grids. We find lower and upper bounds for $p_{cov}(n, p, r_s, k)$ and show that they can give an acceptable estimate of the coverage probability. Let $N(r, x, y)$ be the number of sensors whose distance from the point (x, y) is less than or equal to r. For example, $N(r, 0.5, 0.5)$ denotes the number of sensors whose distance from the top-right corner of the unit square is less than or equal to r. We first prove the following lemma.

LEMMA 1. *Let L be the set of $\sqrt{l} \times \sqrt{l}$ points in a virtual grid on the unit square. Let us also denote by $A(u)$ the event that point u on L is covered by a sensor grid with coverage radius r_s. We then have*

$$Pr\left(\bigwedge_{u \in L} A(u)\right) \geq \prod_{u \in L} Pr(A(u)). \tag{6}$$

PROOF. We use FKG inequality to prove this lemma [Fortuin et al. 1971]. We first show that for any two subsets I and J of L, we have $\Pr(A(I) \bigwedge A(J)) \geq \Pr(A(I)) \times \Pr(A(J))$. Here, $A(I)$ $(A(J))$ is the event that all points in I (J) are covered. Since this is true for any two subsets of L, Equation (6) can be derived by partitioning L and the resulted components, repeatedly, and then using this property at each step.

First note that we can enumerate the nodes in the sensor grid from 1 to n. Accordingly, we can show the status of the network with a n-tuple binary vector, where 0 and 1 are assigned to inactive and active nodes, respectively. Let us denote by T the set of all possible binary n-tuples as the network status, that is, $T = \{\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \{0, 1\}^n\}$. T can be then defined as a *finite distributive lattice* as follows. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in T, we define $\mathbf{x} \vee \mathbf{y}$ as the elementwise "or" of $\mathbf{x}$ and $\mathbf{y}$. That is, if $\mathbf{w} = \mathbf{x} \vee \mathbf{y} = (w_1, w_2, \ldots, w_n)$, then $w_i = x_i \vee y_i$. Similarly, we define $\mathbf{x} \wedge \mathbf{y}$ as the elementwise "and" of $\mathbf{x}$ and $\mathbf{y}$, that is, if $\mathbf{w} = \mathbf{x} \wedge \mathbf{y} = (w_1, w_2, \ldots, w_n)$, then $w_i = x_i \wedge y_i$. With these definitions, it is easy to check that $\vee$ and $\wedge$ are distributive over each other. Note that the lattice defined this way is partially ordered, as we have $(\mathbf{x} \wedge \mathbf{y}) \preceq \mathbf{x}, \mathbf{y} \preceq (\mathbf{x} \vee \mathbf{y})$.

We now define a probability measure $\mu : T \to \mathbb{R}^+$ as follows. For $\mathbf{x} \in T$, $\mu(\mathbf{x}) = p^k(1 - p)^{n-k}$, where $k = \sum_{i=1}^{n} x_i$. Note that $\mu(\mathbf{x})$ in fact indicates the probability that the sensor grid admits the status $\mathbf{x}$ with k active sensors and $n - k$ inactive sensors. It is also trivial to verify that $\mu(\mathbf{x})\mu(\mathbf{y}) \leq \mu(\mathbf{x} \vee \mathbf{y})\mu(\mathbf{x} \wedge \mathbf{y})$, which is required by FKG inequality. Given two subsets I and J of L, we also define functions $f, g : T \to \mathbb{R}^+$ as follows. For every $\mathbf{x} \in T$, $f(\mathbf{x}) = 1$ $(g(\mathbf{x}) = 1)$ if I (J) is covered. By these definitions, f and g are both *increasing functions* over T. Given the lattice T, measure μ, and functions f and g from earlier, the FKG inequality holds as follows.

$$\left(\sum_{\mathbf{x} \in T} \mu(\mathbf{x}) f(\mathbf{x})\right) \cdot \left(\sum_{\mathbf{x} \in T} \mu(\mathbf{x}) g(\mathbf{x})\right) \leq \left(\sum_{\mathbf{x} \in T} \mu(\mathbf{x}) f(\mathbf{x}) g(\mathbf{x})\right) \cdot \left(\sum_{\mathbf{x} \in T} \mu(\mathbf{x})\right). \tag{7}$$

However, $\sum_{\mathbf{x} \in T} \mu(\mathbf{x}) = 1$. Furthermore, $\sum_{\mathbf{x} \in T} \mu(\mathbf{x}) f(\mathbf{x})$, $\sum_{\mathbf{x} \in T} \mu(\mathbf{x}) g(\mathbf{x})$, and $\sum_{\mathbf{x} \in T} \mu(\mathbf{x})$ $f(\mathbf{x}) g(\mathbf{x})$ are in fact equal to $\Pr(A(I))$, $\Pr(A(J))$, and $\Pr(A(I) \bigwedge A(J))$, respectively. Equation (7) can thus be rewritten as $\Pr(A(I) \bigwedge A(J)) \geq \Pr(A(I)) \times \Pr(A(J))$. This completes the proof. □

Now we are ready to prove the lower bound on the coverage probability.

THEOREM 4. *Consider the coverage probability for a finite sensor grid with parameters n, p, and r_s. We then have*

$$p_{cov}(n, p, r_s, 1) \geq \prod_{u \in L} \left[1 - (1-p)^{N(r'_s, x_u, y_u)} \right], \qquad (8)$$

where L is the virtual grid in Lemma 1, and the radius r'_s is given by $r_s - \frac{1}{\sqrt{2l}}$.

