Measuring the Impact of Increased Product Substitution on Pricing and Capacity Decisions under Linear Demand Models

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We compare two alternative measures of product substitutability for linear demand functions that are commonly used in the literature in a variety of price/quantity decision and capacity investment problems, in monopolistic and competitive settings. While the use of the incorrect measure of product substitutability leads to unrealistically high prices and profits as products become more substitutable, the results obtained using the more appropriate measure are in line with what is expected to happen in real markets. Using the appropriate measure of product substitutability, we find that the optimal investment in manufacturing flexibility for a firm tends to decrease as the products become closer substitutes; this is because (1) pricing can be used more effectively to balance supply and demand, and (2) the gains obtained by shifting production to the more profitable product are reduced due to increased correlation between the price potentials of the substitutable products. The optimal investment in flexible capacity may increase, however, if the correlation between the market potentials of the two products significantly decreases, while the correlation between price potentials remains stable as they become less differentiated. The value of flexibility always increases with demand variability. We also show that, as long as the optimal investments in dedicated capacity for both products are positive, the optimal expected prices and production quantities do not depend on the cost of the flexible capacity. Manufacturing flexibility simply allows the firm to achieve those expected values at a lower investment cost.

Keywords: Demand modeling, product substitutability, capacity and flexibility planning, pricing.

1. Introduction

It is of great importance to accurately define the relation between the demand and prices of the products in a market system. The resulting price-demand equations are a key input to a wide variety of analytical models ranging from strategic capacity and flexibility determination to the operational price setting common in revenue management. The validity of the results obtained from these models will of course be no better than that of the underlying demand models considered. To be of real value, these demand models must satisfy two conditions: (1) represent customer behavior accurately, and (2) be simple enough to allow for solution and analysis of the analytical models they are fed into. Consequently, the demand model to choose will depend on the decision problem at hand. While consumer choice models, such as the multinomial logit model, are typically used in the revenue management and product assortment literature (Anderson et al. (1992), McFadden (1986), Hanson and Martin (1996), Swann (1999), Aydin and Ryan (2000), Jonard and Schenk (2003)), more simplistic models may be appropriate for strategic decisions to gain analytical tractability. Our objective is to study simple linear demand models to determine whether they provide a reasonable input to capacity and pricing decision models, and to identify the impact of product substitutability on optimal capacity and flexibility investments. For that purpose, we consider two substitutable products and analyze the effect of product substitutability on optimal prices, capacities and production levels in various decision models in monopolistic and competitive settings. We will use and compare alternative measures of product substitutability that are commonly used in the economics literature.

Understanding the effect of product substitution on capacity, flexibility and pricing decisions is important for many industries in planning their product assortment and/or the design of their production network. For instance, consider a company with a complex production network, such as General Motors, where multiple products are produced in a number of plants. The performance of the system is highly dependent on the allocation of products to plants and the use of flexibility (Jordan and Graves (1995) and Muriel et al. (2006)). Our results show that it significantly affects the optimal capacity and flexibility requirements, and is thus fundamental in deciding which products should share flexible resources.

The linear demand functions are derived from the maximization problem of a representative consumer with a quadratic and strictly concave utility function which is defined as (Singh and Vives (1984))

$$U(Q_1,Q_2) = A_1Q_1 + A_2Q_2 - \left(a_1Q_1^2 + 2bQ_1Q_2 + a_2Q_2^2\right)/2$$

where Q_i is the amount of product *i*, $A_i > 0$, $a_i > 0$ and $a_jA_i - bA_j > 0$ (in order to obtain positive demand quantities, as shown below) for $i = 1,2$ and $i \neq j$. To ensure strict concavity of the utility function we let $a_1a_2 - b^2 > 0$. The parameter *b* captures the interactions among the products and is defined as the measure of product substitutability or complementarity; the products are substitutes, independent or complements when $b > 0$, $b = 0$ or $b < 0$, respectively. The given utility function represents the fact that the value of using both substitutable (complementary) products is less (more) than the sum of the separate values of using each product by itself (Samuelson (1974)). Furthermore, the consumer utility decreases (increases) as products become more substitutable (complementary), i.e. as *b* (-*b*) increases, everything else held constant.

The maximization of $U(Q_1,Q_2) - P_1Q_1 - P_2Q_2$, where P_i is the price of product *i*, gives rise to the following inverse and direct linear demand functions:

$$P_i = A_i - a_iQ_i - bQ_j, \qquad i, j = 1,2 \text{ and } i \neq j$$

$$Q_i = \frac{a_j A_i - bA_j}{a_i a_j - b^2} - \frac{a_j}{a_i a_j - b^2} P_i + \frac{b}{a_i a_j - b^2} P_j, \qquad i, j = 1, 2 \text{ and } i \neq j.$$

Note that the relation $a_i > b$ for $i = 1, 2$ is also required to ensure the following two conditions: 1) demand for a product should be more sensitive to changes in its price than to changes in the price of the other product 2) the total demand cannot increase with an increase in product prices. When $b \geq 0$, the normalized quantity $b^2 / a_1 a_2$ measures the relative degree of substitutability between the two products, ranging from zero when b=0 (independent products) to one when $A_1 = A_2$ and $b = a_1 = a_2$ (perfect substitutes), and is thus suggested as the measure of product substitutability.

