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On systematic design of universally capacity approaching rate-compatible sequences of LDPC code ensembles over binary-input output-symmetric memoryless channels

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***Abstract*—Despite tremendous amount of research on the design of Low-Density Parity-Check (LDPC) codes with belief propagation decoding over different types of Binary-Input Output-Symmetric Memoryless (BIOSM) channels, most results on this topic are based on numerical methods and optimization which do not provide much insight into the design process. In particular, systematic design of provably capacity achieving sequences of LDPC code ensembles over the general class of BIOSM channels, has remained a fundamental open problem. For the case of the Binary Erasure channel, explicit construction of capacity achieving sequences have been proposed based on a property called the flatness condition. In this paper, we propose a systematic method to design universally capacity approaching rate-compatible LDPC code ensemble sequences over BIOSM channels. This is achieved by interpreting the flatness condition over the BEC, as a Successive Maximization (SM) principle that is generalized to other BIOSM channels to design a sequence of capacity approaching ensembles called the parent sequence. The SM principle is then applied to each ensemble within the parent sequence, this time to design rate-compatible puncturing schemes. As part of our results, we extend the stability condition which was previously derived for degree-2 variable nodes to other variable node degrees as well as to the case of rate-compatible codes. Consequently, we rigorously prove that using the SM principle, one is able to design universally capacity achieving rate-compatible LDPC code ensemble sequences over the BEC. Unlike the previous results on such schemes over the BEC in the literature, the proposed SM approach is naturally extendable to other BIOSM channels. The performance of the rate-compatible schemes designed based on our systematic method is comparable to those designed by optimization.**

I. Introduction

Low-Density Parity-Check (LDPC) codes have received much attention in the past decade. During this period there have been great achievements in the area of designing LDPC code ensembles with Belief Propagation (BP) decoding which exhibit an asymptotic performance practically close to the capacity over different types of channels, including the general class of Binary-Input Output-Symmetric Memoryless (BIOSM) channels [1]-[10]. In particular, for the Binary Erasure Channel (BEC), the performance analysis and code design have been addressed in both the asymptotic regime [3]-[8] and for finite block lengths [1], [2]. In [3], [4], [5], Shokrollahi *et al.* proposed a scheme to design sequences of LDPC code ensembles over the BEC, whose performance is proved to achieve the capacity for sufficiently large average check and variable node degrees. A more general category of capacity achieving sequences over the BEC were proposed in [11], [12], [13]. Construction and analysis of capacity achieving ensemble sequences of codes defined on graphs has also been studied in [6], [7], [8] for the BEC. A sequence of degree distributions with rate R is said to be *capacity achieving* over the BEC if the thresholds of the ensembles can be made arbitrarily close to $1 - R$, the capacity upper bound over the BEC, as the average check and variable node degrees tend to infinity. For BIOSM channels, it is easier to consider ensembles for a given channel parameter instead of a given rate. The results however are easily extendable to the case of fixed rate ensembles. We call a sequence of degree distributions capacity achieving over a BIOSM channel, if the rate of the ensembles within the sequence can be made arbitrarily close to the channel capacity while maintaining the reliable communication. The design of provably capacity achieving sequences over general BIOSM channels is still an open problem.

Another important problem of interest in LDPC codes is to design rate-compatible LDPC code schemes. In such a scheme, starting from a given primary ensemble called the *parent code*, we are interested in obtaining a set of codes with higher transmission rates, which can provide reliable transmission when the channel condition improves, by puncturing the parent code. For rate-compatibility, the design should be such that for two consecutive rates, the code with the higher rate can be constructed by puncturing the code with the lower rate. Starting from a parent code with performance close to capacity, the important challenge in a rate-compatible design is to also keep the performance of the punctured codes close to the capacity. More specifically, if the parent code is chosen from a capacity achieving sequence, all punctured codes have to be capacity achieving as average check node degree increases. To formulate the problem mathematically, imagine a parent code with rate R^n from a capacity achieving sequence which can

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provide reliable transmission over a channel with parameter θ^0. Our aim is to provide reliable transmission over a set of channels with parameters θ^j, $j = 1, ..., J$, while increasing the rate by puncturing the parent code in a rate-compatible fashion. For each θ^j, $j = 1, ..., J$, we need to choose a puncturing pattern that maximizes the corresponding reliable transmission rate $R^{n,j}$. Let $c(\theta^j)$ denote the capacity of the channel with parameter θ^j, and assume that $\theta^i < \theta^j$ and $c(\theta^i) > c(\theta^j)$ for $i > j$. We call a rate-compatible scheme *universally capacity achieving*, if $\lim_{n\to\infty} R^{n,j} = c(\theta^j)$ for $j = 1, .., J$. Analysis and design of rate-compatible LDPC codes have been addressed asymptotically in [14]-[18] and for finite block lengths in [19]-[22]. It is worth mentioning here that Raptor codes [23] can also achieve the capacity of the BEC at several rates but in a different framework than puncturing.

Unlike the BEC for which almost all aspects of conventional and rate-compatible codes have been analytically investigated, for the general family of BIOSM channels, the contributions are mostly based on numerical methods and optimization. This usually provides little insight into the design method. In this respect, a fundamental open problem is to prove the existence of capacity achieving sequences of LDPC codes with BP decoding over BIOSM channels as well as to systematically construct such sequences. This can be seen as a sub-problem as well as a building block for the more general problem of designing universally capacity achieving rate-compatible LDPC coding schemes. The analytical results on this topic are very limited in the literature. In [9], it has been shown that capacity approaching LDPC codes over BIOSM channels can be designed using optimization[1]. Several important analytical properties including the so-called stability condition have been proven for BIOSM channels in [9], [10]. It has been shown in [24] that for an ensemble with a rate close to capacity, variable node and check node Generalized Extrinsic Information Transfer (GEXIT) curves have to satisfy a so-called matching condition. In [25], several bounds have been derived for LDPC codes that are universally valid over any BIOSM channel.