PROOF. First note that the choice of the virtual grid L and its size, l, is arbitrary. As a result, for any given r_s, we choose l large enough such that $r_s - 1/\sqrt{2l} > 0$. To prove this theorem, we make use of some results in Kumar et al. [2008]. Lemma 3.1 in Kumar et al. [2008] states that for a given set of points L that consists of all grid points of a $\sqrt{l} \times \sqrt{l}$ virtual grid on a unite square, if L is covered by a network of radius r'_s, the unit square is covered by the same network but with radius $r_s = r'_s + \frac{1}{\sqrt{2l}}$. Hence, $p_{cov}(n, p, r_s, 1) \geq \Pr(L \text{ covered})$. Now we use Lemma 1 to prove the lower bound. Let us denote by $A(u)$ the event that point u is covered by a network with coverage radius r'_s. By Lemma 1, we have $\Pr(\bigwedge_{u \in L} A(u)) \geq \prod_{u \in L} \Pr(A(u))$. Now, note that the probability that a point u with coordination (x_u, y_u) is covered by the set of n nodes with coverage radius r'_s is given by $[1 - (1-p)^{N(r'_s, x_u, y_u)}]$. Thus, we write $p_{cov}(n, p, r_s, 1) \geq \Pr(L \text{ covered}) = \Pr(\bigwedge_{u \in L} A(u)) \geq \prod_{u \in L} \Pr(A(u)) = \prod_{u \in L} [1 - (1-p)^{N(r'_s, x_u, y_u)}]$. □

Now, we prove an upper bound for the coverage probability.

THEOREM 5. *Consider sensor grids with parameters n, p, r_s. Then the coverage probability is upper bounded by*

$$p_{cov}(n, p, r_s, 1) \leq \left[1 - (1-p)^{N(r_s, .5, .5)} \right]^4 \times \left[1 - (1-p)^{N(r_s, .5, 0)} \right]^{4 \lfloor \frac{(1-2r_s)}{2r_s} \rfloor}$$

$$\times \left[1 - (1-p)^{N(r_s, 0, 0)} \right]^{\lfloor \frac{(1-2r_s)}{2r_s} \rfloor^2}, \qquad (9)$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

PROOF. Let $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2), \ldots, X_m = (x_m, y_m)$ be m points on the deployment region S_0. Assume that $d(X_i, X_j) > 2r_s$ for $i \neq j$, where $d(., .)$ is the Euclidean distance between the points. Then the event that X_i is covered is independent of the event that X_j is covered, because there is no sensor node that can cover both points. Hence, the probability that all X_i's are covered is given by

$$\prod_{i=1}^{m} \left[1 - (1-p)^{N(r_s, x_i, y_i)} \right]. \qquad (10)$$

This implies that $p_{cov}(n, p, r_s, 1)$ is upper bounded by $\prod_{i=1}^{m} [1 - (1-p)^{N(r_s, x_i, y_i)}]$. Thus using any set of points on the plane that satisfy $d(X_i, X_j) > 2r_s$, we can find an upper bound for $p_{cov}(n, p, r_s, 1)$. In particular, considering the set of points given by Figure 4, we obtain the upper bound in Equation (9). □

Note that the choice of X_i's in the proof ensures that we consider the edge effects. In fact, in many situations, the coverage probability is dominated by the first and second terms in Equation (9) which are related to edge effects. One may suggest that using a
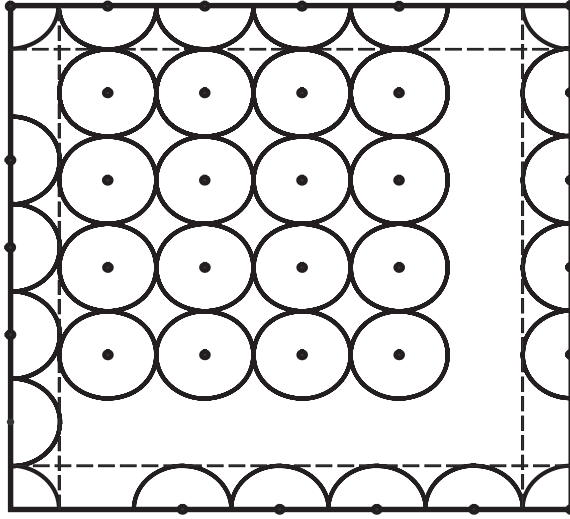
Fig. 4.   Location of the points used for the upper bound of Theorem 5. The centers of the circles are points $X_1 = (x_1, y_1), X_2 = (x_2, y_2), \ldots, X_k = (x_k, y_k)$.

triangular grid instead of non-overlapping balls can result in a more dense packing and, consequently, a better bound. However, using a triangular grid results in fewer nodes on the sides of the square. We have evaluated Equation (10) for the triangular grid as well as some other more complicated layouts. It turn out that the resulting bound is looser for the triangular grid. Moreover, there is only a negligible improvement by using other layouts at the expense of a more complicated expression compared to Equation (9). It is also worth noting that $N(r_s, .5, .5)$, $N(r_s, .5, 0)$, and $N(r_s, 0, 0)$ introduce discontinuities in the upper bound, as predicted by Theorem 3.

Figure 5 compares the results obtained by Theorems 4 and 5 and the simulations for $n = 100$ and $p = 0.2$. We observe that Theorems 4 and 5 provide significantly better estimates of coverage probability compared with the asymptotic analysis in Figure 2. The asymptotic behavior of these bounds can be checked by letting n to grow large. The derivation of the lower bound employs a similar argument, as in the case of Lemma 4.1 in Kumar et al. [2008]. It can be checked that this bound leads to the same asymptotic expression as in Theorem 1; hence it is tight asymptotically. On the other hand, when n gets large, we can reasonably expect the same situation as in the upper bound of Theorem 5. That is, the terms corresponding to the virtual nodes (i.e., nodes on the virtual grid) on the corner and close to the edges will be dominant in the lower bound of Equation (8). This is true because there are fewer sensor nodes around these virtual nodes to cover them, causing the coverage probability $\Pr(A(u))$ for these virtual nodes to decay faster than the rest of the virtual nodes. Regarding the asymptotic behavior of the upper bound of Equation (9), it can be verified that if $\frac{np\pi r^2}{\log(np)} < 1 - \epsilon$ as n tends to infinity, then the upper bound will be $o(1)$.

