On Generalized EXIT Charts of LDPC Code Ensembles over Binary-Input Output-Symmetric Memoryless Channels

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***Abstract*—Generalized Extrinsic Information Transfer (GEXIT) charts were introduced as an extension of EXIT charts which have an extensive use in analysis and design of many iterative schemes including Low-Density Parity-Check (LDPC) codes. While a powerful as well as an insightful concept, their full potential as a designing tool for LDPC code ensembles has not been realized due to some missing steps. This papers aims at filling these gaps by proving some important properties of GEXIT charts and using them to design capacity-approaching LDPC code ensembles. The primary results on GEXIT charts are limited to regular variable and check node degrees. Moreover, variable node GEXIT curves have only been derived for the case where no physical channel is present. In a recent paper, GEXIT curves for irregular variable node and check node degree distributions have been derived. In this paper, we derive GEXIT charts of LDPC code ensembles over binary-input output-symmetric memoryless channels with any channel parameter. For the case of binary symmetric channel, we derive closed form expression for the GEXIT curve of variable nodes. We also propose to use an alternative representation of GEXIT charts in which we plot the inverse of variable node GEXIT curve together with dual GEXIT curve of the check node. We prove that the area theorem still holds in this case. Using these results, we analyze and design capacity-approaching LDPC codes using GEXIT charts.**

I. Introduction

Low-Density Parity-Check (LDPC) codes have been an active area of research in the past decade due to their good performance under iterative message passing algorithms [1]. For regular LDPC codes, there is a fixed but different number of ones in each column and row of the parity check matrix. This constraint makes it impossible to design codes with performance approaching the channel capacity [2], [3], [4]. Therefore, irregular LDPC codes were introduced [5] for which this constraint is not enforced any more. In both cases, the number of ones is in such a way that the parity check matrix is sparse, i.e. the density of ones is low.

For large block sizes, the ensembles of LDPC codes are considered and presented by their degree distribution pairs $(\lambda(x), \rho(x))$. Concentration results [2], [6] indicate that for sufficiently large block lengths, the performance of any ensemble over a binary-input output-symmetric memoryless (BIOSM) channel tends to the average performance of the ensembles. This important result makes it possible to use asymptotic analysis tools such as density evolution (DE) [2] to analyze and design LDPC code ensembles with performance approaching the capacity.

In density evolution technique, assuming that the all-one codeword is transmitted and starting from initial channel density, the evolution of this density is tracked throughout the iterations. At each iteration, the probability of error is computed for the density. This is defined as the probability that the value of the random variable corresponding to the density is negative. Depending on the initial density, which itself directly depends on the channel parameter,[1] two cases can happen. The probability of error of the evolving density either tends to zero (DE converges) or is bounded away from zero after infinite number of iterations. It can be proved that for a large set of channel types including Binary Erasure Channel (BEC), Binary Symmetric Channel (BSC) and Binary Input Additive White Gaussian Noise (BIAWGN) channel, there is a boundary value on the channel parameter such that if the channel parameter is less than that value, the DE converges and if it is more, DE does not converge. We call this boundary value the *threshold* of the ensemble over the given channel [2].

It can be seen that in each iteration of the DE, a possibly large set of values representing the density, should be updated which can be computationally expensive, especially when DE is used to design ensembles. Consequently, many approximation tools and techniques have been developed to reduce the complexity of DE, such as Extrinsic Information Transfer (EXIT) charts [7] and Gaussian approximation [8]. The aim of all these methods is to map a given density to a scalar. This not only reduces the complexity, but also provides a better insight into the dynamic of iterative decoding. Concentrating now on the EXIT chart method, it can be shown that for the BEC, EXIT chart is not an approximation to DE anymore and is actually able to provide exactly the same threshold predicted by DE. Moreover, an area theorem is proved for the EXIT charts of the BEC indicating that the area between the EXIT curve of variable node and inverse EXIT curve of check node can be translated into the distance to capacity. In other

