The Relationship between Discrete Calculus Methods and other Mimetic Approaches

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Background

Hardware:
- GPUs, FPGAs, HPC, Algorithms

Numerical Methods:
- Unstructured Staggered mesh methods, Fractional step methods, Discrete Calculus Methods.

Turbulence Modeling:
- Turbulent Potentials, Eddy Collision Model

Applications:
- Wind Turbines, DNS, Super-hydrophobic surfaces, droplets.
Mimetic Methods

FE: Raviart-Thomas/Nedelec/Whitney  
  • Algebraic Topology  
  • Electromagnetics

FV: Staggered Mesh Methods  
  • Many local conservation properties  
  • Fluid Dynamics

FD: Keller Box  
  • Multi-symplectic  
  • Wave Eqns  

FD: SOM Box  
  • Robust  
  • Heat Eqn

NN: Non-Sibsonian Meshless methods  
  • Time-dependent domains  
  • Solid mechanics
Numerical Methods

Finite Difference

Finite Element

Meshless

SOM

Edge/Face

Staggered

Natural Neighbors

All Consistent Numerical Methods

Mimetic Methods
Question

Is there any relationships between the various mimetic methods?

(1) Yes – many (all ?) can be derived as discrete calculus methods.

(2) Yes – they tend to use the same basis functions.
Incompressible Fluid Dynamics

\[
\begin{align*}
  u_{\tilde{e}}^{n+1} - u_{\tilde{e}}^n &= \int_{t^n}^{t^{n+1}} dt \int (\nabla \cdot F) \cdot dl - G \bar{p}_{\tilde{c}} \\
  DU_{f}^{n+1} &= 0
\end{align*}
\]

Need to relate these two

\[
\begin{align*}
  U_{f} &= \int u \cdot ndA \\
  u_{\tilde{e}} &= \int u \cdot dl
\end{align*}
\]
FE Basis Functions

Face Elements
Nédélec/RT/Whitney

\[ u^h (x) = u^0 + \frac{D}{n} \bar{x} \]

Interpolant with continuity of the normal flux

\[ u^h \cdot n^f = u^0 \cdot n^f + \frac{D}{n} L^f \parallel \]

Constant normal velocity on each face

Constant divergence

\[ \nabla \cdot u^h = D \]
Find the constants given the data (4x4)

\[ u^0 \cdot n^f + \frac{L^f_1}{n} D = \frac{1}{A_{f_1}} U_{f_1} \]

\[ u_{\bar{\varepsilon}} = \int u^h \cdot dl \]

Evaluate the integral

The basis function determines the relationship between the two velocities
StagMesh Interpolation

\[
\int (u_{i,i}x_j + u_j) \, dV = \int (u_i x_j) \, dV = \int (x_j u_i) n_i \, dA
\]

\[U_f = \int \mathbf{u} \cdot n \, dA\]

- Gauss’ Theorem (Again)
- Assume constant flux on face
- Assume constant divergence

\[
\bar{\mathbf{u}}_c V = \sum_{\text{faces}} U_f \left( \mathbf{x}_{f}^{cg} - \mathbf{x}_{c}^{cg} \right)
\]

\[
\bar{\mathbf{u}}_c = \frac{1}{V_c} \mathbf{R} \mathbf{U}_f
\]

SM = implicit basis functions
Average Cell velocity

\[ \bar{\mathbf{u}}_c = \frac{1}{V_c} \mathbf{R} \mathbf{U}_f \]

\[ \mathbf{u}_e = \mathbf{R}^T \bar{\mathbf{u}}_c \quad \text{for incompressible} \]

\[ \mathbf{u}_e = (\mathbf{R}^T \frac{1}{V_c} \mathbf{R}) \mathbf{U}_f \]

- Explicit Formula for the same matrix relationship (Hodge*) as FE
- Symmetric
- Generalizable to polyhedra
- The intermediate is a (cell average) velocity vector. (Momentum, KE)
FD Interpolation

Use 3 face values at each vertex
Average vectors to center
Works on (almost) any polygon

- Assumes constant on face
- Assumes constant divergence

Use Least Squares

\[ \mathbf{\bar{v}}_c = \frac{1}{A_f} \mathbf{U}_f \]

- Also same cell velocity
FE Basis Functions

Edge Elements
Nedelec/RT/Witney

\[ \vec{u}^h(x) = \vec{u}^0 + \frac{1}{n-1} \vec{W} \times \vec{X} \]

Interpolant with continuity of the tangential components

Constant tangential velocity on each edge

\[ \vec{u}^h \cdot t^e = \vec{u}^0 + \frac{1}{n-1} (\vec{w} \times \vec{L}^e_\perp) \cdot t^e \]

Constant vorticity

\[ \nabla \times \vec{u}^h = \vec{w} \]
**StagMesh Interpolation**

\[(a \cdot x)u \rightarrow \int (u \times n + xw \cdot n) dA = \int xu \cdot dl\]

\[u_e = \int u \cdot dl\]