The above parameterization of linear demand functions, in which b (or a function of b) is defined as the measure of product substitutability, has been commonly used in the economics literature (Singh and Vives (1984), Roller and Tombak (1990,1993), Tyagi (1999), Bernhofen (2001), Mukherjee (2004)), but has rarely been used by the operations management community. Recent exceptions are the papers by Goyal and Netessine (2005,2007) and Bish and Suwandechochai (2006).

Letting $\varepsilon_i = \left(a_j A_i - bA_j\right) / \left(a_i a_j - b^2\right)$, $\alpha_i = a_j / \left(a_i a_j - b^2\right)$, $\beta = b / \left(a_i a_j - b^2\right)$ for $i, j = 1, 2$ and $i \neq j$ the direct demand functions can be written as

$$Q_i = \varepsilon_i - \alpha_i P_i + \beta P_j, \qquad i, j = 1, 2 \text{ and } \mathrm{i} \neq \mathrm{j}.$$

In this notation the parameter $\varepsilon_i > 0$ is the demand intercept which represents the potential market size for the product and $\alpha_i > 0$ and β are the price and cross-price sensitivity parameters, respectively, with $\alpha_i > \beta$, $i = 1, 2$. Similarly, the products are substitutes, independent or complements when $\beta > 0$, $\beta = 0$ or $\beta < 0$, respectively. The sum of the intercepts, $\varepsilon_1 + \varepsilon_2$, can be interpreted as the total market size, i.e. the maximum total demand achieved when both prices are zero.

Note that the market size and price and cross-price sensitivity parameters are all related and depend on the product substitutability parameter *b*; as products become more substitutable (i.e., as *b* increases), the customers become more sensitive to changes in prices and the overall market size decreases $\left(\varepsilon_1 + \varepsilon_2 = \frac{a_2 A_1 - b A_2}{a_1 a_2 - b^2} + \frac{a_1 A_2 - b A_1}{a_1 a_2 - b^2} = \frac{(a_2 - b) A_1 + (a_1 - b) A_2}{a_1 a_2 - b^2} \text{ is decreasing in } b\right)$. These properties reflect the widely accepted facts that (1) more differentiated products reach a larger customer base and (2) consumers are less price-sensitive when purchasing a more unique item (e.g. Talluri and Van Ryzin (2005), pp. 395-396). However, in most of the Operations Management and Marketing literature the relationship among these parameters is ignored (except $\alpha_i > \beta$) and the parameter β, which represents the cross-price effect, is used as a measure of the product substitutability (McGuire and Staelin (1983), Choi (1991), Birge et al. (1998), Garcia-Gallego and Georgantzis (2001), Bish and Suwandechochai (2005)). When using β instead of *b* as the measure of product substitutability in the linear demand functions, an increase in product substitutability does not affect the product's own price effect on its demand or the total market size; this contradicts the accepted facts for differentiated products mentioned above.

A common thread in the results derived from such models, which use β as a measure of substitutability, is that optimal prices and profits increase as product substitutability grows. This counterintuitive and unrealistic result, however, is often not questioned. Only Choi (1991) argues that this parameterization does not realistically describe the relationship between demands and prices for substitutable products. He proposes a nonlinear demand model which gives more realistic results, but is difficult to solve and analyze. McGuire and Staelin (1983) also point out that cooperatively set prices in a monopoly setting increase without bound as products become closer to perfect substitutes and suggest a different parameterization of linear demand functions. For completion while maintaining the conciseness of the current paper, we discuss the parameterization of linear demand functions proposed by McGuire and Staelin (1983) in the Appendix. More recently, Biller et al. (2006) question the pitfalls of the β-measure, propose a couple of linear demand alternatives and discuss the problems arising from their use. These results suggest that β is not the right parameter to measure product substitutability.

Moreover, when the more appropriate measure of product substitutability, *b,* is studied in the above papers (by doing the appropriate substitutions for the parameters) one can easily show that product prices decrease as products become more substitutable for all the models considered. The profits may change in either direction, but the profits of all players do not increase at the same time and the changes are reasonable. We should also point out that using the *b*-measure of product substitutability in McGuire and Staelin (1983) and Choi (1991) changes the trends of prices and profits, but does not change the other results derived, such as the choice of industry structure –decentralized vs. integrated- (McGuire and Staelin (1983)) or the effects of power structure between the manufacturers and retailers (Choi (1991)).

