The Utility of Mimetic Methods for Computing Geophysical Flows

Flow Problems in Oil & Gas Industry
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Mimetic Methods

Finite Difference
Finite Volume
SOM
Staggered
Edge/Face
Natural Neighbors
Meshless

All Numerical Methods
Mimetic Description

All Methods: Accuracy, Stability, Convergence

Mimetic Methods: Also have:
- Secondary conservation statements
- Wave propagation properties
- Satisfy constraints (incompressibility)
- Don’t have modes.

They mimic the PDE properties.

Get the Physics and Mathematics of PDES right by using Exact Discretization
Why it Matters

- Very large jumps in material properties
- Very high aspect ratio meshes
- Complex physics

Often nobody will notice (quickly) if the computational solution is wrong.
Mimetic Advection \[ \frac{\partial T}{\partial t} + u \cdot \nabla T = 0 \]

Jason Frank, CWI/University of Amsterdam
Change in mesh size (3x)

Central
Box Method

Very small timestep.
Mimetic Eigenvalues

Costabel/Dauge, Arnold*, Gerritsma

\[ \nabla \times \nabla \times \mathbf{v} - \nabla (\nabla \cdot \mathbf{v}) = \lambda \mathbf{v} \quad \lambda = m^2 + n^2 \]

1,1,2,4,4,5,5,8,…

Linear FE

\[ \mathbf{v}_{node} \]

Raviart-Thomas FE

\[ \mathbf{u} = \mathbf{v} \cdot \mathbf{t}_{edge} \]

*Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability
Mimetic Vector Fields

\[ \nabla \times \nabla \times \mathbf{v} - \nabla (\nabla \cdot \mathbf{v}) = f \]

Linear FE  Raviart-Thomas FE
Mimetic Physics
Discrete Calculus Approach
How to always construct a mimetic method.

Separate Discretization from Approximation

- Do ALL discretization exactly.
- This means that the calculus and the physics remain exact.

- All approximations = interpolation problems.
- Numerical approximation only in material laws (which are engineering approximations already)

PDE -> LA
LA -> square LA
Example PDE

Physical Equation (Heat Equation)

\[
\frac{\partial (\rho cT)}{\partial t} = \nabla \cdot k \nabla T
\]

Components of the Physical Equation

\[
\frac{\partial i}{\partial t} + \nabla \cdot \mathbf{q} = 0 \quad \text{Conservation of Energy}
\]

\[
\mathbf{g} = \nabla T \quad \text{Definition of Gradient}
\]

Physic

Math

Material Approximation

\[
\mathbf{q} = -k \mathbf{g} \quad \text{Fourier’s Law}
\]

\[
i = \rho cT \quad \text{Perfectly Caloric Material}
\]
Mimetic Discretization

Exact Discretization of Physics and Calculus.

\[
\int_{\tilde{c}} idV |^{n+1} - \int_{\tilde{c}} idV |^{n} + \sum_{\tilde{f}} \int_{\tilde{f}} d\bar{t} \int_{\tilde{f}} q \cdot n dA_{\tilde{f}} = 0
\]

\[
\int_{e} g \cdot d\bar{l} = T_{n2} - T_{n1}
\]

\[
I_{\tilde{c}} = \rho c V_{\tilde{c}} T_{n}
\]

Numerically Exact

Numerically Approximate

\[
Q_{\tilde{f}} = -M_1 g_e
\]

\[
I_{\tilde{c}} = M_2 T_n
\]

\[
Q_{\tilde{f}} = -k \frac{A_{\tilde{f}}}{L_e} g_e
\]

\[
I_{\tilde{c}} = \rho c V_{\tilde{c}} T_n
\]
\[ I_{\tilde{c}}^{n+1} - I_{\tilde{c}}^n + DQ_{\tilde{j}} = 0 \]

\[ I_{\tilde{c}} = \rho c V_{\tilde{c}} T_n \]

\[ Q_{\tilde{j}} = \int dt \int_{\tilde{j}} q \cdot n dA_{\tilde{j}} \]

\[ Q_{\tilde{j}} = -k \frac{A_{\tilde{j}}}{L_e} g_e \]

\[ \int_{e} g \cdot d\mathbf{l} = g_e \]
Results

- Log scale
Summary

- Interesting Developments in Numerical Methods.
- Some of it happening here in the Netherlands
- These methods are very useful for complex physics.