- **Stokes’ Theorem**
- Assume constant along edge
- Assume constant vorticity

\[\vec{u}^{cg}_{f} \times \hat{n}_{f} A_{f} = \sum_{edges} u_{e} (\vec{x}^{cg}_{e} - \vec{x}^{cg}_{f})\]

\[\int \nabla \times \vec{u} \vec{X} dV = -\int \vec{u} \times \hat{n} dA\]

SM = Rampant use of Stokes’ Theorem
Summary: Basis Functions

- FE uses Explicit Basis Functions
- SM uses Stokes’ Theorem
  This approach can be applied to arbitrary polygons.
- SOM uses Discrete Inner Products
  highly discontinuous/anisotropic materials and arbitrary polygons

Many approaches to achieving the same underlying interpolation (Hodge *).
Dual mesh is not unique.
FV = top hat
FE = tent functions (unique for Galerkin)
FE / FV relationship

FV = one CV

FE = weighted average of CV

FE Discrete Calculus

Weighted Exact Discretization

\[ \int w(\nabla \cdot u) dV = \int w u \cdot n dA - \int (\nabla w) \cdot u dV = \int u \cdot (-\nabla w) dV \]

Smeared Flux

\[ D^w = \sum_{faces} U^w_f \]

Compare

\[ D = \int \nabla \cdot u dV = \sum_{faces} \int u \cdot n dA = \sum_{faces} U_f \]
FE Exact Discretization

\[ \int \nabla \cdot \mathbf{u} \, dV = \sum_{\text{faces}} \int \mathbf{u} \cdot \mathbf{n} \, dA \]

\[ \int (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA = \sum_{\text{edges}} \int \mathbf{v} \cdot d\mathbf{l} \]

\[ \int \nabla T \cdot d\mathbf{l} = T_2 - T_1 \]

\[ \int w \nabla \cdot \mathbf{u} \, dV = \sum_{\text{cells}} \int \mathbf{u} \cdot (-\nabla w) \, dV \]

\[ \int (\nabla \times \mathbf{v}) \cdot (-\nabla w) \, dV = \sum_{\text{faces}} \int \mathbf{v} \cdot (\mathbf{n} \times \nabla w) \, dA \]

\[ \int \nabla T \cdot (\mathbf{n} \times \nabla w) \, dA = \sum_{\text{edges}} \int T (\nabla w) \cdot d\mathbf{l} \]
Higher-Order Exact Gradient

\[
\int_{\text{edge}} \nabla T \cdot dl = T_{n2} - T_{n1}
\]

\[
\int_{\text{edge}} x \nabla T \cdot dl = (x_{n2}T_{n2} - x_{n1}T_{n1}) - t_e \int_{n1}^{n2} Tdl
\]

\[
\int_{\text{face}} n \times \nabla T dl = \sum_{\text{edges}} t_e \int Tdl
\]

20 outputs / tet

10 inputs / tet
Higher-Order Exact Curl

\[ \int (\nabla \times \mathbf{v}) \cdot n \, dA = \sum_{edges} \int \mathbf{v} \cdot d\mathbf{l} \]

\[ \int \mathbf{x}(\nabla \times \mathbf{v}) \cdot n \, dA = \sum_{edges} \int \mathbf{xv} \cdot d\mathbf{l} + \int \mathbf{n} \times \mathbf{vdA} \]

\[ \int \nabla \times \mathbf{vdV} = \sum_{faces} \int \mathbf{n} \times \mathbf{vdA} \]

\[ 15 \text{ outputs / tet} \]

\[ 20 \text{ inputs / tet} \]

Note: \( C^{[2]} G^{[2]} = 0 \)

Mimetic

\[ \begin{bmatrix}
\int_{face} (\nabla \times \mathbf{v}) \cdot n \, dA \\
\int_{face} \mathbf{x}(\nabla \times \mathbf{v}) \cdot n \, dA \\
\int_{cell} (\nabla \times \mathbf{v}) \, dV
\end{bmatrix} = \begin{bmatrix}
C^{[2]} \\
C^{[2]} \\
C^{[2]}
\end{bmatrix} \]

\[ \begin{bmatrix}
\int_{edge} \mathbf{v} \cdot d\mathbf{l} \\
\int_{edge} \mathbf{xv} \cdot d\mathbf{l} \\
\int_{face} \mathbf{n} \times \mathbf{vdA}
\end{bmatrix} \]
Review

**FE:**
Explicit basis functions
Fixed geometry – precise proofs
Implicit dual mesh
Semi-implicit Hodge*

**StagMesh:**
Implicit basis functions
Arbitrary polygons.
Explicit dual mesh
Explicit Factored Hodge*
Summary

- Underlying assumptions about the solution (basis functions) are often the same.

- The Test Functions are different. Test functions affect the metric (geom).

- Higher Order is achieved by noticing that you can exactly discretize different ways and with different moments.

- Anyone can do it.

www.ecs.umass.edu/mie/tcfd/Publications.html