We also like to talk about the time complexity of computing the bounds in Theorems 4 and 5. The upper bound of Theorem 5 can be computed in time $O(n)$. This is because we need to find the neighbors for $1/(2r_s)^2$ points, and finding the number of neighbors for each point takes a constant amount of time. However, if $1/(2r_s)^2 > n$, then $p_{cov} = 0$. Thus, the complexity is $O(1/(2r_s)^2) = O(n)$. For the lower bound in Theorem 4, note that we only need to find the number of neighbors for every node of L. Given the sensor grid and the virtual grid L, finding the number of neighbors for each node of L takes a
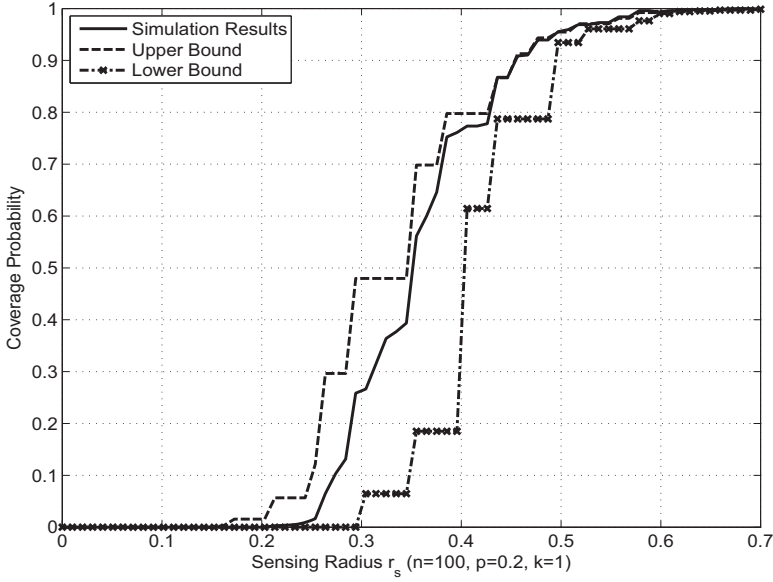
Fig. 5.   Comparison of finite-size analysis with the simulation results for the coverage probability of sensor grids with $n = 100$ and $p = 0.20$.

constant amount of time. Also note that for r'_s to be positive, L needs to contain more nodes than the sensor grid, hence $l > n$. Therefore, the lower bound can be computed with complexity $O(l)$.

Theorems 4 and 5 can be easily generalized for k-coverage. Since the proof is very similar, we just state the result in one theorem.

THEOREM 6.   *Consider the k-coverage probability for a sensor grid with parameters n, p, and r_s, and assume that L and r'_s are as defined in Theorem 4. Then we have*

$$p_{cov}(n, p, r_s, k) \geq \prod_{u \in L} \left[1 - \sum_{i=0}^{k-1} \binom{N(r'_s, x_u, y_u)}{i} p^i (1-p)^{N(r'_s, x_u, y_u)-i} \right], \tag{11}$$

and

$$p_{cov}(n, p, r_s, k) \leq \left[1 - \sum_{i=0}^{k-1} \binom{N(r_s, .5, .5)}{i} p^i (1-p)^{N(r_s, .5, .5)-i} \right]^4$$

$$\times \left[1 - \sum_{i=0}^{k-1} \binom{N(r_s, .5, 0)}{i} p^i (1-p)^{N(r_s, .5, 0)-i} \right]^{4 \lfloor \frac{(1-2r_s)}{2r_s} \rfloor}$$

$$\times \left[1 - \sum_{i=0}^{k-1} \binom{N(r_s, 0, 0)}{i} p^i (1-p)^{N(r_s, 0, 0)-i} \right]^{\lfloor \frac{(1-2r_s)}{2r_s} \rfloor^2}. \tag{12}$$

Figure 6 compares the results obtained by Theorem 6 and the simulations for $k = 2$, $n = 100$, and $p = 0.2$. We observe that the two results are very close.
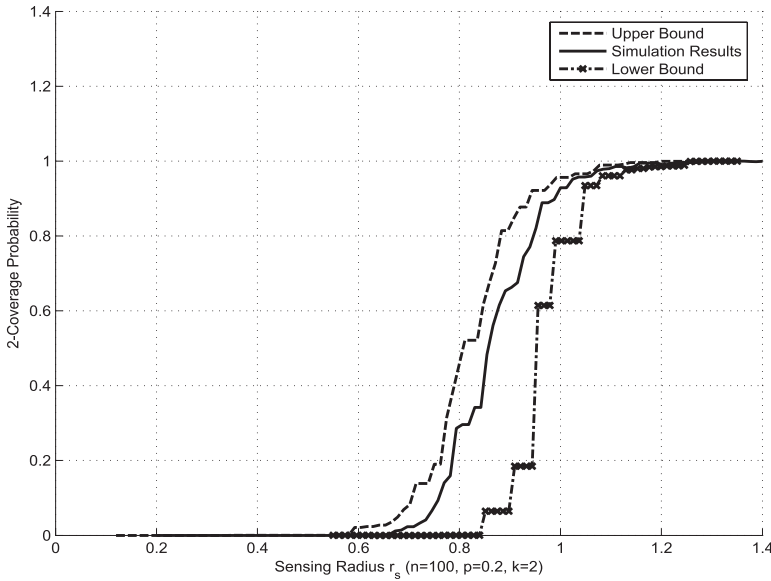
Fig. 6.   Comparison of finite-size analysis with the simulation results for the two-coverage probability ($k = 2$) of sensor grids with $n = 100$ and $p = 0.20$.

3. SMALL-SCALE ANALYSIS FOR RANDOM SENSOR NETWORKS

In this section, we try to establish a framework for analysis of finite sensor networks with random node deployment. As we mentioned earlier, the exact analysis of network properties is usually very difficult or at least results in very complicated formulas. Thus, we will try to find simple lower and upper bounds which are sufficiently close together that can be used to find a good estimate of the exact value of the desired property. Here, we consider coverage and connectivity in finite sensor networks.