Birge et al. (1998), Bish and Suwandechochai (2005, 2006) and Biller et al. (2006) study the impact of parameter β on the optimal capacity investment decisions and show that the (total) capacity investment level of each firm increases with this parameter. However, the study of parameter *b* results in decreased (total) capacity investment levels as products become closer substitutes (see section 3 and Bish and Suwandechochai (2006) who also study both measures of product substitutability).

Our study of linear demand models for two substitutable products within different system structures in competitive and monopoly settings shows that the use of β as a measure of product substitutability results in drastically higher prices and profits as product substitutability increases in all the settings studied, reinforcing the previously mentioned findings. We show, however, that the use of the right measure of product substitutability, where all of the coefficients in the linear demand functions depend appropriately on the level of product substitutability, can provide results in line with what we expect to happen in practice. Linear demand models with the correct measure of product substitutability provide analytical tractability while leading to realistic conclusions.

The value of capacity flexibility for independent products has been extensively studied in the operations management literature (e.g. Fine and Freund (1990), Jordan and Graves (1995), Van Mieghem (1998), Van Mieghem and Dada (1999), Bish and Wang (2004), Biller et al. (2006)). More recently, attention has shifted to capacity investment decisions for substitutable products. Chod and Rudi (2005), Bish and Suwandechochai (2005, 2006), and Goyal and Netessine (2005, 2007) study the choice between flexible and dedicated technology, not allowing investments in a mix of the two, for two substitutable products. Chod and Rudi (2005) define the parameter β as the measure of substitutability but do not analyze the sensitivity of decisions with respect to this parameter. They show that the investment of a monopolist in flexible capacity increases in both demand variability and correlation. Bish and Suwandechochai (2006) study both measures of substitutability, β and b, and show that the optimal flexible capacity level increases with β while decreases with b. The study by Goyal and Netessine (2005, 2007) supports the results of Chod and Rudi (2005) and further explores the impact of product substitutability on optimal capacity investments, using the parameter b as the measure of product substitutability, for competitive

(2007) and monopoly (2005, 2007) settings. They show that as products become closer substitutes, the optimal investment in capacity decreases under both dedicated and flexible technologies. Moreover they conclude that "as products become more substitutable firms are more favorably inclined towards investing in flexible technology". This is in contrast with the findings of Biller et al. (2006) in their study of the β measure and other alternative linear demand functions, which shows that the need for manufacturing flexibility decreases as products become closer substitutes because demand can be shifted through pricing to make better use of the available dedicated capacity.

In this study, we analyze the impact of product substitution on the optimal mix of dedicated and flexible capacities the firm should invest in, using the right measure of product substitutability, i.e. the parameter b. Supporting the results of the existing literature, we show that the total capacity investment increases with demand variability and correlation between market potentials. While the investment in flexible capacity increases with demand variability, it decreases with the correlation between demand or price intercepts. Furthermore, we show that the investment in flexible capacity may decrease or increase as the products become closer substitutes, depending on the underlying assumptions on the characteristics of the market. This explains the conflicting results found in the literature.

The paper is organized as follows. In section 2, we study the pricing/quantity decisions of the firm(s) in monopoly and competitive settings with unlimited capacities and known linear demand functions. In section 3, we assume random demand/price intercepts and study the capacity investment and production/pricing decisions of a firm in two stages: In the first stage, the capacity investment decisions are made under uncertain demand curves. In the second stage, the demand curve for each product is realized and the firm makes production and pricing decisions under the capacity and flexibility constraints associated with the prior investment decisions. We first study the problem for a monopoly firm, and then discuss the oligopoly case where n symmetric firms compete in quantities (Cournot competition) in both markets. The final section discusses conclusions and extensions.

Throughout the paper, we increase the level of product substitutability (i.e., decrease product differentiation) by increasing β or b. To simplify the exposition, we assume without loss of generality that variable production costs are zero (e.g. Singh and Vives (1984), Chod and Rudi (2005)).

2. Unconstrained Pricing/Quantity Decisions

In this section we study different price and quantity decision models in monopoly and competitive settings for the two alternative measures of product substitutability/complementarity. In Sections 2.1 and 2.2 we consider two substitutable products and analyze how optimal price and production decisions change as products become more substitutable, i.e. as β or b becomes more positive. In Section 2.3 we study the case of complementary products and discuss how these decisions are affected as products become more complementary, i.e. as β or b becomes more negative.

