Capacitated Multicommodity Network Flow Problems with Piecewise Linear Concave Costs[*]

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Abstract

Lagrangian techniques have been commonly used to solve the capacitated multicommodity network flow problem with piece-wise linear concave costs (CMNFP). In this paper, we show that the resulting lower bounds are no better than those obtained by the linear programming relaxation and focus on developing algorithms based on the latter. For that purpose, we characterize structural properties of the optimal solution of the linear programming relaxation and propose a heuristic solution approach that uses these properties to transform the fractional solution into an integer one. Our computational experiments show the effectiveness of the algorithm.

Keywords: Capacitated Concave Cost Multicommodity Network Flows, Integer Programming, Linear Programming Relaxation, Production, Inventory and Transportation Control, Logistics.

1   Introduction

Large scale multicommodity network flow problems have received considerable attention from the research community. This is hardly surprising given the variety and the importance of the underlying practical applications of the problem, telecommunication networks, production-distribution systems, rail and urban road systems being a few examples. Consequently

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there exists a vast body of literature on manufacturing and transportation logistics as well as on the telecommunication applications pertaining to this problem. See Tomlin (1966), Kennington (1978), Assad (1978), Magnanti and Wong (1984), Minoux (1989), Gavish (1991) and Balakrishnan et al. (1991) for extensive reviews of the problem's applicability under a variety of cost structures and formulations.

Our motivation in this paper is to coordinate production, inventory management and transportation decisions in capacitated systems that have built in economies of scale. Economies of scale are common in production and transportation and can be generally represented by piece-wise linear concave functions. It is these existing economies of scale that motivate the shipper to coordinate production, inventory strategies and the timing of transportation decisions so as to minimize system-wide costs. In addition, in most practical situations facilities face capacity constraints, or an upper bound on the amount of material they can process. In transportation, equipment capacity and location shipping and receiving constraints are typically experienced. The coordination problem can be modeled as a capacitated multicommodity network flow problem with piece-wise linear concave costs (CMNFP). Timing considerations can be included by constructing an expanded network where each node represents a facility at a particular point in time, see for instance Ahuja et al. (1993). To represent cost and capacity considerations at the production stage in the expanded network model, each production location at a particular time period is represented by two nodes connected by an arc between them. This arc corresponds to the production flows and bears the associated production cost and the capacity restrictions.

In what follows, we review the literature that deals specifically with piece-wise linear and concave multicommodity network flow problems. Balakrishnan and Graves (1989) and Amiri and Pirkul (1997) approach the Multicommodity Network Flow Problem with Piece-wise Linear Concave Cost (MNFP) using Lagrangian relaxation techniques. The former strictly tackle the uncapacitated problem, although the formulation presented could be used to include capacitated cases, whereas the latter explicitly consider arc capacities. The algorithms developed in both papers are tested on similarly structured, randomly generated

instances, where capacities are never tight. Both give outstanding results. This is not surprising, since Chan et al. (2001) show that the linear programming relaxation of the formulation used is very strong for uncapacitated instances; moreover, the relaxation is proven to be tight for certain simple network structures. They also derive structural properties of the linear programming relaxation and use them to develop an effective linear programming based algorithm. In this paper, we show that the Lagrangian relaxations of the MNFP, with or without capacity constraints, do not improve upon the linear programming relaxation and propose a linear programming based solution procedure that extends that of Chan et al. (2001) to include capacity constraints.

The capacitated problem (CMNFP) generalizes the capacitated multicommodity fixed charge network design problem (CMNDP), in which the cost of using an edge is simply a fixed charge independent of the quantity shipped. The fixed-charge network design model is NP-Complete, even in the uncapacitated case, see Johnson et al. (1978). Nonetheless, efficient heuristic algorithms have been developed for the uncapacitated case (e.g. Balakrishnan et al. (1989)). Again, the reason for their outstanding performance can be attributed to the quality of the linear programming relaxation. In the capacitated case, however, even finding a feasible solution to the problem may pose a challenge. The higher complexity of the capacitated problem is demonstrated in a stream of recent work in this area; see Gendron et al. (1999), which describes the different approaches to solving this problem – Lagrangian relaxation, cutting plane methods and heuristics – and discusses how combining these methods may lead to effective solution schemes. The advantage of lagrangian relaxation methods is that they can find strong lower bounds and high-quality solutions in a fraction of the time required to solve the linear programming relaxation; see Gendron et al. (1999), which presents an extensive computational study that shows the effectiveness of bundle-based approaches to optimize the lagrangian dual.

A very specific network structure with general piecewise linear costs is considered in Croxton et al. (2003) to model the selection of different transportation modes and shipment routes in merge-in-transit operations. In this case, a set of capacitated warehouses coordi-

nates the flow of goods from a number of suppliers to multiple retailers with the objective of reducing costs through consolidation. They use a simplex-based cutting plane approach, for their arc based formulation, generating cuts with disaggregation techniques to solve the problem.

In the remainder of the paper, we first introduce the problem formulation and show the relative performance of various relaxations. In Section 4, we study the linear programming relaxation and find interesting properties that lead to the development of a heuristic algorithm, described in Section 5. The strength of the relaxation and the performance of the algorithm are analyzed through computational experiments in Section 6.

2   Model

Consider a directed network $G = (V, E)$ and let the cost of any arc e, $e \in E$, be $F_e(z_e)$, a piecewise linear and concave cost function which is non-decreasing in the total amount, z_e, of flow on arc e. Each arc e has a limited capacity of C_e units. Demands in the network are known deterministically. Each demand is referred to as a separate commodity, say commodity k, and characterized by an origin node $O(k)$, a destination node $D(k)$, and a specified weight w_k. Let K be the total number of commodities or demands in the system. Our objective is to find routes for these K commodities through the network to satisfy all the demands at minimum cost, by taking advantage of the economies of scale.

In what follows, we formulate the problem as a mixed integer linear program. As presented in Balakrishnan and Graves (1989), the piecewise linear concave cost structure allows for such a formulation of the problem. Let R be the number of different slopes in the cost function. To avoid cumbersome notation, we assume without loss of generality that R remains the same over all arcs in the network. Let M_e^{r-1}, M_e^r, $r = 1, \ldots, R$, denote the lower and upper limits, respectively, on the interval of quantities corresponding to the rth slope of the cost function associated with arc e. Note that $M_e^0 = 0$ and M_e^R can be set to the total quantity of all commodities that may use arc e (in the uncapacitated case) or to the capacity C_e. We associate with each of these intervals, say r, a variable cost per unit, denoted by