[1] In this paper we consider channels that are parameterized by one variable such as ϵ and we assume that the quality of channel degrades as ϵ increases.

words, to obtain capacity achieving codes over the BEC, it is enough to find a degree distribution for which the area between the curves tends to zero. This situation is called *matching condition*. This is consistent with the previously obtained results by Shokrollahi [9], and also in [10], known as flatness condition, used to design capacity achieving sequences for the erasure channel. Unfortunately, such an area theorem does not hold for the EXIT charts of other channels. Consequently, EXIT charts can not be used to design provably capacity achieving sequences of LDPC code ensembles over general BIOSM channels.

Recently, a new tool named Generalized EXIT (GEXIT) chart has been proposed in [11]. In this tool, similar to EXIT charts, intermediate densities in DE are mapped to a scalar. However, the mapping has been cleverly defined such that the resulting curves for variable and check nodes of a regular LDPC code ensemble fulfill the area property. This suggests that GEXIT charts have the potential to be used in designing capacity achieving sequences for channels other than the BEC.

In [11], the GEXIT curves have only been obtained for regular LDPC code ensembles. Moreover, it is assumed that the (physical) channel parameter takes its largest possible value (worst channel quality). This in fact makes the computed curve independent of the channel parameter. This is equivalent to the assumption that no physical channel is present. Consequently, the full potential of GEXIT charts in designing capacity achieving sequences of LDPC code ensembles has not been realized. In a recent paper [12], an analytical relationship between the GEXIT curves of irregular variable and check node degrees, and that of regular variable node and check node degrees is obtained. It is shown that this relationship is similar to the one that exists between the EXIT curve of regular and irregular variable and check node degrees.

In this paper, we develop GEXIT curves of variable nodes for any given channel parameter. Using this result and through a tedious derivation, we obtain a closed form formulation for the GEXIT curves of variable nodes over the BSC. It is important to note that in [11], the matching condition for a given ensemble has been suggested to hold for check node GEXIT curve and the inverse of the so called dual GEXIT curve of the variable node. However, taking the channel parameter into account, it might not be as easy to obtain a closed form formulation for the dual GEXIT curve of the variable node. In order to employ our obtained formulation for variable node GEXIT curve in the analysis and design of codes, we prove that similar matching results hold for dual GEXIT curve of the check node and the inverse of variable node GEXIT curve. Using these results, we analyze and design capacity-approaching LDPC codes using GEXIT charts.

The organization of the paper is as follows. In the next section, some definitions and notations are reviewed. In Section III, we develop the GEXIT curve of variable nodes for an arbitrary channel parameter. In particular, closed form formulation for the GEXIT curve of variable nodes of the BSC is derived. In section IV, alternative representation of matching condition is proposed. In Section V, some examples on analysis and design of capacity approaching codes in BSC and BIAWGN channels are proposed. Finally, Section VI concludes the paper. The details of derivations and proof of propositions have been omitted due to lack of space.

II. Review and Notations

A. Ensembles of LDPC Codes

An LDPC code ensemble is represented by its degree distributions $\lambda(x) = \sum_{i=2}^{D_v} \lambda_i x^{i-1}$, $\rho(x) = \sum_{i=2}^{D_c} \rho_i x^{i-1}$, where λ_i and ρ_i represent the fraction of edges emanating from variable and check nodes of degree i, respectively. Moreover D_v and D_c represent the maximum variable and check node degrees. The rate of the ensemble, $R(\lambda, \rho)$, can be computed as:

$$R(\lambda, \rho) = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx} = 1 - \frac{\sum_{i=2}^{D_c} \frac{\rho_i}{i}}{\sum_{i=2}^{D_v} \frac{\lambda_i}{i}}.$$

B. Notations

It is assumed that all channels considered in this paper are Binary-Input, Output-Symmetric and Memoryless (BIOSM). Instead of considering a single BIOSM channel, families of BIOSM channels parameterized by a real-valued parameter such as ϵ are usually considered and are denoted by BIOSM(ϵ). If $\boldsymbol{X} = \{\pm 1\}$ denotes the channel input alphabet and $\boldsymbol{Y}$ denotes channel output alphabet, then the channel can be characterized by its transition probability density $P_{Y|X}(y|x)$.