3.1. Preliminaries

We consider a wireless sensor network that consists of n nodes and assume that the nodes are placed on a plane based on a given probability distribution. For example, in wireless sensor networks, it is usually assumed that the nodes are randomly and uniformly deployed over a given field [Akyildiz et al. 2002]. We assume that each node has a fixed communication radius. Two nodes are connected (can communicate with each other) if they are within communication range of each other. Throughout the article, we assume $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ−algebra on $\mathbb{R}^2$ and m is the Lebesgue measure on $\mathcal{B}(\mathbb{R}^2)$. Note that we just use measure-theoretic definitions to take care of technicalities, but it is not necessary for the reader to be familiar with them. The reader can simply assume that for a set F in $\mathbb{R}^2, m(F)$ is the area of F. $B(X, R)$ is the closed ball with radius R centered at X in $\mathbb{R}^2$. $S(X, L)$ is the closed square with side L centered at X in $\mathbb{R}^2$. In particular, $S_0 = S(O, 1)$ is the closed square with unit area centered at the origin. If u and v are two nodes of a network located in $\mathbb{R}^2$, then $d(u, v)$ is the Euclidean distance between the location of the points. For any set $F \in \mathcal{B}(\mathbb{R}^2)$, we define $v(F) = m(F \cap S_0)$. Clearly, v defines a measure on $\mathcal{B}(\mathbb{R}^2)$.

Wireless networks are sometimes modeled with the probability space of graphs that we represent with $g(n, r) = g(n, r(n))$. In this model, it is assumed that n nodes are uniformly and randomly distributed over $S_0 = S(O, 1)$. If two nodes u and v satisfy $d(u, v) \leq r(n)$, then the edge $\{u, v\}$ belongs to edges of the graph. A more general model
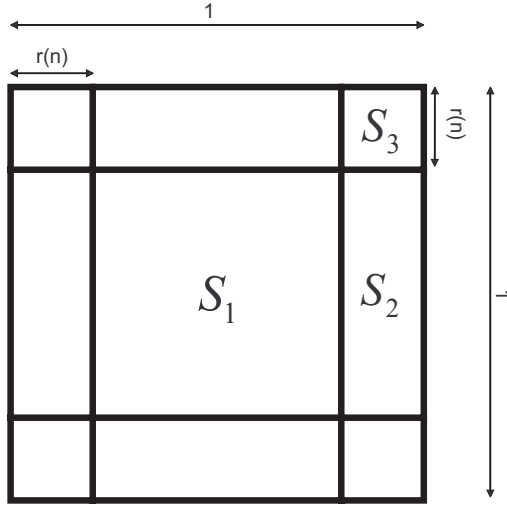
Fig. 7.   The field S_0 and its subdivisions S_1, S_2, and S_3.

is the model $g(n, r, p)$, in which two nodes are connected with probability $0 < p \leqslant 1$ if their distance is less than r. In this model, p models link failures that are common in wireless networks. Note that here we are using p as a different notation from the previous section. Asymptotic properties of $g(n, r)$ have been studied extensively. Here we are interested in these properties when n is not necessarily large. It is worth noting that the assumption that the nodes are distributed on a square is made for simplicity. These arguments can easily be generalized to other models for the deployment region as well as the case where nodes are distributed nonuniformly over the deployment region. For the purpose of analysis, we divide the square S_0 into different parts shown in Figure 7.

3.2. Asymptotic versus Finite Analysis

In this section, we present some evidence to show that previous asymptotic results diverge significantly from actual values for finite networks. To show this, we consider connectivity. We first provide the asymptotic probability of disconnectivity for $g(n, r, p)$ and compare it to simulation results. The following result is proved in Gupta and Kumar [1998], where a slightly different model is considered. However, the results can be trivially extended to $g(n, r)$.

THEOREM 7 (GUPTA AND KUMAR 1998). *Let $c_n = n\pi r^2 - \log(n)$, then $g(n, r)$ is connected with high probability if $\lim_{n \to \infty} c_n = \infty$. On the other hand, if $\lim_{n \to \infty} c_n = c < \infty$, then for large n, $g(n, r)$ is disconnected with a strictly positive probability $1 - p_{asymp}(c)$.*

This theorem states that if $\lim_{n \to \infty} c_n = c < \infty$, the network connectivity probability will be bounded away from one. In fact, $p_{asymp}(c)$ is the limit for the probability that the network is connected when n goes to infinity. To find $p_{asymp}(c)$, Penrose [1997] proved that $g(n, r)$ is connected if and only if the longest edge of its corresponding minimal spanning tree (MST) is smaller than r. On the other hand, if we denote the longest edge of the MST by M_n, it is shown [Penrose 1997] that the distribution of $n\pi M_n^2 - \log n$ converges to the following double exponential distribution.

$$\lim_{n \to \infty} P\big[n\pi M_n^2 - \log n \leq \alpha\big] = exp(-e^{-\alpha}) \quad \text{for } \alpha \in \mathbb{R}. \tag{13}$$
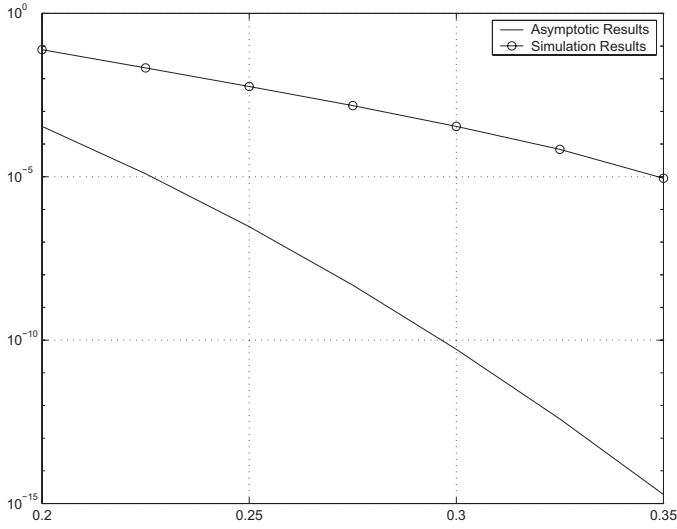
Fig. 8. Comparison of asymptotic results with the small-scale simulation results for the probability of disconnectivity of $g(n = 100, r)$.

