Numerical Methods

The **Foundation** on which CFD rests.
Revolution

**Math:**
- Accuracy
- Stability
- Convergence
- Consistency

**Physics:**
- Conservation
- Spurious Modes
- Wave propagation
- Maximum/minimum Constraints

Mimetic methods *mimic* the physics.
Relationship

Finite Difference

SOM

Finite Volume

Edge/Face

Natural Neighbors

Meshless

Staggered

Mimetic Methods

All Numerical Methods
Why it Matters

- Finding numerical issues in complex physics is hard.
- Adding new physics is uncertain.
- Measure twice, cut once.
Mimetic Advection \( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = 0 \)

Jason Frank, CWI/Amsterdam
1D: Change in mesh size (3x)

Central
Box Method

Not upwinding, not time stepping.
Mimetic Eigenvalues

Gerritsma, Costabel/Dauge, Arnold*

\[ \nabla \times \nabla \times \mathbf{v} - \nabla (\nabla \cdot \mathbf{v}) = \lambda \mathbf{v} \]

\[ \lambda = m^2 + n^2 \]

1, 1, 2, 4, 4, 5, 5, 8, ...

*Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability
Mimetic Vector Fields

$$\nabla \times \nabla \times \mathbf{v} - \nabla (\nabla \cdot \mathbf{v}) = f$$

Linear FE          Nedelec FE
Mimetic Physics
Do ALL discretization exactly.
This means that the calculus and the physics remain exact.

Numerical approximation only in material laws
Which are engineering approximations already.
Numerical approximation goes with physical approximation.
Exact PDE Discretization

Infinite number of equations
Infinite number of unknowns

How could you possibly make this finite without loosing some information?

Discretization MUST be approximate!

Answer: Collect those infinite unknowns into a finite number of groups.

Use integral quantities as the primary unknowns.
Example  
Shallow Water Equations

\[ \frac{\partial h}{\partial t} + \nabla \cdot (uh) = 0 \]
\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -g \nabla h - 2\Omega \times u - bu \]
Basic Equations

Vertically averaged Navier-Stokes + Hydrostatic

Conservation of Mass

\[ \frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0 \]

Conservation of Momentum

\[ \frac{\partial hu}{\partial t} + \nabla \cdot (uhu) = -\nabla \left( \frac{g}{2} h^2 \right) - 2\Omega \times (hu) - b(hu) \]

Constant Density

No Mesh Motion

No Bottom Topography
Mimetic Discretization

Exact Discretization of Physics

\[ \int_c h dA^{n+1} - \int_c h dA^n + \sum_f \int dt \int_f h u \cdot n dS = 0 \]

\[ \int_{\tilde{e}} h u \cdot d\tilde{l} |^{n+1} - \int_{\tilde{e}} h u \cdot d\tilde{l} |^n + \int dt \int_{\tilde{e}} \nabla \cdot (u q) \cdot d\tilde{l} = - (\int p_{\tilde{v}_2} dt - \int p_{\tilde{v}_1} dt) \]

Cells and time

Dual edges and time

Exact discrete matrix system

\[ H_{c}^{n+1} - H_{c}^{n} + D \overline{Q}_f = 0 \]

\[ U_{\tilde{e}}^{n+1} - U_{\tilde{e}}^{n} + C( u h u ) = - G \left( \frac{g}{2} h_v^2 \right) \]

Numerically Exact

But Not Solvable
Solvable Discretization

**Exact discrete matrix system**

\[ H_{c}^{n+1} - H_{c}^{n} + D\overline{Q}_f = 0 \]

\[ U_{\tilde{e}}^{n+1} - U_{\tilde{e}}^{n} + C(\textbf{u}h\textbf{u}) = -G\left(\frac{g}{2} h_{\tilde{e}}^2\right) \]

**Numerical Approximation (interpolation in space/time)**

\[ H_{c} = \int_{i^{n+1}} h dA \]

\[ \overline{p}_\tilde{v} = \int_{i^{n}} p_{\tilde{v}} dt \]

\[ U_{\tilde{e}} = \int_{\tilde{e}} h\textbf{u} \cdot d\textbf{l} \]

\[ \overline{Q}_f = \int dt \int_{f} h\textbf{u} \cdot n dS \]

\[ \frac{\overline{Q}_f}{S_f} = \frac{1}{2} \left( \frac{U_{\tilde{e}}^{n}}{L_e} + \frac{U_{\tilde{e}}^{n+1}}{L_e} \right) \]

**Hydrostatic assumption**

\[ \frac{g}{2} \left[ \frac{1}{2} \left( \frac{H_{c}^{n}}{A_c} + \frac{H_{c}^{n+1}}{A_c} \right) \right]^2 = \frac{g}{2} \overline{h}_{\tilde{v}}^2 = \overline{p}_\tilde{v} \]

**Flux to Momentum**

Just one (very simple) possible example
Dual Mesh Viewpoint

\[ U_{\tilde{e}}^{n+1} - U_{\tilde{e}}^n + C(uhu) = -G\left(\frac{g}{2}h_n^2\right) \]

\[ H_{c}^{n+1} - H_{c}^n + D\bar{Q}_f = 0 \]

\[ U_{\tilde{e}} = \int_{\tilde{e}} hu \cdot dl \]

\[ \frac{\bar{Q}_f}{S_f} = \frac{1}{2} \left( \frac{U_n^{e}}{L_e} + \frac{U_{n+1}^{e}}{L_e} \right) \]

\[ \bar{p}_{\tilde{v}} = \int_{t^n}^{t^{n+1}} p_{\tilde{v}} dt \]

\[ \frac{g}{2} \left( \frac{H_n^{c}}{A_c} + \frac{H_{n+1}^{c}}{A_c} \right)^2 = \frac{g}{2} \bar{h}_v^2 = \bar{p}_{\tilde{v}} \]

\[ H_c = \int_c hdA \]
Vorticity Conservation

Momentum Eqn

\[ U_{\tilde{e}}^{n+1} - U_{\tilde{e}}^n + C(uhu) = -G\left(\frac{g}{2} h_n^2\right) \]

Transform the Gradient

\[ U_{\tilde{e}}^{n+1} - U_{\tilde{e}}^n + C(uhu) = -G\left(\frac{g}{2} h_n^2\right) = -\frac{1}{2} (\tilde{h}_{n1} + \tilde{h}_{n2}) G(gh_{\tilde{n}}) \]

Sum over the edges (Curl)

\[ \sum_{\tilde{e}} \frac{U_{\tilde{e}}^{n+1}}{h_{\tilde{e}}} - \sum_{\tilde{e}} \frac{U_{\tilde{e}}^n}{h_{\tilde{e}}} + \frac{1}{h_{\tilde{e}}} C(uhu) = -\sum_{\tilde{e}} G(gh_{\tilde{n}}) = 0 \]

Discrete Definition of Vorticity is found

\[ \int_{\tilde{c}} \omega dA = \int_{\tilde{c}} \nabla \times u dA \approx \sum_e \frac{1}{h_e} \int_{\tilde{e}} hu \cdot dl = \sum_e \frac{U_{\tilde{e}}}{h_e} \]
Method Choices

Choice of the Dual Mesh.

Choice of which mesh to use.
Node centered pressure.
Cell centered pressure.

Choice of interpolation.
 polynomial reconstruction in cells.
reconstruction in dual cells
weighted interpolations (FE).
Results
References


Summary

- Numerical Methods are changing.
- Exact Discretization and Approx Solution.
- Works on all types of PDEs