Throughout the paper, we consider the probability distributions of extrinsic log-likelihood ratios (L-densities). For example L-densities for BEC(p), BSC(δ) and BIAWGN(σ) are as follows:

$$\begin{aligned} c_p^{BEC(p)}(y) &= p \triangle_0 (y) + (1-p) \triangle_\infty (y), \\ c_\delta^{BSC(\delta)}(y) &= \delta \triangle_{-\log\frac{1-\delta}{\delta}} (y) + (1-\delta) \triangle_{\log\frac{1-\delta}{\delta}} (y), \\ c_\sigma^{BIAWGN(\sigma)}(y) &= \sqrt{(\sigma^2/8\pi)} e^{-\frac{(y-2/\sigma^2)^2}{8/\sigma^2}}. \end{aligned}$$

Note that for simplicity, we may parameterize a BIAWGN channel with the mean of its Gaussian L-density, q, i.e., BIAWGN(q) where $q = 2/\sigma^2$.

It is shown in [6] that if $c(y)$ is an L-density, it has the symmetry property, i.e., $c(-y) = c(y).\exp(-y)$. Consequently, its entropy can be obtained using the following formula:

$$H(c) = \int_{-\infty}^{\infty} c(y) \log_2(1 + e^{-y}) dy.$$

C. Definition of GEXIT Function

Let X be a vector of length n. Let the channel from X to Y be memoryless, where Y_i is the result of passing X_i through the channel. Assume that all individual channels are parameterized in a smooth way by a common parameter ϵ, i.e., $\epsilon_i = \epsilon_i(\epsilon)$, $i \in [n]$. Let Ω be a further observation of X so that $p_{\Omega|X,Y}(\omega|x,y) = p_{\Omega|X}(\omega|x)$. Then the i^{th} and the average GEXIT function satisfying the general area theorem are defined by [11]:

$$g_i(\epsilon) = \frac{\partial H(X_i|Y,\Omega)}{\partial \epsilon_i} \frac{d\epsilon_i}{d\epsilon},$$

and

$$g(\epsilon)=\frac{1}{n}\sum_{i=1}^{n}\frac{\partial H(X_i|Y,\Omega)}{\partial\epsilon_i}\frac{d\epsilon_i}{d\epsilon}=\frac{1}{n}\sum_{i=1}^{n}g_i(\epsilon), \tag{1}$$

respectively.

Based on the results of [11], given two families of L-densities $\{c_{\epsilon_i}\}$ and $\{a_\epsilon\}$ parameterized by ϵ, the GEXIT function can be represented as follows:[2]

$$G(c_{\epsilon_i},\,a_\epsilon)=\frac{\int_z\int_w a_\epsilon(z)\frac{\partial c_{\epsilon_i}(\omega)}{\partial\epsilon}\times\log_2\left(1+e^{-z-\omega}\right)d\omega dz}{\int_w\frac{\partial c_{\epsilon_i}(\omega)}{\partial\epsilon}\times\log_2\left(1+e^{-\omega}\right)d\omega}, \tag{2}$$

and the GEXIT kernel is defined as:

$$l^{c_{\epsilon_i}}(z)=\frac{\int_\omega\frac{\partial c_{\epsilon_i}}{\partial\epsilon}\times\log_2\left(1+e^{-z-w}\right)d\omega}{\int_\omega\frac{\partial c_{\epsilon_i}}{\partial\epsilon}\times\log_2\left(1+e^{-w}\right)d\omega}. \tag{3}$$