Thus, we have

$$p_{asymp}(c) = \lim_{n \to \infty} P[M_n \le r] = \lim_{n \to \infty} P\left[n\pi M_n^2 - \log n \le n\pi r^2 - \log n\right] = e^{-e^{-c}}. \qquad (14)$$

Therefore, asymptotically, the probability that $g(n, r)$ is connected is given by

$$p_{asymp} = e^{-ne^{-n\pi r^2}}.$$

In Figure 8, we compare the probability of having a disconnected graph for $n = 100$ and for both exhaustive simulations and the asymptotic results. The probability of disconnectivity is shown as a function of r, the communication radius. The experiment shows that these results may differ by ten orders of magnitude. This illustrates that the asymptotic method fails to provide a good approximation for small-scale networks.

A natural question to ask is what makes the results for the asymptotic analysis so different from the finite case? As you can see in Figure 7, S_0 is formed by three regions—S_1, and boundary regions S_2 and S_3. One important phenomenon in asymptotic analysis is that boundary effects can be neglected. Loosely speaking, the asymptotic analysis of the network properties is usually dominated by what happens in region S_1 in Figure 7. This can considerably simplify the analysis and results in simple and closed-form formulas for network properties. In fact, we saw an example of this phenomenon in the asymptotic formula for connectivity in Equation (14). However, in small-scale networks, boundary effects cannot be neglected. In other words, nodes in the corners of the field can play an important role in some network properties.

Another important issue in the analysis of finite networks is the effect of constant factors. In asymptotic analysis, we usually neglect constant factors. However, in the small-scale analysis, we must consider them. This is in fact a distinction of any finite analysis from the asymptotic analysis and is not specific to geometric graphs.

3.3. Small-Scale Analysis for Coverage

In this section, we study the coverage probability, $p_{cov}(n, r)$, for finite sensor networks modeled by $g(n, r)$. We prove lower and upper bounds for the coverage probability.

We start with the lower bound, which gives the worst-case performance, as well as a guarantee of the coverage probability.

THEOREM 8. *Consider the coverage probability of a sensor network modeled by $g(n, r)$. Then we have*

$$p_{cov}(n, r) \geq 1 - \sum_{u \in L}[1 - \nu(B(X(u), r'))]^n, \tag{15}$$

where L is the set of $\sqrt{l} \times \sqrt{l}$ points in a virtual grid on the unit square and the radius r' is given by $r - \frac{1}{\sqrt{2l}}$.

PROOF. We briefly describe the proof. As in the case of grid deployment, showing that L is covered guarantees the coverage of the entire region. Let $A(u)$ be the event that the virtual grid point u is covered. Using union bound, we have

$$\Pr(L \text{ not covered}) = \Pr\left[\bigcup_{u \in L} \overline{A(u)}\right] \leq \sum_{u \in L} \Pr(\overline{A(u)}) = \sum_{u \in L}[1 - \nu(B(X(u), r'))]^n. \tag{16}$$

Therefore, $p_{cov}(n, r) = 1 - \Pr(L \text{ not covered}) \geq 1 - \sum_{u \in L}[1 - \nu(B(X(u), r'))]^n$. □

Now we prove an upper bound for the coverage probability.

THEOREM 9. *The coverage probability of a unit square for a sensor network modeled by $g(n, r)$ has an upper bound given by*

$$p_{cov}(n, r) \leq \left[1 - \left(1 - \frac{\pi r^2}{4}\right)^n\right]^4 \times \left[1 - \left(1 - \frac{\pi r^2}{2}\right)^n\right]^{4\lfloor\frac{(1-2r)}{2r}\rfloor} \times [1 - (1 - \pi r^2)^n]^{\lfloor\frac{(1-2r)}{2r}\rfloor^2}. \tag{17}$$

PROOF. We adapt the proof of Theorem 5 to prove Theorem 9. Consider k points $U_1, U_2, \ldots, U_k$ on the unit square and assume that these k points are at least apart by $2r$ units from one another. Similar to the proof of Theorem 5, we can observe that $p_{cov}(n, r)$ is upper bounded by the probability that all the k points are covered, which is given by $\prod_{i=1}^{k}(1 - [1 - \nu(B(U_i, r))]^n)$. Using the set of points depicted in Figure 4, we find the upper bound given by Equation (17). □

Figure 9 compares the bounds predicted by Theorems 8 and 9 with the simulated coverage probability value. The asymptotic result from Kumar et al. [2008] is also presented. Clearly, the bounds are more useful than the asymptotic result in the sense that they give a better estimate of the coverage probability.

3.4. Small-Scale Analysis for Connectivity

In this section, we study the connectivity properties of finite sensor networks modeled by $g(n, r, p)$. We find lower and upper bounds for the probability $p_{disc}(n, r, p)$ that $g(n, r, p)$ is disconnected. Let $p_{low}(n, r, p)$ and $p_{upp}(n, r, p)$ be the lower and upper bounds on $p_{disc}(n, r, p)$, respectively. Here we consider the case where $p_{disc}(n, r, p)$ is small, that is, $p_{disc}(n, r, p) < 0.1$. In practice, this is usually the range that is important, since we want the network to be connected with high enough probability. Using these bounds, we then provide a simple formula to estimate $p_{disc}(n, r, p)$. As we will see by simulations that the proposed formula gives a very good estimate for $p_{disc}(n, r, p)$. First, note that a connected component of a graph g is defined as a connected subgraph that is isolated from the rest of g.
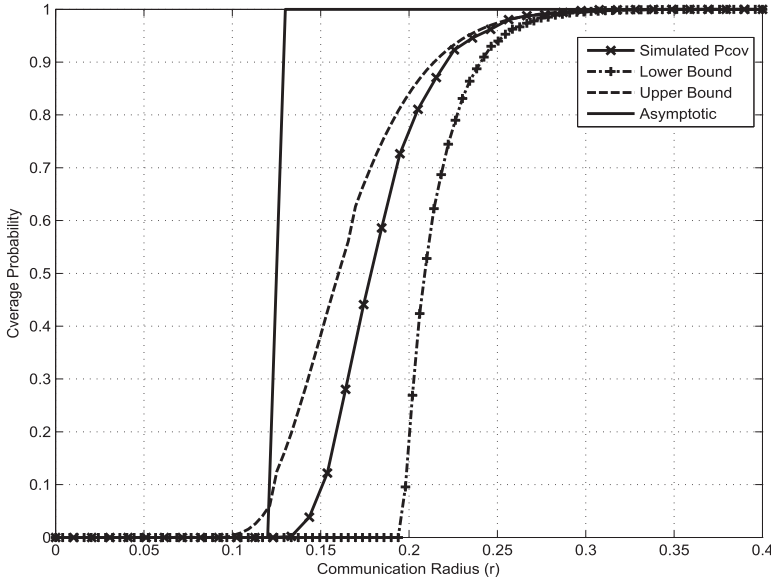
Fig. 9. Simulation results and upper and lower bounds of coverage probability for a random wireless network of size $n = 100$.