It has been shown in [14], [16] that there is an upper bound on the puncturing ratio of LDPC codes over BIOSM channels, above which the code can not provide reliable transmission for any channel parameter. Moreover, it has been shown that over the BEC, the random puncturing maintains the ratio of rate to capacity at the same value as that of the parent code. Several important bounds on the performance of punctured LDPC codes have been derived in [18]. For the case of maximum-likelihood decoding, capacity achieving codes have been designed based on puncturing in [17]. Among important results on the optimization-based design of rate-compatible codes over BIOSM channels, we can mention [15] for the asymptotic regime and [19], [20], [21] for finite block lengths.

In this paper, we systematically design sequences of universally capacity approaching rate-compatible LDPC code ensembles over BIOSM channels. We then provide some evidence suggesting that the designed sequences could in fact be universally capacity achieving. Starting from the conventional (unpunctured) case, we extend some of the properties of capacity achieving sequences over the BEC [11], to BIOSM channels. Among such properties, only the stability condition [9] has been shown to be extendable to BIOSM channels other than the BEC. We will analyze the case where the stability condition is satisfied with equality, i.e; the fraction of degree 2 edges (λ_2) is set equal to its upper bound, and show that this imposes an upper bound on the fraction of degree 3 edges (λ_3). Using a similar approach for the other degrees, we propose *Successive Maximization* (SM) of λ_i values as a systematic approach to design a sequence of LDPC code ensembles with performance approaching the capacity as the average check node degree increases. We then conjecture that such sequences might in fact be capacity achieving over BIOSM channels. For the rate-compatible LDPC codes on BIOSM channels, we first prove a property similar to the stability condition [9]. We show that for a given parent code (which provides reliable transmission over a channel with parameter θ), there is an upper bound for the fraction of punctured degree-2 variable nodes (Π_2) above which the probability of error of the punctured code is bounded away from zero and below which the probability of error tends to zero if it is made sufficiently small. We then consider the special case of the BEC and show that similar upper bounds can be obtained for variable nodes of all degrees in addition to degree-2 nodes. Using such upper bounds, we prove that applying the SM principle results in a universally capacity achieving rate-compatible scheme over the BEC. Moreover, for such a scheme, if puncturing fractions $\Pi_i^{n,j}$ are used to puncture the parent sequence (λ^n, ρ^n) over the channel with parameter θ^j, where i is the variable node degree, the values of $\Pi_i^{n,j}$ are independent of n. This result is consistent with the one obtained in [16], [14] based on a completely different approach. We then extend a weaker version of the results on the BEC to general BIOSM channels. Assuming that similar upper bounds on the puncturing ratios of other variable nodes (in addition to the upper bound on Π_2) exist, we show that the SM principle can be applied to puncturing fractions of variable nodes to systematically design a coding scheme whose performance universally approaches the capacity in a rate-compatible fashion. This proposes a significantly different approach than the existing optimization-based methods in the literature. Our numerical results indicate that if the parent ensemble is chosen from the capacity approaching sequences designed based on the SM principle, the performance of the resulting rate-compatible schemes is similar to that of the existing optimization-based results in the literature. Moreover, we show that for a sequence of parent code ensembles (λ^n, ρ^n) designed based on the SM principle, the values of puncturing fractions $\Pi_i^{n,j}$ for degree 2 variable nodes ($i = 2$) are independent from the parent ensemble (n) and only depend on the original channel parameter (θ^0) and the one for which the puncturing pattern is designed (θ^j). Our numerical results suggest that this property may in fact hold for other values of i. The importance of this property is that

[1]We distinguish between "capacity approaching" and "capacity achieving" sequences. The former term is used when the performance of the ensemble sequence can be shown (probably numerically) to approach capacity without any guarantee to achieve it. The latter term is used if the performance provably tends to capacity as the average node degrees tend to infinity.

for a given channel parameter θ^j, the computed values of Π_i can universally be applied to any ensemble designed based on the SM method for a given original channel parameter θ^0 with an arbitrary check node distribution.

The paper organization is as follows. The next section is devoted to notations and some definitions. In Section III, after a short review on the construction methods of capacity achieving sequences over the BEC, we explain our approach (Successive Maximization) to devise capacity approaching sequences for other channels. In Section IV, we focuss on the puncturing of a given ensemble within a sequence that is designed based on the methodology of previous sections. We also provide some properties of rate-compatible codes for the BEC and BIOSM channels. Moreover, we show that a similar SM principle to that of Sections III can be used to devise a universally capacity approaching rate-compatible scheme. In Section V, we show examples of our designs and Section VI concludes the paper. The proof of theorems have not been presented due to lack of space and can be found in [26].