The pricing/quantity decision can be made before or after product demands are realized. In both cases, the optimization problem at hand is essentially deterministic. Since the profit function is linear in the (random) demand/price intercept, maximizing the expected profits in the former case is equivalent to solving the deterministic problem with the mean demand intercepts. In the latter case, the deterministic problem is solved for each demand realization and the expected profits calculated using the resulting prices and production quantities.

2.1 Price Decision Models

In this section we study pricing decisions of the firms under unlimited capacity for three different models: Monopoly, Bertrand and price Stackelberg. The profit function Π_i of the firm which offers product i, is given by:

$$\Pi_i\left(P_i, P_j\right) = P_i \cdot \left(\varepsilon_i - \alpha_i P_i + \beta P_j\right), \qquad i, j = 1,2 \text{ and } i \neq j .$$

In the monopoly model, there is a single firm producing both products. The monopolist sets the prices of the products to maximize the overall profits, $\Pi_1 + \Pi_2$. For the competitive models we assume that there are two firms each producing one of the products, with firm *i* offering product *i*, *i*=1,2. In the Bertrand model each firm chooses its profit-maximizing price given the price of the rival firm and assumes that its price does not change the price of the rival firm. In the price Stackelberg model, one of the firms is the price leader and the other one is the follower. The leader defines its price and the follower sets the price after observing the leader's price. In our model we assume firm 1 is the leader and firm 2 is the follower.

Table 1: Optimal prices and production levels; functions of β

MODEL	Price	Quantity
Monopoly	$P_i = \dfrac{\alpha_j \varepsilon_i + \beta \varepsilon_j}{2\left(\alpha_i \alpha_j - \beta^2\right)}$	$Q_i = \dfrac{\varepsilon_i}{2}$
Bertrand	$P_i = \dfrac{2\alpha_j \varepsilon_i + \beta \varepsilon_j}{4\alpha_i \alpha_j - \beta^2}$	$Q_i = \dfrac{2\alpha_i \alpha_j \varepsilon_i + \alpha_i \beta \varepsilon_j}{4\alpha_i \alpha_j - \beta^2}$
Price Stackelberg (Leader)	$P_1 = \dfrac{2\alpha_2 \varepsilon_1 + \beta \varepsilon_2}{4\alpha_1 \alpha_2 - 2\beta^2}$	$Q_1 = \dfrac{2\alpha_2 \varepsilon_1 + \beta \varepsilon_2}{4\alpha_2}$
Price Stackelberg (Follower)	$P_2 = \dfrac{\varepsilon_2}{4\alpha_2} + \dfrac{\beta \varepsilon_1 + \alpha_1 \varepsilon_2}{4\alpha_1 \alpha_2 - 2\beta^2}$	$Q_2 = \dfrac{\varepsilon_2}{4} + \dfrac{\alpha_2\left(\beta \varepsilon_1 + \alpha_1 \varepsilon_2\right)}{4\alpha_1 \alpha_2 - 2\beta^2}$

Table 1 presents the well-known expressions for the optimal price and production quantity of each firm for all three models as functions of β (Singh and Vives (1984), Birge et al. (1998)). The results show that when β is used to represent the degree of product substitutability both product prices increase in all models as products become closer substitutes. This contradicts with our intuition and the observed fact of higher prices for highly differentiated products. In addition, the higher prices for more substitutable products lead to higher (Bertrand and price Stackelberg) or unchanged (monopoly) production levels, and hence to unrealistically higher profits for both firms. These results are driven by the fact that for a fixed set of prices the total system demand increases and customers' sensitivity to price decreases as products become more substitutable according to the β measure: $Q_1 + Q_2 = \varepsilon_1 + \varepsilon_2 - (\alpha_1 - \beta)P_1 - (\alpha_2 - \beta)P_2$.

Table 2: Optimal prices and production levels; functions of *b*

MODEL	Price	Quantity
Monopoly	$P_i = \frac{A_i}{2}$	$Q_i = \frac{a_j A_i - bA_j}{2\left(a_i a_j - b^2\right)}$
Bertrand	$P_i = A_i - \frac{a_i\left(2a_j A_i + bA_j\right)}{4a_i a_j - b^2}$	$Q_i = \frac{a_j\left(2a_i a_j - b^2\right)A_i - a_i a_j bA_j}{\left(4a_i a_j - b^2\right)\left(a_i a_j - b^2\right)}$
Price Stackelberg (Leader)	$P_1 = \frac{A_1}{2} - \frac{a_1 bA_2}{4a_1 a_2 - 2b^2}$	$Q_1 = \frac{A_1}{4a_1} + \frac{a_2 A_1 - bA_2}{4\left(a_1 a_2 - b^2\right)}$
Price Stackelberg (Follower)	$P_2 = \frac{3A_2}{4} - \frac{bA_1}{4a_1} - \frac{a_1 a_2 A_2}{4a_1 a_2 - 2b^2}$	$Q_2 = \frac{a_1\left(4a_1 a_2 - 3b^2\right)A_2 - b\left(2a_1 a_2 - b^2\right)A_1}{4\left(2a_1 a_2 - b^2\right)\left(a_1 a_2 - b^2\right)}$