THEOREM 10. *Consider a wireless sensor network modeled by* $g(n, r, p)$. *Then we have*

$$p_{disc}(n, r, p) \geq n \int_{S_0} (1 - v(B(X, r))p)^{n-1} dm(X)$$

$$- \binom{n}{2} \int_{S_0} \int_{S_0} (1 - v(B(X, r))p - v(B(Y, r))p$$

$$+ v(B(X, r) \cap B(Y, r))p^2)^{n-2} dm(X) \times m(Y), \tag{18}$$

and

$$p_{disc}(n, r, p) \leq \sum_{k=1}^{n/2} \binom{n}{k} p_{comp}(\{v_1, v_2, \dots, v_k\}) = n \int_{S_0} (1 - v(B(X, r))p)^{n-1} dm(X)$$

$$+ \sum_{k=2}^{n/2} \binom{n}{k} p_{comp}(\{v_1, v_2, \dots, v_k\}), \tag{19}$$

where $p_{comp}(\{v_1, v_2, \dots, v_k\})$ *is the probability that the vertices in* $\{v_1, v_2, \dots, v_k\}$ *construct a connected component in* $g(n, r, p)$.

PROOF. Let $p_1(n, r, p)$ be the probability that there exists at least one isolated node (i.e., a vertex with no neighbors) in $g(n, r, p)$. Let $v_1, v_2, \dots, v_n$ be the n vertices of $g(n, r, p)$. Then $p_{disc}(n, r, p) \geq p_1(n, r, p)$. Applying the inclusion-exclusion lemma,

we obtain

$$p_{disc}(n, r, p) \geq \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \Pr\{v_1, v_2, \ldots, v_k \text{ are isolated vertices}\}$$

$$\geq n \Pr\{v_1 \text{ is isolated}\} - \binom{n}{2} \Pr\{v_1 \text{ and } v_2 \text{ are isolated}\}.$$

Note that $\Pr\{v_1 \text{ is isolated }\} = \int_{S_0} (1 - \nu(B(X, r))p)^{n-1} dm(X)$. Now define $Circ(a, b, r) = \{(x, y) : (x - a)^2 + (y - b)^2 \leq r^2\}$. Then we have

$\Pr\{v_1 \text{ and } v_2 \text{ are isolated vertices}\}$

$$= \int_{S_0} \int_{S_0 \setminus Circ(X,r)} \left(1 - \nu(B(X, r))p - \nu(B(Y, r))p + \nu(B(X, r) \cap B(Y, r))p^2\right)^{n-2} dm(X)$$

$$\times m(Y) + (1 - p)$$

$$\times \int_{S_0} \int_{Circ(X,r)} \left(1 - \nu(B(X, r))p - \nu(B(Y, r))p\right.$$

$$+ \nu(B(X, r) \cap B(Y, r))p^2\big)^{n-2} dm(X) \times m(Y)$$

$$\leq \int_{S_0} \int_{S_0} \left(1 - \nu(B(X, r))p - \nu(B(Y, r))p + \nu(B(X, r) \cap B(Y, r))p^2\right)^{n-2} dm(X) \times m(Y).$$

Combining these equations, we conclude the lower bound. For the upper bound, note that $p_{disc}(n, r, p)$ is equal to the probability that $g(n, r, p)$ has at least one component of size less than $n/2$. This is given by Equation (19). □

Note that the bounds for $p_{disc}(n, r, p)$ may not satisfy the simplicity requirement. Particularly in the upper bound, except, for the first few terms, finding the rest of them is computationally infeasible. We now try to give an estimation of $p_{disc}(n, r, p)$ based on these bounds. Let us denote the kth term in the upper bound by a_k. We recall the assumption that $p_{disc}(n, r, p)$ is not very large; specifically, we assumed $p_{disc}(n, r, p) < 0.1$. An important observation here is that, by this assumption, the a_k coefficients decay very fast, and hence, the term $\sum_{k=1}^{n/2} a_k$ is dominated by a_1. This can be seen by both numerical simulations and intuitive analytical arguments. In fact, as it is shown in Penrose [2003], as n tends to infinity, the impact of the terms a_k, $k > 1$ fades. Figure 10 compares a_1 and a_2 for $g(n = 100, r, p = 0.5)$. As we see, a_2 is at least one order of magnitude smaller than a_1. Using the same approach, we find out that a similar argument is true about the first and second terms in the lower bound of Equation (18). However, the first term is shared by both the lower and upper bounds. Based on these observations, we approximate the probability of disconnectivity as follows.

$$p_{disc}(n, r, p) \simeq n \int_{S_0} (1 - \nu(B(X, r))p)^{n-1} dm(X). \tag{20}$$