II. Definitions and Notations

In this section we present some definitions and properties which will be frequently used throughout the paper. We mainly follow the notations and definitions of [10], [16]. As our focus is on symmetric channels and a BP decoder, throughout the paper, without loss of generality, we assume that the all-one code word is transmitted. Moreover, we assume that the messages in the BP algorithm are in the log-likelihood ratio domain. We represent a (λ, ρ) LDPC code ensemble with its edge-based check and variable node degree distributions as $\rho(x) = \sum_{i=2}^{D_c} \rho_i x^{i-1}$ and $\lambda(x) = \sum_{i=2}^{D_v} \lambda_i x^{i-1}$, with constraints

$$\sum_{i=2}^{D_c} \rho_i = 1 \text{ and } \sum_{i=2}^{D_v} \lambda_i = 1, \tag{1}$$

where the coefficient of x^i represents the fraction of edges connected to the nodes of degree $i+1$, and D_v and D_c represent the maximum variable node degree and the maximum check node degree, respectively. Average check node and variable node degrees are given by: $\overline{d}_c = 1/(\sum_{i=2}^{D_c} \rho_i/i)$ and $\overline{d}_v = 1/(\sum_{i=2}^{D_v} \lambda_i/i)$, respectively. The code rate R satisfies

$$R = 1 - \overline{d}_v/\overline{d}_c. \tag{2}$$

We also define node-based degree distributions as $\overline{\rho}(x) = \sum_{i=2}^{D_c} \overline{\rho_i} x^{i-1}$ and $\overline{\lambda}(x) = \sum_{i=2}^{D_v} \overline{\lambda}_i x^{i-1}$, with constraints

$$\sum_{i=2}^{D_c} \overline{\rho_i} = 1 \text{ and } \sum_{i=2}^{D_v} \overline{\lambda_i} = 1, \tag{3}$$

where the coefficient of x^i represents the fraction of nodes having degree $i+1$. We represent a BIOSM channel with parameter θ by $C(\theta)$ and define $c(\theta)$ as the Shannon capacity of that channel. We also assume that the channel is physically degraded when θ increases. For a sequence of degree distributions $(\lambda^n(x), \rho^n(x))$, λ_i^n and ρ_i^n indicate the ith coefficient of the nth member of the sequence for variable node and check node degree distributions, respectively. Similar to [5], we limit ourselves to check node degree distributions for which T_i's, the Taylor series expansion coefficients of $1-\rho^{-1}(1-x)$ around $x=0$, are positive. For example, check regular ensembles exhibit such a property.

Consider now the density evolution in the belief propagation algorithm for the channel $C(\theta)$ where we track the evolution of the initial channel density P_0 throughout iterations. Based on [9], [10], Q_l, the outgoing density from check nodes at iteration l can be written as

$$Q_l = \Gamma^{-1}\rho(\Gamma(P_{l-1})), \tag{4}$$

where P_{l-1} is the density from iteration $l-1$ entering the check nodes and Γ is the check node operator defined in [9], [10]. Also, P_l the outgoing density from variable nodes at iteration l can be written as

$$P_l = P_0 \otimes \lambda(Q_l), \tag{5}$$

where $\otimes$ is the convolution operation, and the power of a density in variable node and check node degree distributions has been defined in [10]. Note that there is a one-to-one mapping between density P_0 and parameter θ. We now review the following important definitions and theorems from [9], [10]. Let P be a symmetric density (as defined in [9]). For such a density we define $\mathbb{P}(P)$ and $\mathfrak{P}(P)$ as:

$$\mathbb{P}(P) = 0.5 \int_{-\infty}^{\infty} P(x) e^{-(|x/2|+x/2)} dx, \tag{6}$$

and

$$\mathfrak{P}(P) = \int_{-\infty}^{\infty} P(x) e^{-(x/2)} dx. \tag{7}$$

The first integral is the probability that the corresponding random variable is negative. The unusual form of the integral makes it possible to take care of the impulse densities at zero. The second integral is the Bhattacharyya constant. For any given density P, the Bhattacharyya constant tends to zero if and only if (iff) $\mathbb{P}(P)$ tends to zero. Let $p_l = \mathfrak{P}(P_l)$ and $q_l = \mathfrak{P}(Q_l)$. Corresponding to (4) and (5), we then have the following relationships [10]:

$$q_l \leq 1 - \rho(1 - p_{l-1}), \tag{8}$$

$$p_l = \mathfrak{P}(P_0)\lambda(q_l). \tag{9}$$

From (8) and (9) we can see that:

$$p_l \leq \mathfrak{P}(P_0)\lambda(1 - \rho(1 - p_{l-1})). \tag{10}$$

It is important to note that for the BEC, (8) and (10) are satisfied with equality. Moreover, $\mathfrak{P}(P_0)$ is equal to the average erasure probability for the BEC.

The stability of an ensemble is defined as follows [9]. A given degree distribution (λ, ρ) is stable iff there exists $\xi > 0$ such that if $\mathbb{P}(P_l) < \xi$ then $\lim_{l\to\infty} \mathbb{P}(P_l) = 0$. In that respect, it is proven in [9], [10] that if $\lambda'(0)\rho'(1) > 1/\mathfrak{P}(P_0)$ then $\mathbb{P}(P_l)$ is bounded away from zero for every l and if $\lambda'(0)\rho'(1) < 1/\mathfrak{P}(P_0)$, then the ensemble is stable.

We call an ensemble (λ, ρ) convergent over $C(\theta)$, if starting from the initial density P_0, $\lim_{l\to\infty} \mathbb{P}(P_l) = 0$. The threshold of an ensemble over $C(\theta)$ is the supremum value of θ for which the ensemble is convergent.

Consider now the $(k+2)$-tuple $(\lambda_2, \lambda_3, ..., \lambda_k, D_v, \rho(x); \theta)$ which corresponds to a degree distribution $(\lambda(x), \rho(x)) = (\sum_{i=2}^{k} \lambda_i x^{i-1} + (1-\sum_{i=2}^{k} \lambda_i) x^{Dv-1}, \rho(x))$ over $C(\theta)$ where $D_v > k$, and $0 \leq \lambda_i < 1, \forall i \in \{2, ..., k, D_v\}$. We call this setting a *code-channel pair*. With slight negligence, we call a code-channel pair convergent if the ensemble is convergent over the channel.