In contrast, when *b* is used as a measure of product substitutability, product prices do not increase with product substitutability (Table 2). In the monopoly model, the optimal price of each product does not change with substitutability. This is intuitive because in a monopoly setting one would expect the price of a product (1) not to increase when it has a closer substitute, and (2) to be identical in the two extreme cases, i.e. when the product has no substitutes ($b = 0$) and when the product has a perfect substitute ($A_1 = A_2$ and $b = a_1 = a_2$), since both settings are practically equivalent to having a unique product in the system.

The optimal system-wide production in the monopoly model decreases with product substitutability as a result of the decrease in total market size. Moreover, as products become closer to perfect substitutes ($A_1 \approx A_2$ and $b \to a_1 \approx a_2$), the total production level for the two products converges to the production level of one of the products when they are independent (b=0). The same result follows for profits, since prices do not change with product substitutability. Then, having two very close substitutes converges to the case of having only one (commodity) product, which is intuitive. This is in contrast with our findings for the monopoly model with the β-measure where, as products become closer to perfect substitutes, both prices and profits increase to infinity.

In competitive models, as products become less differentiated, the buyers of each product become more sensitive to its price, i.e., the product's own price sensitivity increases; hence the competition strengthens and prices decrease. See Example 1 in the Appendix for illustration. Under the assumption of zero marginal costs, the optimal prices and profits decrease to zero as products become closer to perfect substitutes. This is in line with the theory that when two firms producing homogeneous products with the same marginal cost compete in price, at equilibrium both prices are the same and equal to the marginal cost (Mansfield and Yohe (2004)). In the Bertrand model, the production quantities can move in either direction with an increase in product substitutability. This can be easily explained in a symmetric setting where $A_1 = A_2 = A$, $a_1 = a_2 = a$. For small values of b, an increase in b significantly reduces the demand intercepts,

$A/(a+b)$, but has a relatively small effect on each product's own price sensitivity, $a/\left(a^2-b^2\right)$, resulting in lower production levels. Conversely, for higher values of b, an increase in b has a more pronounced effect on the products' own price effects than on the intercepts. This leads to considerably reduced prices and increased production levels as the two products become even closer substitutes. The optimal responses in the price Stackelberg model exhibit similar behavior, except that the production level for the leader always decreases with an increase in product substitutability (see Example 1 in Appendix). In the price Stackelberg model with symmetric demands, the follower always has higher benefits: lower prices leading to higher production levels and higher profits. Thus, for identical firms under the described linear demand model, each firm prefers following rather than leading and any sequential order is preferred over moving simultaneously (Gal-Or (1985), Dowrick (1986) and Damme and Hurkens (2004)).

In all the settings analyzed, using the parameter b as a measure of product substitutability provides a more accurate representation of the effect of changes in the level of substitutability on the closed market system. The results of the numerical example in the Appendix (see Tables A1 and A2) highlight the importance of using the right parameter to measure the degree of product substitutability. Table A1 shows the rapid increase in prices and profits as the products become closer substitutes when β is used as the measure of product substitutability; in particular, the monopoly case experiences a 50-fold increase in prices and profits when comparing $\beta=0$ with $\beta=49$ in a symmetric case where $\varepsilon_1=\varepsilon_2=2500$ and $\alpha_1=\alpha_2=50$. Using the b-measure, however, as the monopoly firm produces closer substitutes the price remains stable but the system profits steadily decrease almost by half as a result of decrease in the total market size. For the competitive models, when β is used as the measure of product substitutability, as the products become close to perfect substitutes, each firm increases its profits by at least a factor of 3.8 (e.g.

the follower in the Stackelberg model increases its profits by a factor of almost 6). Under the *b*-measure, in contrast, the competing firms see a steep reduction in their profits by offering close substitutes.