Consequently, the GEXIT curve is given in parametric form by $\{H(c_{\epsilon_i}),\,G(c_{\epsilon_i},\,a_\epsilon)\}$, where

$$H(c_{\epsilon_i})=\int_{-\infty}^{\infty}c_{\epsilon_i}(\omega)\log(1+e^{-\omega})d\omega.$$

D. Dual GEXIT Curve

Let GEXIT curve be given in the parametric form by $\{H(c_{\epsilon_i}),\,G(c_{\epsilon_i},\,a_\epsilon)\}$. According to [11] dual GEXIT curve is defined in parametric form as $\{G(a_\epsilon,\,c_{\epsilon_i}),\,H(a_\epsilon)\}$, where

$$G(a_\epsilon,\,c_{\epsilon_i})=\frac{\int_z\int_w c_{\epsilon_i}(\omega)\frac{\partial a_\epsilon(z)}{\partial\epsilon}\times\log_2\left(1+e^{-z-\omega}\right)d\omega dz}{\int_z\frac{\partial a_\epsilon(z)}{\partial\epsilon}\times\log_2\left(1+e^{-z}\right)dz}, \tag{4}$$

and

$$H(a_\epsilon)=\int_{-\infty}^{\infty}a_\epsilon(z)\log(1+e^{-z})dz. \tag{5}$$

E. GEXIT Charts

In [7], EXIT chart for a given LDPC code ensemble is realized if we plot the EXIT curve of variable node against the inverse of check node. Similar concept can be used for GEXIT curves. In particular, in [11], GEXIT chart is realized by plotting inverse of dual GEXIT curve of variable node against the GEXIT curve of check node.

III. Derivation of Variable Node GEXIT Function for an Arbitrary Channel Parameter

A. General Case

As discussed in Section I, variable node GEXIT functions have been obtained in [11] assuming that no physical channel is present or equivalently, the channel parameter has been set to its worst value. Consequently, Equation (2) has been derived based on such an assumption. In the following proposition, we have derived a formulation similar to (2) using (1) for an arbitrary channel parameter.

Proposition 1: The GEXIT function $g_i(\epsilon)$ over a BIOSM channel with parameter q denoted by its L-density $b_q(y)$ is represented as:

$g_i(\epsilon)=$

$$\int_z\int_\omega\int_y a_\epsilon(z)b_q(y)\frac{\partial c_{\epsilon_i}(\omega)}{\partial\epsilon}\times\log_2\left(1+e^{-z-y-\omega}\right)dyd\omega dz, \tag{6}$$

where in (6), $a_\epsilon(z)$ denotes extrinsic L-density, and $c_\epsilon(\omega)$ represents i^{th} extrinsic channel L-density. Normalized GEXIT function is defined as follows similar to (2):

$$\frac{\int_z\int_\omega\int_y a_\epsilon(z)b_q(y)\frac{\partial c_{\epsilon_i}(\omega)}{\partial\epsilon}\times\log_2\left(1+e^{-z-y-\omega}\right)dyd\omega dz}{\int_\omega\frac{\partial c_{\epsilon_i}(\omega)}{\partial\epsilon}\times\log_2\left(1+e^{-\omega}\right)d\omega}. \tag{7}$$

Note that if in the above formulation we set $b_q(y)$ to Δ_0, the equation is reduced to (2).

Example 1 (Variable node GEXIT curve in the BEC): For variable node of degree d_v and transmission over BEC(p), from equation (7) by substituting the values of L-densities from Table I and evaluating the integrals we can see that parameterized GEXIT curve is given by:

$$\{H(c_\epsilon),\,G(c_\epsilon(\omega),\,b_p(y),\,a_\epsilon(z))\}=\{\epsilon,\,p\,\epsilon^{dv-1}\}, \tag{8}$$

which is the same as EXIT curve.