III. Capacity achieving sequences

For the moment, consider the case of the BEC. We first recall that any ensemble sequence designed in [3], [4], [5] consists of a set of ensembles with a fixed rate R. Then it is proven that the thresholds associated to such ensembles can be made arbitrarily close to $\theta = c^{-1}(R)$. In this paper for the sake of simplicity, we consider a slightly different case where the channel parameter θ is fixed and we design ensembles to have a variable rate R_n which can be made arbitrarily close to $c(\theta)$. Consequently, the definition of capacity achieving sequences can be extended to the case of fixed θ. More specifically, a sequence of degree distributions (λ^n, ρ^n) is called capacity achieving over a BIOSM channel $C(\theta)$, if the corresponding ensembles are convergent over $C(\theta)$ and if their rates R_n, can be made arbitrarily close to $c(\theta)$ for sufficiently large average check node degrees as n tends to infinity. Concentrating again on the BEC, note that the channel parameter θ is the same as the channel probability of erasure.

The derivation of capacity achieving sequences for the BEC proposed in [3], [4], [5] is based on the *flatness condition*. Based on [3], if a sequence of LDPC code ensembles satisfies the conditions below, it can be proved to be capacity achieving.

$$\frac{d^i}{dx^i}[\theta\lambda^n(1-\rho^n(1-x))-x]|_{x=0} = 0, 1 \leq i \leq D_v - 2, \quad (11)$$

where D_v, the maximum variable node degree, is determined by θ and $\rho(x)$ such that (see [11], [12]):

$$\sum_{i=2}^{Dv-1} T_i \leq \theta < \sum_{i=2}^{Dv} T_i. \quad (12)$$

If we apply the flatness condition, we have:

$$\lambda_i^n = T_i/\theta = \lambda_i^{n*}, 2 \leq i \leq Dv-1, \quad (13)$$

where we define[2]

$$\lambda_i^{n*} \triangleq T_i/\mathfrak{P}(P_0). \quad (14)$$

One can easily verify that $\lambda_2^{n*} = 1/(\mathfrak{P}(P_0)\rho'(1))$. In other words, the value of λ_2^n has been set to its maximum value dictated by the stability condition. In [11], [12], [13], we presented an alternate approach for the design of capacity achieving sequences over the BEC. In this method, the values of λ_i^n are computed based on the following principle: Starting from $i = 2$, set the value of λ_i^n to a maximum value $\widetilde{\lambda_i^n}$ such that the ensemble remains convergent for sufficiently large D_v. It is shown in [12] that the values of λ_i^n computed based on this principle are the same as those derived based on (11). In other words, for the BEC, we have $\widetilde{\lambda_i^n} = \lambda_i^{n*}$. To have an intuition of why such a process results in good ensembles whose rates achieve the capacity in the limit, note that based on (2), maximizing the rate is equivalent to maximizing $\overline{d_v}^{-1}$. Based on the definition of $\overline{d_v}^{-1}$, this implies that we should assign higher percentages to the lower degree coefficients as far as the constructed ensemble remains convergent. We remind the reader that the structure of the ensembles proposed by Shokrollahi is in such a way that all variable node degrees from 2 to D_v have to be present. In [11] a super-set of such sequences has been proposed which includes Shokrollahi's sequences as a special case. It is shown in [11] that if each ensemble within the sequence only contains all variable node degrees from degree 2 to $k < D_v$ and degree D_v, where k is a strictly increasing function of D_v (and ultimately a function of $\rho(x)$ and θ), the sequence comprising such ensembles is also capacity achieving[3]. In this paper, we deal with such sequences which are more general.

For the case of BIOSM channels, the flatness condition can not be defined similar to the BEC case as the density evolution equation is not in polynomial form anymore. We, however, expect that applying the new interpretation of flatness condition, i.e., obtaining a sequence of upper bounds $\widetilde{\lambda_i^n}$ and setting $\lambda_i^n = \widetilde{\lambda_i^n}$, may result in a systematic approach for devising capacity achieving sequences for other BIOSM channels. Note that such upper bounds have to fulfill a threshold property similar to that of λ_i^{n*} over the BEC as follows: If we set $\lambda_i^n = \widetilde{\lambda_i^n}$ for $i < k$, the value of $\widetilde{\lambda_k^n}$ has to be in such a way that while the probability of error is bounded away from zero for any ensemble with $\lambda_k^n > \widetilde{\lambda_k^n}$, for any ensemble with $\lambda_k^n < \widetilde{\lambda_k^n}$, the probability of error has to tend to zero as the number of iterations tends to infinity, if it is made sufficiently small, regardless of the value of other λ_i^n's ($i > k$). Unlike the BEC case, there is no proof that such upper bounds exist for other BIOSM channels (with the exception of the bound on λ_2), and even if they do exist, their values may not be easily obtained analytically (except for $i = 2$, where we have $\widetilde{\lambda_2^n} = \lambda_2^{n*}$). In the next section, we prove the existence of a *positive* upper bound on λ_3 that fulfils the aforementioned properties and conjecture that similar upper bounds exist for other λ_i values. This makes it possible to apply the SM principle as a design tool for ensemble sequences. Numerical evidence presented in Section V confirms that the resulting sequences are at least capacity approaching and may in fact be capacity achieving.

Recall that the stability condition theorem in [9], [10] remains silent about the case where $\lambda'(0)\rho'(1)$ is exactly equal to $1/\mathfrak{P}(P_0)$. Here, we show that when this is the case, a similar upper bound exists for λ_3, i.e., if $\lambda_2 = 1/(\mathfrak{P}(P_0)\rho'(1))$, there exists a threshold value for λ_3 below which the ensemble is convergent and above which it is not.