2.2 Quantity Competition Models

We now turn our attention to two different quantity competition models, Cournot and quantity Stackelberg, and show that the study of quantity competition leads to very similar results to that of price competition. In the Cournot model the firms decide on the profit maximizing quantities simultaneously, while in the quantity Stackelberg model first the leader chooses its production and then its competitor, the follower, chooses its production level after observing the leader's decision. The profit function of firm *i* is given by:

$$\Pi_i\left(Q_i, Q_j\right) = Q_i \cdot \left(A_i - a_i Q_i - bQ_j\right), \quad i, j = 1,2 \text{ and } i \neq j$$

or equivalently, for the second parameterization of the linear demand functions,

$$\Pi_i\left(Q_i, Q_j\right) = Q_i \cdot \left(\frac{\alpha_j \varepsilon_i + \beta \varepsilon_j}{\alpha_i \alpha_j - \beta^2} - \frac{\alpha_j}{\alpha_i \alpha_j - \beta^2} Q_i - \frac{\beta}{\alpha_i \alpha_j - \beta^2} Q_j\right), \quad i, j = 1,2 \text{ and } i \neq j .$$

Table 3: Optimal prices and production levels for quantity competition models; functions of β

MODEL	Price	Quantity
Cournot	$P_i = \frac{\alpha_j\left(2\alpha_i\alpha_j - \beta^2\right)\varepsilon_i + \alpha_i\alpha_j\beta\varepsilon_j}{\left(4\alpha_i\alpha_j - \beta^2\right)\left(\alpha_i\alpha_j - \beta^2\right)}$	$Q_i = \frac{\left(2\alpha_i\alpha_j - \beta^2\right)\varepsilon_i + \alpha_i\beta\varepsilon_j}{4\alpha_i\alpha_j - \beta^2}$
Quantity Stackelberg (Leader)	$P_1 = \frac{\varepsilon_1}{4\alpha_1} + \frac{\alpha_2\varepsilon_1 + \beta\varepsilon_2}{4\left(\alpha_1\alpha_2 - \beta^2\right)}$	$Q_1 = \frac{\varepsilon_1}{2} + \frac{\alpha_1\beta\varepsilon_2}{4\alpha_1\alpha_2 - 2\beta^2}$
Quantity Stackelberg (Follower)	$P_2 = \frac{\alpha_1\left(4\alpha_1\alpha_2 - 3\beta^2\right)\varepsilon_2 + \beta\left(2\alpha_1\alpha_2 - \beta^2\right)\varepsilon_1}{4\left(2\alpha_1\alpha_2 - \beta^2\right)\left(\alpha_1\alpha_2 - \beta^2\right)}$	$Q_2 = \frac{3\varepsilon_2}{4} + \frac{\beta\varepsilon_1}{4\alpha_1} - \frac{\alpha_1\alpha_2\varepsilon_2}{4\alpha_1\alpha_2 - 2\beta^2}$

Table 3 lists the optimal prices and production quantities as a function of β for the two models. When β is used as the measure of product substitutability, we observe that the prices of both products increase as products become closer substitutes resulting in higher profits for both competitors. At the same time, although the production quantities of each product can move in either direction, the system-wide production increases with product substitutability.

In contrast, when parameter b is used as the measure of substitutability for quantity competition models, both the price and production quantity for each product decrease with product substitutability (the only exception is the production level for the leader in the quantity Stackelberg model which may change in either direction; see Table 4) resulting in lower profits for each manufacturing firm. Moreover as products converge to perfect substitutes the solution converges to the situation where firms produce a homogeneous product.

Table 4: Optimal prices and production levels for quantity competition models; functions of b

MODEL	Price	Quantity
Cournot	$P_i = \frac{a_i\left(2a_jA_i - bA_j\right)}{4a_ia_j - b^2}$	$Q_i = \frac{2a_jA_i - bA_j}{4a_ia_j - b^2}$
Quantity Stackelberg (Leader)	$P_1 = \frac{2a_2A_1 - bA_2}{4a_2}$	$Q_1 = \frac{2a_2A_1 - bA_2}{4a_1a_2 - 2b^2}$
Quantity Stackelberg (Follower)	$P_2 = \frac{A_2}{4} + \frac{a_2\left(a_1A_2 - bA_1\right)}{4a_1a_2 - 2b^2}$	$Q_2 = \frac{A_2}{4a_2} + \frac{a_1A_2 - bA_1}{4a_1a_2 - 2b^2}$

Observe that for identical firms, each firm prefers leading rather than following and simultaneous movement is preferred to following, but not to leading. This is in contrast with the case where the firms' strategies are defined in terms of prices, for which we previously showed that firms prefer following in a sequential game and any sequential order is preferred over moving simultaneously.

As in price competition models, while the results for the β–measure are inconsistent with common sense, the results when using b to measure product substitutability are in line with our intuition (see Tables A1 and A2 for a numerical illustration).

2.3 Linear Demand Models for Complementary Products

Economides and Viard (2006) (also see Stanford Knowledgebase (2004)) suggest that the sales of a product increase as the number of complementary products created for it increases, and the firm has an incentive to make the products more compatible. Davis and Murphy (2000) also mention that the demand for a product increases in the presence of a complementary product. Venkatesh and Kamakura (2003) study pricing of two complementary products under a monopoly and show that the optimal prices of products are monotonically increasing in the degree of product complementarity. The monopolist gains by charging higher prices for complementary products while stimulating more consumers to buy both products. They measure the degree of complementarity/substitutability as the relative increase in reservation price gained through purchasing both products, as compared to the sum of the reservation prices of the two products.