TABLE I
VARIABLE NODE L-DENSITIES IN BEC(P)

L-densities	Value
$c_\epsilon(\omega)$	$\epsilon\,\triangle_0\,(\omega)\,+\,(1-\epsilon)\,\triangle_\infty\,(\omega).$
$a_\epsilon(z)$	$\epsilon^{dv-1}\,\triangle_0\,(z)\,+\,1-\epsilon^{dv-1}\,\triangle_\infty\,(z).$
$b_p(y)$	$p\,\triangle_0\,(y)\,+\,(1-p)\,\triangle_\infty\,(y).$

B. Derivation of GEXIT Function for the BSC

In this subsection, we use the result of last subsection to obtain the closed form relationship for the GEXIT function of variable nodes for the BSC.

Proposition 2: For a variable node of degree d_v and transmission over $BSC(\delta)$, the closed form GEXIT function $g_i(\epsilon)$ is given by

$$\begin{aligned}&\sum_{j=\pm1}j\sum_{i=0}^{d_v-1}\begin{pmatrix}d_v-1\\ i\end{pmatrix}\varepsilon\,(1-\varepsilon)^{\,d_v-1-i}\times\\ &\{\delta\log_2\left(1+\tfrac{1-\delta}{\delta}\left(\tfrac{\varepsilon}{1-\varepsilon}\right)^{(d_v-1-2i)-j}\right)\\ &+(1-\delta)\log_2\left(1+\tfrac{\delta}{1-\delta}\left(\tfrac{\varepsilon}{1-\varepsilon}\right)^{(d_v-1-2i)-j}\right)\\ &+\varepsilon^{\left(\frac{-j-1}{2}\right)}(1-\varepsilon)^{\left(\frac{j-1}{2}\right)}\left(\tfrac{\varepsilon}{1-\varepsilon}\right)^{(d_v-1-2i)}\frac{1}{\ln 2}\times\\ &\left(\frac{\delta}{1+\frac{\delta}{1-\delta}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{(d_v-1-2i)-j}}+\frac{1-\delta}{1+\frac{1-\delta}{\delta}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{(d_v-1-2i)-j}}\right)\}.\end{aligned} \tag{9}$$

[2] Throughout the paper, we may drop the integral limits for more clarity. In such cases, the integral limits are assumed to be $-\infty$ and ∞ unless otherwise stated.

C. GEXIT Chart for the BIAWGN Channel

In the case of the BIAWGN channel, there is no closed form formula for GEXIT function and calculation of GEXIT function can be done only by numerical methods. Since in [11] the formulations of GEXIT function of variable nodes have been obtained in the absence of physical channel, in this subsection we derive the formulations in the presence of physical channel where the physical channel is BIAWGN(q).

Using an approach similar to [11], Equation (6) is modified to

$$g_i(\epsilon) = \int_z \int_y a_\epsilon(z) b_q(y) l^{c_\epsilon}(z, y)\, dy\, dz,$$

where $l^{c_\epsilon}(z, y)$ is the GEXIT kernel given in [11].

We now propose the following proposition which gives a closed form equation for the starting point of the variable node GEXIT curve over the BIAWGN channel.

Proposition 3: Let $g_i(\epsilon)$ be the variable node GEXIT function over BIAWGN(q). Then

$$\lim_{\epsilon \to 0} g_i(\epsilon) = \frac{2}{\sqrt{\pi\, q}} \int_{-\infty}^{\infty} e^{-\frac{(y-q)^2}{4\,q}} \frac{e^{-2y}}{(1+e^{-y})^2} dy.$$

IV. Alternative Representation of GEXIT Chart

In this section we propose a new representation of GEXIT chart which is slightly different from [11]. Consider a degree distribution pair $(\lambda,\, \rho)$ and transmission over BIOSM channel characterized by its L-density c so that density evolution converges to $\triangle_\infty$. Let $\{a_\alpha\}_{\alpha=-1}^{\infty}$ and $\{b_\alpha\}_{\alpha=0}^{\infty}$ denote the interpolated families[3]. We remind that according to Lemma 17 in [11], the following two GEXIT curves do not cross and faithfully represent density evolution:

-GEXIT curve of check nodes:

$$\{H(a_\alpha),\, G(a_\alpha,\, b_{\alpha+1})\},$$

-Inverse of dual GEXIT curve of variable nodes:

$$\{H(a_\alpha),\, G(a_\alpha,\, b_\alpha)\}.$$

Now if we replace GEXIT curve of check node by dual GEXIT curve of check node and also replace the inverse dual GEXIT curve of variable node by inverse of GEXIT curve of variable node, we can state the following proposition:

Proposition 4: Consider a degree distribution pair $(\lambda,\, \rho)$ and transmission over a BIOSM channel characterized by its L-density c so that density evolution converges to $\triangle_\infty$. Let $\{a_\alpha\}_{\alpha=-1}^{\infty}$ and $\{b_\alpha\}_{\alpha=0}^{\infty}$ denote the interpolated families as defined in [11]. Then the two following curves parameterized by:

$$\begin{aligned} &\{G(b_{\alpha+1},\, a_\alpha),\, H(b_{\alpha+1})\}, \text{ dual of check node,} \\ &\{G(b_\alpha,\, a_\alpha),\, H(b_\alpha)\}, \text{ inverse of variable node,} \end{aligned} \qquad (10)$$

do not cross and faithfully represent density evolution. Moreover, the area under the dual GEXIT curve of check node is equal to $1 - \int \rho$ and the area to the left of the inverse of variable node GEXIT curve is equal to $H(c) \int \lambda$. It follows that $R(\lambda,\, \rho) \leq 1 - H(c)$ and the equality holds (we achieve the capacity) if the the two curves match.

V. Some Examples on GEXIT Charts for BSC and BIAWGN Channels

In previous sections, we obtained formulations for the GEXIT curves of irregular check node and variable node degrees for any given physical channel parameter. Moreover, we introduced a new representation of GEXIT charts in which the check node dual GEXIT curve can be plotted against the inverse of variable node GEXIT curve. This makes it possible to verify the matching condition for irregular LDPC code ensembles and design irregular ensembles. These facts are examined in this section through some examples.

Example 2 ((3, 6) ensemble and transmission over the BSC):
Consider a (3, 6) regular ensemble and transmission over BSC(0.07). The corresponding GEXIT chart is shown in Fig. 1 where dashed and solid curves correspond to the dual GEXIT curve of check node and inverse GEXIT curve of variable node, respectively[4]. Note that as opposed to variable node GEXIT curves in [11], the starting point of the dashed curve is not at coordinates (1,1) and depends on the channel parameter. As can be seen, the two curves do not match which is expected as the ensemble is regular.

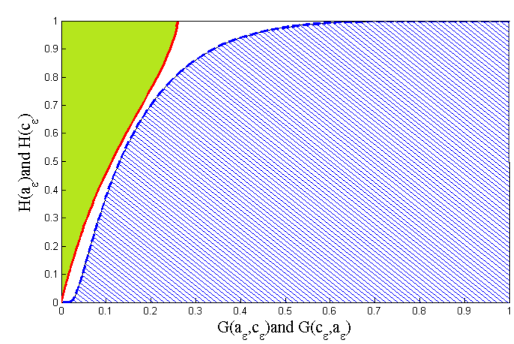

Fig. 1. GEXIT Chart for (3, 6) regular LDPC code ensemble for BSC(0.07).

Example 3 (Capacity approaching LDPC code ensemble over the BSC): The Shannon limit for the BSC and rate one-half is $\delta_{opt} = 0.110028$. The designed ensemble whose degree distribution can be found in [2] has a threshold equal to $\delta^* = 0.106$ which is pretty close to the Shannon limit. In Fig. 2, the corresponding GEXIT chart is plotted. As can be seen, the matching condition holds in this case.

Example 4 (Irregular ensemble over the BIAWGN channel):

[3]According to Definition 9 in [11].

[4]We maintain the same settings for the curves in the rest of figures.