Theorem 1: Consider the code-channel pair $(\lambda_2^*, \lambda_3, D_v, \rho(x); \theta)$ where D_v can be arbitrarily large

[2] Note that the definition of (14) will be used for any type of BIOSM channel throughout the paper.

[3] Shokrollahi's sequences correspond to those of [11] with $k(D_v) = D_v - 1$.

and let $\lambda_3^U = 3/(\overline{d_c}(1-c(\theta))) - (3/2)\lambda_2$. There exists a threshold value $\widetilde{\lambda_3}$ in the interval $[\lambda_3^* , \lambda_3^U]$ such that if $\lambda_3 < \widetilde{\lambda_3}$, the ensemble is convergent for sufficiently large value of D_v and if $\lambda_3 > \widetilde{\lambda_3}$, the probability of error is bounded away from zero regardless of the value of D_v.

We expect the result of Theorem 1 to be generalized to $\lambda_k, k > 3$, if $\lambda_i = \widetilde{\lambda_i}, 2 \le i \le k-1$. This, however, remains to be proved.

IV. UNIVERSALLY CAPACITY APPROACHING RATE-COMPATIBLE LDPC CODES

In previous sections, we considered sequences of degree distributions (λ^n, ρ^n) and our intention was to design them such that their rates approach the capacity as n tends to infinity. In this section, we consider the problem of puncturing a degree distribution for a given n. For simplicity, we sometime drop the index n and refer to the ensemble as the *parent* ensemble. We use the notations (λ^p, ρ) and R^p for the parent ensemble and its rate, respectively. We show the fraction of the punctured bits (variable nodes) by Π. The resulting code rate in this case is equal to $R^p/(1-\Pi)$. If the puncturing is performed randomly, we refer to it as *random puncturing*. Otherwise, the puncturing is called *intentional* [15]. In intentional puncturing, variable nodes of degree i can potentially have different puncturing fractions Π_i. The overall puncturing fraction Π can then be expressed as

$$\Pi = \sum_{2}^{Dv} \Pi_i \overline{\lambda}_i^p \tag{15}$$

where $\{\overline{\lambda_i}^p\}$ is the node-based degree distribution of variable nodes for the parent ensemble.

In many situations, it is necessary to obtain more than one rate by puncturing. In this case, for a simple implementation, the puncturing pattern should be in such a way that for 2 consecutive rates, the punctured code with a higher rate can be constructed by puncturing the code with the lower rate. A puncturing pattern with this property is called *rate-compatible*. Let the set of channel parameters θ^j be ordered reversely by channel degradation (i.e., θ^0 is for the worst channel condition which corresponds to the parent code). For any $C(\theta^j)$, consider the set $\Phi^j = \{\Pi_i^j, 2 \le\ i \le D_v\}$.[4] For a rate-compatible scheme, we must have $\Pi_i^m \le \Pi_i^n$ for any $m < n$ and any i. In the rest of the paper, with slight abuse of language, we call a puncturing scheme rate-compatible if these conditions are satisfied.

To analyze the asymptotic behavior of a punctured ensemble, we model the puncturing of LDPC codes over a channel $C(\theta)$ as the transmission of the unpunctured bits over $C(\theta)$ while sending the punctured bits on an erasure channel with erasure probability of 1. Let $\mathcal{E}$ be the set of all edges in the graph. Also let $\mathcal{E}_i^{punc}$ be the set of edges in the graph which are connected to the variable nodes of degree i which are punctured. Also let $\mathcal{E}^{punc}$ be the union of sets $\mathcal{E}_i^{punc}$. Similarly define $\mathcal{E}^{un}$ and $\mathcal{E}_i^{un}$ for unpunctured edges. We define

$$\lambda^{punc}(x) = \sum \lambda_i^{punc} x^{i-1},$$

where

$$\lambda_i^{punc} = \frac{|\mathcal{E}_i^{punc}|}{|\mathcal{E}^{punc}|}.$$

Notation $|.|$ denotes the cardinality of the set. We also define φ^{punc} as the fraction of punctured edges:

$$\varphi^{punc} = \frac{|\mathcal{E}^{punc}|}{|\mathcal{E}|}.$$

The polynomial $\lambda^{un}(x)$ and φ^{un} can be defined similarly for unpunctured edges.

We can now derive the density evolution equations for our setting. Similar to the previous section, let Q_l be the probability density function of outgoing message of the check nodes at iteration l. We define P_l^{punc} and P_l^{un} as the density at the output of the punctured and unpunctured variable nodes, respectively. We then have

$$\begin{aligned} P_l^{punc} &= P_o^{punc} \otimes \lambda^{punc}(Q_l), \\ P_l^{un} &= P_o^{un} \otimes \lambda^{un}(Q_l), \\ P_l' &= \varphi^{punc} P_l^{punc} + \varphi^{un} P_l^{un}, \\ Q_l &= \Gamma^{-1}(\rho(\Gamma(P_{l-1}'))), \end{aligned} \tag{16}$$

in which $P_0^{punc} = \Delta_0$ where Δ_x is the Dirac delta function at x [10]. A punctured scheme is convergent if the probability of error tends to zero as the number of iterations tends to infinity. Consider a sequence of degree distributions $(\lambda^n(x), \rho^n(x))$. Consider also a set of channels with parameters $\theta^j, j = 0, 1, ..., J$, ordered increasingly by their quality. Now assume that the parent ensemble sequence $(\lambda^n(x), \rho^n(x))$ is punctured by the set $\phi^{n,j} = \{\Pi_i^{n,j}, 2 \le i \le D_v\}$ to create higher rate ensemble sequences that are convergent over the corresponding channels. This scheme is *universally capacity achieving* if $\lim_{n\to\infty} R^{n,j} = c(\theta^j)$ for all values of j. A universally capacity achieving scheme is called rate-compatible if the puncturing patterns $\phi^{n,j}$ are rate-compatible for every value of n.