For linear demand functions, if b is used to measure the degree of product complementarity, then the results for the various price/quantity decision models studied in the previous sections support the above assertions; that is, as products become more complementary the market potential for each product increases leading to an increase in production (sales) of each product and resulting profits. Furthermore, the optimal product prices tend to increase with product complementarity.

When β is used to measure the degree of product complementarity, on the other hand, the market potential (demand intercept) of a product does not change when it has a complement; this contradicts the perception that additional customer segments are reached when offering complementary products. Furthermore, the expressions given in Tables 1 and 3 show that the profit maximizing price and sales of a product both decrease in the presence of a complementary product, either produced by the firm or a competitor, which is not in line with what we expect to happen in practice (see Table A3 for numerical illustration). These contradictory results show that the parameter β does not properly capture the complementarity effect between the products. Also note that $\alpha_j\varepsilon_i + \beta\varepsilon_j > 0$ is required for i=1,2 and $i \neq j$ when $\beta < 0$, so that the inverse demand functions have positive intercepts.

We must point out, however, that under the *b*-measure the increase in production and profits may become unrealistically large at high levels of complementarity. For the unconstrained pricing models shown in Table 2, for example, observe that when $a_1 = a_2$ the production of each product and the firms' profits increase unboundedly as products become more complementary, i.e. as $-b \rightarrow a_1(= a_2)$. For quantity competition models (Table 4 and Table A4 in Appendix), this undesirable effect is much less pronounced.

Observe also that when products become closer to perfect complements one would expect the sales for the products to become more similar and converge to the same value (Wang (2006)), which is not satisfied for the price and quantity Stackelberg models, and satisfied for the others only when the demand functions for the products are identical. Note that although one would expect the own and cross price effects to be similar for almost perfect complements ($-b \approx a_1 \approx a_2$), the price intercepts can take any values depending on the characteristics of the products.

In conclusion, our analysis for complementary products shows that b is a much better measure of complementarity than β. It needs to be used with caution, though, for strong complements, i.e. when $-b \approx a_1 \approx a_2$, since it may lead to unrealistically high prices and sales, particularly for price decision models, and to significantly different sales of products with distant price intercepts for quantity competition models.

3. Capacity Investment Decisions under Uncertainty

In this section, we address the capacity investment problem of a monopolist that faces uncertain demand for two products, and analyze the impact of product substitutability on the firm's decisions. For that purpose, we use the b-measure, since it was shown to be a more appropriate measure of substitutability for linear demand functions. We consider a two-stage decision problem that has been extensively studied in the operations management literature under various assumptions (Fine and Freund (1990), Chod and Rudi (2005), Goyal and Netessine (2005, 2007), Biller et al. (2006), Bish and Suwandechochai (2005,2006). In the first stage, the firm needs to determine the levels of flexible and dedicated capacities to install under high demand uncertainty. The uncertainty in demand at this stage is modeled by assuming random price/demand intercepts, as done in much of the previous work (e.g., Fine and Freund (1990), Van Mieghem and Dada (1999)). In the second stage, demand is realized and profit maximizing prices and production quantities are determined. While the firm's decisions are made in two stages, we formulate and solve the two stages simultaneously using a scenario-based stochastic programming approach as in Fine and Freund (1990) and Biller et al. (2006). Our goal is to understand how product substitution impacts the optimal mix of dedicated and flexible capacity the firm should invest in.

Flexible capacity has been shown to be a very valuable tool for the firm to balance supply and demand, and thus increase its profits. The gains stem from the ability to shift production either to the product with higher demand (Fine and Freund (1990)) or to that with higher premium (Van Mieghem (1998)). As a result, it is highly valuable when the correlation between the demands or the prices of the products is low. As products become closer substitutes (as *b* increases), however, customers become more sensitive to changes in product prices. Furthermore, a unit increase in the price of one product results in a larger portion of demand diverted to the other product, since *b/a* increases. Hence, a smaller increase in a product's price results in a larger portion of demand shifted to its substitute. In other words, it becomes inexpensive to shift demand from one product to the other through pricing. Thus, the benefits of using flexible (costly) capacity to shift production to the product with higher market potential are significantly reduced. Consequently, for high levels of substitution, the value of flexibility lies in the ability to shift production to the higher premium product, the product with higher price intercept. Observe, however, that as the products become less differentiated they should command similar prices in the marketplace and, having similar premiums, the benefits of capacity flexibility are very limited. Our computational work fully supports these arguments and shows that the need for flexibility diminishes as the products become less differentiated, unless we assume that the correlation between price intercepts remains low.