Figure 11 shows the upper bound, lower bound, and the simulation result for the probability of disconnectivity of $g(n, r, p)$, for $n = 100$, and $p = 0.5$. As it can be seen, the three curves almost overlap. Based on our simulations, similar results are achieved if we use different choices of parameters.
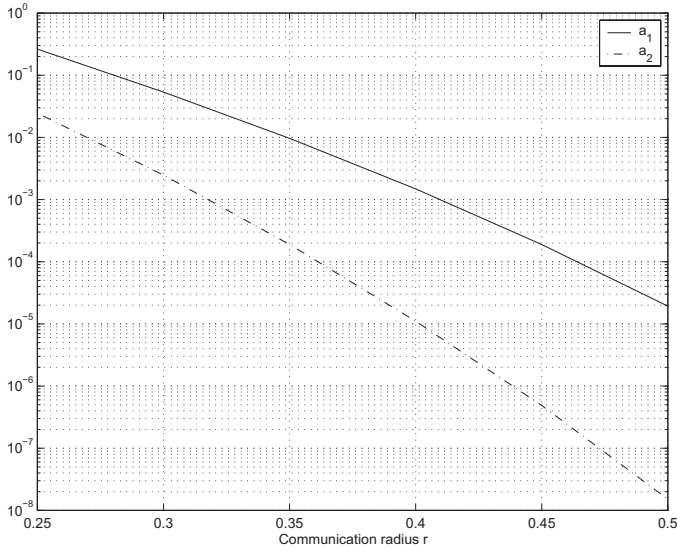
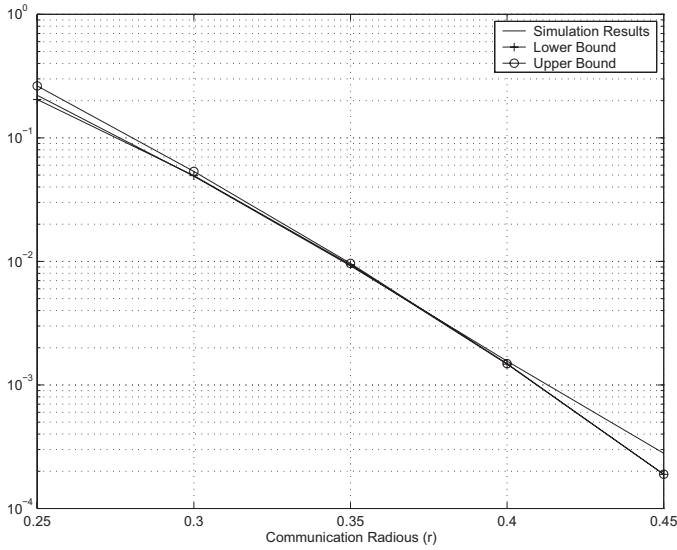Fig. 10.   Comparison of a_1 and a_2 in Equation (19).

Fig. 11.   Disconnectivity probability of $g(100, r, .5)$: lower bound, upper bound, and the simulation results.

It is worth noting that the methodology used here can be used to study k-connectivity. In summary, we find the following approximation of the probability that $g(n, r)$ is not k-connected.

$$p_{k,disc}(n, r) \simeq \sum_{j=0}^{k-1} n \binom{n}{j} \int_{S_0} [v(B(X, r(n)))]^j \times \left(1 - v(B(X, r(n)))\right)^{n-j-1} dm(X). \qquad (21)$$

Our simulations for different values of k confirm the validity of Equation (21). Here, due to the space limitations we omit those results.

4. CONCLUSIONS

In this article, we took some initial steps towards analyzing finite wireless sensor networks. We provided some compelling evidence to show that asymptotic results are not suitable for analyzing practical finite sensor networks. We studied connectivity and coverage of finite unreliable sensor grids as a special case. We showed that the connectivity, as well as all the graph-theoretic quantities, are piecewise constant functions of the transmission radius in such networks. We also proved that the coverage has a similar behavior. Moreover, we obtained lower and upper bounds for the coverage and *k*-coverage probability of the grids and verified their preciseness through simulations. Next, we extended our study to finite sensor networks with random node deployment. Specifically, we considered coverage and connectivity of such networks. We derived lower and upper bounds for their coverage and showed how they could be used to estimate the coverage probability of the network. We also obtained a formula for connectivity of wireless sensor networks and verified its accuracy through simulations. The formula was then extended to include *k*-connectivity. A common characteristic of all these bounds is the ease of computations, making them very attractive.

This article also opens up many research possibilities that offer some potentials for further study. In the past, many other important properties of wireless sensor networks have been studied for large-scale networks. It is an important task to extend these results for networks with practical sizes, that is, small-scale networks. Small-scale analysis can also reveal the effects of network parameters on network characteristics. The next step would be to derive more accurate bounds for network parameters, such as coverage, connectivity, and MAC-layer capacity, and further use the small-scale framework in the design, analysis, and evaluation of communication algorithms for wireless networks.

REFERENCES

AKYILDIZ, I. F., SU, W., SANKARASUBRAMANIAM, Y., AND CAYIRCI, E. 2002. A survey on sensor networks. *IEEE Communi. Mag.* 102–114.

BAI, X., KUMAR, S., XUAN, D., YUN, Z., AND LAI, T. H. 2006. Deploying wireless sensors to achieve both coverage and connectivity. In *Proceedings of the ACM International Symposium Mobile Ad Hoc Networking and Computing (MobiHoc)*. 131–142.

BALAKRISHAN, H., BARRETT, C. L., KUMAR, V. S. A., MARATHE, M. V., AND THITE, S. 2004. The distance-2 matching problem and its relationship to MAC-layer capacity of ad hoc wireless networks. *IEEE J. Select. Areas Commun. 22*, 1069–1079.

BALISTER, P., BOLLOBAS, B., SARKAR, A., AND KUMAR, S. 2007. Reliable density estimates for coverage and connectivity in thin strips of finite length. In *Proceedings of the 13th Annual ACM International Conference on Mobile Computing and Networking (MobiCom'07)*. 75–86.

BALISTER, P. AND KUMAR, S. 2009. Random vs. deterministic deployment of sensors in the presence of failures and placement errorss. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies*. 2896–2900.

BALISTER, P., ZHENG, Z., KUMAR, S., AND SINHA, P. 2009. trap coverage: Allowing coverage holes of bounded diameter in wireless sensor networks. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom)*. 136–144.

BOLLOBÁS, B. 2001. *Random Graphs* 2nd Ed. Cambridge University Press, Cambridge, U.K.