We now prove an important theorem in puncturing a given degree distribution. Consider a parent ensemble (λ^p, ρ) with threshold equal to θ^0. The code-channel pair $(\lambda^p, \rho; \theta)$ is convergent for any $\theta \le \theta^0$. Let P_0 be the channel density associated with θ. We define parameter $\widetilde{\Pi_2}$, corresponding to the parent code-channel pair, as:

$$\widetilde{\Pi_2} = \frac{[1 - \mathfrak{P}(P_0)\rho'(1)\lambda_2^p]}{[1 - \mathfrak{P}(P_0)]\rho'(1)\lambda_2^p}. \tag{17}$$

Note that if the pair is stable, $\widetilde{\Pi_2} \ge 0$.

Theorem 2: Let (λ^p, ρ) be a parent code convergent over $C(\theta)$ with $\lambda_2^p \ne 0$. Suppose that this code is punctured based on the set $\Phi = \{\Pi_i; i = 2, ..D_v\}$ (note that $C(\theta)$ is uniquely associated with the channel density P_0). There exists a threshold value $\widetilde{\Pi_2}$, given by (17), such that if $\Pi_2 > \widetilde{\Pi_2}$, then for any l, $\mathbb{P}(P_l^{punc})$ and $\mathbb{P}(P_l^{un})$ are bounded away from zero and if $\Pi_2 < \widetilde{\Pi_2}$, there exists a strictly positive constant ξ such that if $\mathbb{P}(P_l^{punc}), \mathbb{P}(P_l^{un}) < \xi$ for some l, then $\lim_{l\to\infty} \mathbb{P}(P_l^{punc}) = 0$ and $\lim_{l\to\infty} \mathbb{P}(P_l^{un}) = 0$.

[4]For $\lambda_i = 0$, we assume $\Pi_i = 0$.

This property is similar to the stability condition [9] for conventional LDPC codes which provides an upper bound on the fraction of degree 2 variable nodes.

Corollary 1 (Independency of Π_2 from n for puncturing schemes with $\Pi_2 = \widetilde{\Pi_2}$): Consider a sequence of ensembles $(\lambda^n(x), \rho^n(x))$ which are convergent over $C(\theta^0)$ and let P_0^0 be the associated channel density. Now consider an improved channel $C(\theta^j)$, $j > 0$ and let P_0^j be the associated channel density. If for any ensemble within the sequence, the value of λ_2 satisfies the stability condition corresponding to θ^0 with equality, i.e., if $\lambda_2^n = \lambda_2^{n*}$, then the value of the upper bound $\widetilde{\Pi_2}$ corresponding to θ^j obtained in Theorem 2, is independent of n (in fact independent from the parent ensemble sequence (λ^n, ρ^n)).

Corollary 2 (Rate-compatibility of Π_2 for puncturing schemes with $\Pi_2 = \widetilde{\Pi_2}$): Consider a sequence of ensembles $(\lambda^n(x), \rho^n(x))$ which are convergent over $C(\theta^0)$ and let P_0^0 be the associated channel density. Now consider an improved channel $C(\theta^j)$, $j \geq 0$ and let P_0^j be the associated channel density. If for any ensemble within the sequence, the value of λ_2 satisfies the stability condition corresponding to θ^0 with equality, i.e., if $\lambda_2^n = \lambda_2^{n*}$, then the value of the upper bound $\widetilde{\Pi_2}$ is a decreasing function of θ^j.

Now recall from Section III that over the BEC, similar upper bounds to that of stability condition were obtained for other variable node degrees. In the following, we show a similar behavior for the case of rate-compatible codes over the BEC.

Proposition 1: Let (λ^p, ρ) be a convergent parent code over the BEC with channel parameter ϵ^0. Suppose that the parent code is punctured to be used over a channel with parameter $\epsilon^j < \epsilon^0$. Let p_0^{un} be the Bhattacharyya constant for this channel, i.e., $p_0^{un} = \epsilon^j$. Also assume that $\lambda_i^p \neq 0, 2 \leq i \leq n \leq D_v$. Define

$$\Pi_i^* = \frac{1 - p_0^{un}\lambda_i^p/T_i}{(1 - p_0^{un})\lambda_i^p/T_i}. \tag{18}$$

Then if $\Pi_i = \Pi_i^*$ for $2 \leq i < n$, there exists an upper bound $\Pi_n^* = \frac{1-p_0^{un}\lambda_n^p/T_n}{(1-p_0^{un})\lambda_n^p/T_n}$ on Π_n above which the resulting punctured ensemble is not convergent over $C(\epsilon^j)$ and below which the ensemble is convergent over $C(\epsilon^j)$ if the probability of erasure can be made sufficiently small.

We now would like to prove that the construction of universally capacity achieving rate-compatible LDPC codes over the BEC can be achieved by applying the SM principle to the values of Π_i's, i.e., starting from a parent sequence and for each ensemble member of the sequence, we maximize Π_2 as far as the ensemble remains convergent and continue this procedure successively for other Π_i values. This will be performed for each of the J target channel parameters and we demonstrate that the resulting puncturing patterns are in fact rate-compatible. We also show that if the original parent sequence is capacity achieving, so will be all the J sequences of punctured ensembles.

Theorem 3: Let the parent ensemble sequence (λ^n, ρ^n), constructed based on the method of [11], be capacity achieving over the BEC with parameter ϵ^0. For the set of channel erasure values ϵ^j ($\epsilon^1 > \epsilon^2 > ... > \epsilon^J$), we puncture each ensemble within the parent sequence based on the SM principle. The resulting scheme is then universally capacity achieving rate-compatible.