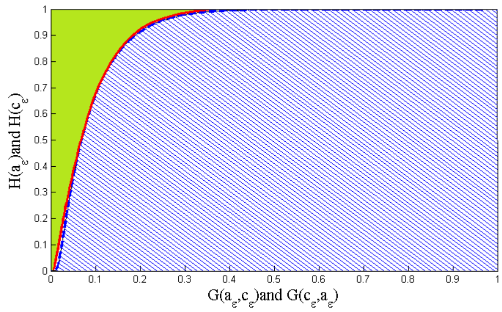

Fig. 2. GEXIT Chart for an irregular LDPC code ensemble over the BSC.

In Fig. 3, GEXIT chart for a highly irregular LDPC code ensemble with maximum variable node degree of 20 taken from Table II of [2] has been plotted. As can be seen, the curves almost completely match showing that the code preforms very closely to capacity.

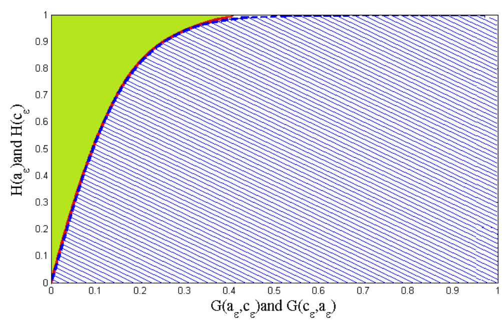

Fig. 3. GEXIT Chart of an irregular LDPC code ensemble with $D_v = 20$ in BIAWGN channel. in BIAWGNC

Example 5 (Code design using GEXIT chart for D_v=20 over the BIAWGN channel):
In this example, we design a degree distribution with rate one half and $D_v = 20$. To be able to compare the result with Example 4, we use the same check node degree distribution as that of Example 4. Using the formulation for GEXIT curve over the BIAWGN channel in Section III.C, we were able to design the following degree distribution:

$$\begin{aligned}\lambda(x) &= 0.2300\,x + 0.2102\,x^2 + 0.0245\,x^3 + 0.0020\,x^4 \\ &\quad + 0.0954\,x^5 + 0.0631\,x^6 + 0.0488\,x^7 + 0.0185\,x^8 \\ &\quad + 0.0395\,x^{18} + 0.2680\,x^{19}, \\ \rho(x) &= 0.64854\,x^7 + 0.34747\,x^8 + 0.00399\,x^9.\end{aligned}$$

The threshold of this rate one half ensemble is 0.3004 dB which is .01dB better than 0.3104 dB, the threshold of the ensemble in Example 4 designed in [2]. These thresholds are less than 0.12 dB away from the Shannon limit.

VI. Conclusion

Primary results on GEXIT charts are limited to regular check node and variable node degrees in the absence of physical channel. In this paper, we derived GEXIT curves of variable nodes over BIOSM channels for any arbitrary channel parameter. We then derived a closed form formula for the GEXIT curves of variable nodes over the BSC. The matching condition in [11] has been proved to hold for the check node GEXIT curve and the inverse of dual GEXIT curve of the variable node. To be able to use the GEXIT curve of variable nodes instead of dual GEXIT curve of variable nodes, we proved that similar matching results hold for dual GEXIT curve of the check node and the inverse of variable node GEXIT curve. Putting our results together with the one in [12] on irregular codes, we were able to plot the GEXIT chart of any given irregular ensemble for any arbitrary channel parameter. In particular, we plotted the GEXIT chart for capacity approaching ensembles over the BSC and BIAWGN channel and verified that matching condition holds in that case. It is important to note that using GEXIT charts, the problem of designing LDPC code ensembles is in fact reduced to a linear programming optimization problem. Consequently, the proposed result on GEXIT charts was used to design a capacity approaching ensemble over the BIAWGN channel.

Acknowledgment

This work was supported by Iran Telecommunication Research Center (ITRC) under project number 19-01-04.

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