Solution of Incompressible Flow Problems on Moving Staggered Meshes

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Moving Mesh Applications

Collapse of a Water Filament
(Re=3, We=3)

- Large Deformations
- Critical (Surface) Physics
Moving Mesh Applications

- Fixed Cost Mesh Adaptation
- Capture Critical (Transient) Physics
Moving Mesh Issues

- How to prescribe the mesh motion and local reconnection.
  - Lagrangian
  - Adaptive
  - Quality Preserving

- Geometry is time dependent.

- Conservation statements can apply to entire vectors - not components.
Unstructured Mimetic Methods

- WANT \[ \int_{L} u \cdot dl \]
- HAVE \[ \int_{A} u \cdot n dA \]

Faces to Dual Edges (*fe’)

Lowest order

\[ \int_{L} u \cdot dl = \frac{L}{A} \int_{A} u \cdot n dA \]

Only reasonable for orthogonal dual meshes
Unstructured Mimetic Methods

First order for general dual meshes

Faces to Cells\(^3\) (\(*fc^3\) : reconstruction)

\[
\mathbf{u}_{cell\_CG} V \approx \int_V (u_i + x_i u_{j,j}) dV = \int_V (x_i u_j)_{,j} dV = \sum_{faces} \int_A x_i u_j n_j dA \approx \sum_{faces} \hat{\mathbf{x}}_{\text{face\_CG}} U
\]

Cells\(^3\) to dual edges (\(*c^3e'\) : recovery)

\[
\int_{L_1} \mathbf{u} \cdot d\mathbf{l} = \mathbf{u}_{cell\_cg} \cdot t'_1 = \mathbf{u}_{cell\_cg} \cdot \hat{\mathbf{x}}_{\text{face\_CG}}
\]

\[
u_e = (X^T V^{-1} X) U_f
\]

Symmetric pos. def.
1st order in general.
Moving Staggered Mesh Equations

- Normals, edges, etc are now time dependent.

Reynolds Transport Theorem

\[ \frac{\partial}{\partial t} \int_{CV(t)} dV = \int_{CS(t)} \mathbf{u}_{mesh} \cdot n dA \]

\[ \frac{\partial}{\partial t} \int_{CV(t)} \rho dV + \int_{CS(t)} \rho (\mathbf{u} - \mathbf{u}_{mesh}) \cdot n dA = 0 \]

\[ \frac{\partial}{\partial t} \int_{CV(t)} \rho \mathbf{u} dV + \int_{CS(t)} \rho \mathbf{u} (\mathbf{u} - \mathbf{u}_{mesh}) \cdot n dA = - \int_{CS(t)} p n dA + \int_{CS(t)} \mu \frac{\partial \mathbf{u}}{\partial n} dA \]
**Constraints**

- Consistency requires that the mesh flux exactly satisfy the volume equation
  \[ U_{\text{mesh}} = \mathbf{u}_{\text{face CG}} \cdot \left\{ \frac{1}{2} (\mathbf{n}^{n+1} A^{n+1} + \mathbf{n}^n A^n) - \frac{\Delta t^2}{12} \sum_{\text{edges}} (\mathbf{u}_{\text{mesh n1}} \times \mathbf{u}_{\text{mesh n2}}) \right\} \]

- Divergence constraint satisfied by using a discrete curl
  \[ U = \sum_{\text{edges}} \pm s \]
  \[ s = \int_{\text{edge}} \mathbf{\Psi} \cdot dl \]
Momentum Equation

- Integrate momentum equation over the **Cell** CV.

\[
\frac{\mathbf{u}_c^{n+1}V^{n+1} - \mathbf{u}_c^nV^n}{\Delta t} + \sum_{\text{faces}} \mathbf{u}_f (U - U_{\text{mesh}}) = -V \nabla p + \sum_{\text{faces}} \nu \nabla \mathbf{u}_f \cdot \mathbf{N}_f
\]

\[
\mathbf{u}_c V = \sum_{\text{faces}} \hat{\mathbf{x}}_{\text{face}_C G} U
\]

Evolution equation for integral along the edge

\[
\frac{X^{T^{n+1}}V^{-1}X^{n+1}U^{n+1} - X^{T^{n+1}}V^{-1}X^nU^n}{\Delta t} + X^{T^{n+1}}V^{-1}a_c = -G p^{n+1}
\]

Unknowns at faces
Eqn. on dual edge
Momentum Equation

- Two step symmetric Hodge* operator

\[ s(\text{edges}) \rightarrow^{\text{curl}} U(\text{faces}) \rightarrow^{\text{div}} V \nabla \cdot \mathbf{u}(\text{cells}) = 0 \]

\[ \downarrow^X \]

\[ V \mathbf{u}(\text{cells}) \]

\[ \downarrow^{X^T} \]

\[ w(\text{dual } \text{face})^{\text{curl}} \leftarrow u_e(\text{dual } \text{edges}) \]
Equation System

- Saddle Point Problem

\[
\begin{pmatrix}
\left(\frac{1}{\Delta t} M^{n+1} - X^{n+1} A^{n+1}\right) & G \\
D & 0
\end{pmatrix}
\begin{pmatrix}
U^{n+1} \\
p^{n+1}
\end{pmatrix}
= \begin{pmatrix}
X^{n+1} r^n \\
0
\end{pmatrix}
\]

\[U = Cs\] where \[DC = 0\] and so \[C^T G = 0\]

- Symmetric Pos. Def.

\[C^T \left(\frac{1}{\Delta t} M^{n+1} - X^{n+1} A^{n+1}\right) C_s = C^T X^{n+1} r_c^n\]

- Reduced number of unknowns.
- Easier to solve.
- BCs on U and p
Validation

Fr = 0.7

Sloshing Tank

Acceleration due to gravity

Time period of standing waves (s)

Analytical

Numerical

Time

Displacement
Validation

Droplets hitting a wall

Droplet
Horizontal Velocity

We=2.0  Re=6.6
Validation

With perturbation analyses good to 0.02%
Conclusion

- Conservative moving mesh Mimetic methods are possible.
- Mimetic methods make incompressibility easy to handle.
- It is useful to define vector quantities.

◆ Conjecture: a good Hodge* operator should be sym. pos. def. and factor into the operator that defines vectors.

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