This result is consistent with the one obtained in [16] stating that random puncturing of a parent ensemble over the BEC, preserves the distance to capacity. The approach taken in [16] is, however, different and is based on the fact that one can model the puncturing of an ensemble over the BEC, as the concatenation of the original BEC channel with another BEC channel with erasure rate equal to puncturing. Similar to the flatness condition, the approach of [16] is not extendable to other BIOSM channels. The importance of our approach is that in principle, it may be extendable to other BIOSM channels where we can expect that applying the SM principle to compute Π_i values, might also result in (a scheme performing close to) a universally capacity achieving rate-compatible scheme. Unlike the BEC case, however, the upper bounds on Π_i have to be estimated numerically (similar to the procedure we use to compute the upper bounds of $\lambda_i; i > 2$, for the unpunctured case) except for Π_2 whose upper bound is given by Theorem 2. Applying this procedure to the capacity approaching ensembles designed based on the method of Section III as parent ensembles, we have in-fact been able to design universally capacity approaching rate-compatible ensembles over other BIOSM channels.

It is important to note that the values of $\Pi_i^{j,n}$ in Theorem 3 do not depend on i and n. While independency of i is a special property for the BEC, based on Corollary 1 these values are independent from n for $i = 2$ over any BIOSM channel. Our numerical results show that for a given $i > 2, i \neq D_v$ and j, the values of $\Pi_i^{j,n}$ are very close for different values of n, suggesting a general independency from n.

V. Design examples

We consider check regular sequence with $\rho(x) = x^{Dc-1}$ for channel parameter $\sigma = .9557$. Let k be the number of constituent variable node degrees. We start with $D_c = 5$ and $k = 3$, and for $D_c > 5$, we set $k = 2^{Dc-6} + 2$. This means that the number of constituent variable node degrees for an ensemble with check node degree D_c is roughly twice that of an ensemble with check node degree $D_c - 1$. As can be seen in Table I, the performance of the ensembles consistently improves as the average check node degree increases. For the rate-compatible codes, we consider the sequence of Table I and puncture the first three ensembles for a set of four channels with noise powers smaller than that of the parent ensemble. The details are provided in Table II, where we define the puncturing polynomial $\Pi(x) = \sum_{i=2}^{Dv} \Pi_i x^{i-1}$ to represent the puncturing fractions. In Fig. 1, we have plotted the distance to capacity $(R^j/c(\theta^j))$ of the ensembles of Table II versus

TABLE I
PERFORMANCE OF A CHECK REGULAR SEQUENCE DESIGNED BASED ON THE SM METHOD OVER A BIAWGN CHANNEL WITH $\sigma = .9557$.

D_c	$R_{AWGN}/c(.9557)$	$\aleph$
5	.8902	3
6	.9386	3
7	.9520	4
8	.9653	6
9	.9756	10
10	.9884	18

TABLE II
THE VALUES OF Π_i USED TO PUNCTURE THE FIRST 3 ENSEMBLES OF THE SEQUENCE OF TABLE I.

$\bar{d}_c$	5	6	7
Parent ensemble	$\lambda(x) = .4322x + .3534x^2 + .2144x^5$	$\lambda(x) = .3457x + .2974x^2 + .3569x^6$	$\lambda(x) = .2881x + .2556x^2 + .0380x^3 + .4183x^9$
$\sigma =$ 0.7410	$\Pi(x) = .2948x + .2037x^2 + .2231x^5$	$\Pi(x) = .2948x + .2124x^2 + .2357x^6$	$\Pi(x) = .2948x + .1921x^2 + .3240x^3 + .2862x^9$
$\sigma =$ 0.6300	$\Pi(x) = .4115x + .2697x^2 + .3473x^5$	$\Pi(x) = .4115x + .2755x^2 + .3483x^6$	$\Pi(x) = .4115x + .2590x^2 + .4472x^3 + .3638x^9$
$\sigma =$ 0.5609	$\Pi(x) = .4703x + .2949x^2 + .4676x^5$	$\Pi(x) = .4703x + .2997x^2 + .4142x^5$	$\Pi(x) = .4703x + .2862x^2 + .5063x^3 + .4094x^9$
$\sigma =$ 0.4675	$\Pi(x) = .5308x + .3026x^2 + .7460x^5$	$\Pi(x) = .5308x + .3065x^2 + .5015x^5$	$\Pi(x) = .5308x + .2997x^2 + .5772x^3 + .5950x^9$

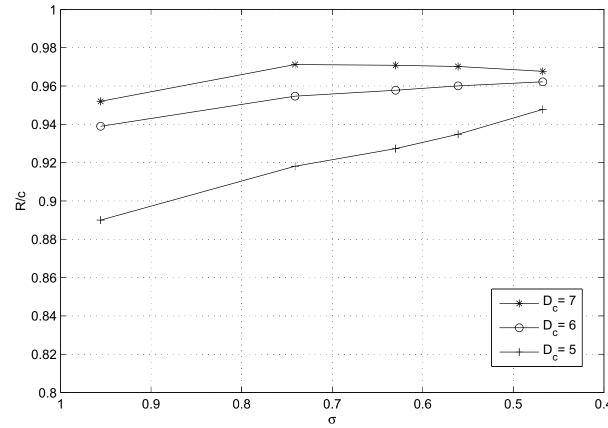

Fig. 1. The ratio of rate to capacity (R/C) for the rate-compatible ensemble sequence of Table II.

TABLE III
THE VALUES OF Π_i'S USED TO PUNCTURE THE ENSEMBLE C_{SM}.