BOOTH, L., BRUCK, J., FRANCESCHETTI, M., AND MEESTER, R. 2003. Covering algorithms, continuum percolation and the geometry of wireless networks. *Ann. Appl. Probab. 13, 2*.

BOOTH, L., BRUCK, J., COOK, M., FRANCESCHETTI, M., AND MEESTER, R. 2005. Continuum percolation with unreliable and spread out connections. *J. Stat. Physics 118*, 3–4, 721–734.

DESAI, M. AND MANJUNATH, D. 2002. On the connectivity in finite ad hoc networks. *IEEE Communi. Lett. 6,* 10, 437–439.

DOUSSE, O., FRANCESCHETTI, M., AND THIRAN, P. 2006. On the throughput scaling of wireless relay networks. *IEEE Trans. Inform. Theory 52,* 6, 2756–2761.

DOUSSE, O. AND THIRAN, P. 2004. Connectivity vs capacity in dense ad hoc networks. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom)*.

DOUSSE, O., THIRAN, P., AND HASLER, M. 2002. Connectivity in ad-hoc and hybrid networks. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom)*.

DUBHASHI, D., HÄGGSTRÖM, O., AND PANCONESI, A. 2003. Connectivity properties of bluetooth wireless networks. http://www.math.chalmers.se/~olleh/papers.html.

ESLAMI, A., NEKOUI, M., AND PISHRO-NIK, H. 2010. Results on finite wireless networks on a line. *IEEE Trans. Communi. 58,* 8, 2204–2211.

FORTUIN, C. M., KASTELEYN, P. W., AND GINIBRE, J. 1971. Correlation inequalities on some partially ordered sets. *Commun. Math. Physics 22,* 2, 89–103.

FRANCESCHETTI, M. AND MEESTER, R. 2008. *Random Networks for Communication: From Statistical Physics to Information Systems*. Cambridge University Press, Cambridge, U.K.

FREANCESCHETTI, M. AND MEESTER, R. 2006. Critical node lifetimes in random networks via the Chen-Stein method. *IEEE Trans. Inform. Theory 52,* 2, 2831–2837.

GHASEMI, A. AND NADER-ESFAHANI, S. 2006. Exact probability of connectivity in one-dimensional ad hoc wireless networks. *IEEE Commun. Lett. 10*, 251–253.

GORE, A. 2006. Comments on the connectivity in finite ad hoc networks. *IEEE Communi. Lett. 10,* 2, 88–90.

GROSSGLAUSER, M. AND TSE, D. 2001. Mobility increases the capacity of ad-hoc wireless networks. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom)*.

GUPTA, P. AND KUMAR, P. 1998. Critical power for asymptotic connectivity in wireless networks. *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming*, W. M. McEneaney, G. Yin, and Q. Zhang, Eds. Birkhäuser, Berlin.

GUPTA, P. AND KUMAR, P. R. 2000. The capacity of wireless networks. *IEEE Trans. Inform. Theory 46,* 2, 388–404.

GUPTA, P. AND KUMAR, P. R. 2003. Towards an information theory of large networks: an achievable rate region. *IEEE Trans. Inform. Theory 49*, 1877–1894.

JANSON, S. 1986. Random coverings in several dimensions. *Acta Mathematica 156*.

KARMACHANDANI, N., MANJUNATH, D., YOGESHWARAN, D., AND IYER, S. K. 2006. Evolving random geometric graph models for mobile wireless networks. In *Proceedings of the 4th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WIOPT'06)*.

KUMAR, S., LAI, T. H., AND BALOGH, J. 2004. On k-coverage in mostly sleeping sensor network. In *Proceedings of the ACM International Conference on Mobile Computing and Networking (MobiCom)*.

KUMAR, S., LAI, T. H., AND BALOGH, J. 2008. On k-coverage in mostly sleeping sensor network. *Wireless Netw.* 277–294.

LI, J., BLAKE, C., DE COUTO, D. S. J., LEE, H. I., AND MORRIS, R. 2001. Capacity of ad hoc wireless netw. In *Proceedings of the 7th ACM International Conference on Mobile Computing and Networking*. 61–69.

LI, X. Y., WAN, P., WANG, Y., AND YI, C. W. 2003. Fault tolerant deployment and topology control in wireless networks. In *Proceedings of the ACM Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc'03)*.

LIU, B., LIU, Z., AND TOWSLEY, D. 2003. On the capacity of hybrid wireless networks. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom)*.

MEESTER, R. AND ROY, R. 1996. *Continuum Percolation*. Cambridge University Press, Cambridge U.K.

PENROSE, M. 2003. *Random Geometric Graphs*. Oxford University Press, Oxford, U.K.

PENROSE, M. D. 1997. The longest edge of the random minimal spanning tree. *Ann. Appl. Probab. 7,* 2, 340–361.

PEREVALOV, E. AND BLUM, R. 2003. Delay limited capacity of ad hoc networks: Asymptotically optimal transmission and relaying strategy. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom)*.

PISHRO-NIK, H., CHAN, K. S., AND FEKRI, F. 2004. On connectivity properties of large-scale sensor networks. In *Proceedings of the 1st Annual IEEE International Conference on Sensor and Ad Hoc Communications and Networks*. 467–472.

SHAKKOTTAI, S., SRIKANT, R., AND SHROFF, N. 2003. Unreliable sensor grids: Coverage, connectivity and diameter. In *Proceedings of the Annual Joint Conference of the IEEE Computer and Communications Societies (InfoCom'03)*.

WAN, P. AND YI, C. W. 2004. Asymptotic critical transmission radius and critical neighbor number for k-connectivity in wireless ad hoc networks. In *Proceedings of the ACM Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc'04)*.

XUE, F. AND KUMAR, P. R. 2004. The number of neighbors needed for connectivity of wireless networks. *Wireless Netw. 10,* 2, 169–181.

YEN, L.-H. AND YU, C. W. 2004. Link probability, network coverage, and related properties of wireless ad hoc networks. In *Proceedings of the 1st International Conference on Mobile Ad-Hoc and Sensor Systems (MASS'04)*.