$\sigma = 0.7410$	$\Pi(x) = 0.2948x + .1921x^2 + .3376x^3 + .2222x^9$
$\sigma = 0.6300$	$\Pi(x) = 0.4114x + .2600x^2 + .4462x^3 + .3470x^9$
$\sigma = 0.5609$	$\Pi(x) = 0.4703x + .2871x^2 + .5063x^3 + .3628x^9$
$\sigma = 0.4675$	$\Pi(x) = 0.5308x + .3007x^2 + .5820x^3 + .4200x^9$

the channel noise standard deviations (σ). As can be seen, the performance of the punctured codes for a given parent ensemble is similar to or better than the parent ensemble. In fact, we expect the punctured ensemble to perform almost the same as the parent ensemble, similar to the case of the BEC. To justify the improvement resulting from puncturing, we note that although the parent ensembles have been constructed based on the SM method, for finite values of D_c, they are not necessarily optimal in that they may not provide the best possible rate for the given channel parameter. This leaves the door open for further improvement with puncturing. From Fig. 1, it is also observed that for any given channel parameter, the performance of punctured ensembles approaches the capacity as the average check node degree increases. Based on Table II, the designed sequence also fulfills the rate-compatibility property. Note however that unlike, for example the approach of [15], we did not impose any constraint to guarantee rate-compatibility and our empirical results suggest that this property is inherent in the proposed method. For the case of Π_2, we analytically proved this fact in Corollary 2. To compare the performance of the schemes designed based on the SM principle and those obtained by optimization, we consider the ensemble used in [15] as a reference. This ensemble (C) has been optimized for the rate one half and has a threshold of $\sigma = .9557$. We can assume that ensemble C has been optimized for the highest rate when the channel parameter σ is set to .9557. The degree distribution of C is:

$$\lambda_C(x) = .25105x + .30094x^2 + .00104x^3 + .43853x^9$$

$$\rho_C(x) = .63676x^6 + .36324x^7$$

Keeping the check node degree distribution of ensemble C intact, we design an ensemble C_{SM} with the same number of constituent variable nodes using the SM method:

$$\lambda_{SM}(x) = .2717x + .2442x^2 + .0371x^3 + .4471x^9.$$

We then apply the SM method again, this time to puncture C_{SM}. The puncturing polynomials for the same four channels considered in Table II are given in Table III. The distance to capacity (R/C) for the parent ensemble and its punctured versions is reported in Fig. 2. As can be seen in Fig. 2, the scheme performs very closely to the scheme obtained by optimization-based puncturing of the ensemble C. In fact, the proposed scheme even slightly outperforms the scheme of [15] on channels with $\sigma = .6300$ and $\sigma = .7410$. The proposed scheme performs inferior only on the best channel parameter ($\sigma = .4675$) and even for this channel parameter, the performance gap is less than .08 dB.[5] We have also demonstrated the performance of random puncturing of the ensemble C for comparison. Also note again that unlike [15], we did not impose any constraint to guarantee rate-compatibility. This reduces the design complexity significantly.

It is interesting to see that based on Tables II and III, except for $i = D_v$, the values of Π_i are almost independent (for Π_2 provably independent based on Corollary 1) of the parent ensembles and only depend on the channel parameter for which the puncturing is applied. In other words, for a given channel parameter θ^j, the computed values of Π_i can universally be applied to any ensemble designed based on the SM method for a given original channel parameter θ^0.

[5]Note that our parent code itself performs close to .1dB worse than C and the gap in performance is always less than this gap for different puncturing rates.

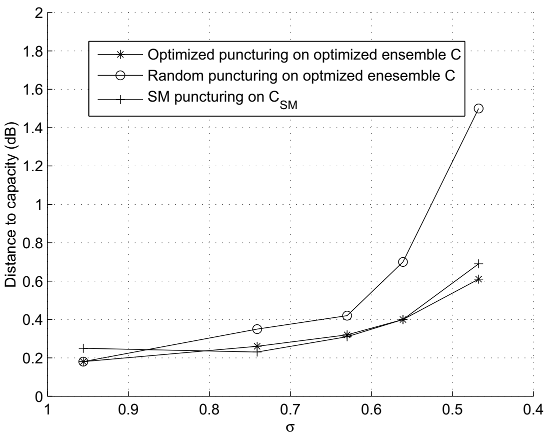

Fig. 2. Performance comparison of schemes constructed based on the proposed SM method and those constructed based on optimization method of [15].

VI. Conclusion

In this paper, we proposed the concept of successive maximization for the systematic design of universally capacity approaching rate-compatible LDPC code ensemble sequences. This was achieved by interpreting the flatness condition over the BEC as a Successive Maximization principle that was generalized to other BIOSM channels to design a sequence of capacity approaching parent ensembles. The SM principle was then applied to each parent ensemble, this time to design rate-compatible puncturing schemes. As part of our results, we were able to extend the stability condition which was previously derived for degree-2 variable nodes to other variable node degrees as well as to the case of rate-compatible codes. Consequently, we rigorously proved that using the SM principle, one is able to design universally capacity achieving rate-compatible LDPC code ensemble sequences over the BEC. Unlike the previous results on such schemes over the BEC in the literature, the proposed SM approach can be naturally extended to other BIOSM channels. Using such an extension, we designed rate-compatible codes over BIAWGN channels whose performance universally approaches the capacity as the average check node degree increases. We demonstrated that for finite values of D_c, the performance of the ensembles designed by our method is comparable to those designed based on optimization. One major step in the continuation of this work is to analytically compute the values of $\widetilde{\lambda}_i$. This can pave the road toward the analytical proof that the proposed sequences can in fact achieve the capacity of BIOSM channels.

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