3.1 Model and Assumptions

Our two-stage planning model can be described as follows. The price and demand intercepts, (A_1, A_2) and $(\varepsilon_1, \varepsilon_2)$ respectively, are assumed to be discrete random variables where $\varepsilon_i = (a_j A_i - bA_j)/(a_i a_j - b^2)$, $i, j = 1,2$, $j \neq i$, and b is defined as the measure of product substitutability. At the time of the capacity investment decisions we assume that the monopolist expects S possible price/demand realizations or scenarios, denoted by A_{is} and ε_{is}, i=1,2, s=1,…,S, each with an occurrence probability q_s with $\sum_{s=1}^{S} q_s = 1$. We also assume that the possible price intercepts satisfy the conditions $a_j A_{is} - bA_{js} > 0$ for $i, j = 1,2$, $j \neq i$ and $s = 1,\ldots,S$, to ensure positive demand intercepts. In this first stage, the firm decides the level of dedicated capacity for product i, K_i, i=1,2, and the level of flexible capacity, K_f, given unit investment costs c_i, i=1,2,f, where $c_1, c_2 < c_f$ and $c_f < c_1 + c_2$. After observing the realized scenario s at the second stage, the firm decides on profit maximizing quantities, Q_{is}, which also determine the optimal prices, P_{is}, for the products.

Using this notation, the two-stage decision problem can be formulated as a single problem as follows:

$$\underset{\substack{Q_{1s}, Q_{2s}, s=1,\ldots,S \\ K_1, K_2, K_f}}{Max} \quad \sum_{s=1}^{S} q_s \left[Q_{1s} \cdot \left(A_{1s} - a_1 Q_{1s} - bQ_{2s} \right) + Q_{2s} \cdot \left(A_{2s} - a_2 Q_{2s} - bQ_{1s} \right) \right] - \sum_{i=1,2,f} c_i K_i$$

subject to

$$0 \leq Q_{1s} \leq K_1 + K_f \qquad \forall s \qquad (1)$$

$$0 \leq Q_{2s} \leq K_2 + K_f \qquad \forall s \qquad (2)$$

$$Q_{1s} + Q_{2s} \leq K_1 + K_2 + K_f \qquad \forall s \qquad (3)$$

$$A_{1s} - a_1 Q_{1s} - bQ_{2s} \geq 0 \qquad \forall s \qquad (4)$$

$$A_{2s} - a_2 Q_{2s} - bQ_{1s} \geq 0 \qquad \forall s \qquad (5)$$

$$K_1, K_2, K_f \geq 0 \qquad (6)$$

This is a concave quadratic problem that can be easily solved with off-the-shelf solvers such as CPLEX. In the following section we derive properties of the optimal prices and production quantities analytically. In Section 3.3 we carry out extensive numerical experiments to determine the effect of increased product substitutability on the optimal investments in flexible and dedicated capacities, since it is difficult to find closed form solutions to the decision variables.

3.2 Properties of the Optimal Solution

The effect of product substitutability on expected product prices and production quantities can be derived from the Kuhn-Tucker optimality conditions and is given in the Theorem below.

Theorem 1: Let K_1^*, K_2^* and K_f^* denote the optimal dedicated and flexible capacity investments and P_i^{s*}, Q_i^{s*}, for *i*=1,2, the optimal prices and production levels under each demand scenario s, for *s*=1,2,…*S*, in the solution to the stochastic program. Let $E\left[P_i^*\right]$ and $E\left[Q_i^*\right]$ denote the associated optimal expected price and production quantity and $E\left[A_i\right]$ the mean price intercept for product *i*, *i*=1,2.

1. If the firm makes an investment to produce both products, i.e. either $K_1^*, K_2^* > 0$ or $K_f^* > 0$, then

$$E\left[P_i^*\right] = \frac{E\left[A_i\right] + \tilde{c}_i}{2} \qquad \text{for } i=1,2$$

$$E\left[Q_i^*\right] = \frac{a_j E\left[A_i\right] - bE\left[A_j\right] - a_j \tilde{c}_i + b\tilde{c}_j}{2(a_i a_j - b^2)} \qquad \text{for } i,j=1,2 \text{ and } i \neq j$$

where $\tilde{c}_i = c_i$ if $K_i^* > 0$ and $\tilde{c}_i = c_f$ if $K_i^* = 0$.

2. If the firm invests in a single product, say product *i*, then the expected price and production for this product are $E\left[P_i^*\right]=\frac{E\left[A_i\right]+c_i}{2}$ and $E\left[Q_i^*\right]=\frac{E\left[A_i\right]-c_i}{2a_i}$, respectively.

See Appendix for the proof.

The interesting case is the one in which at optimality the firm invests in production capacity, dedicated or flexible, for both products; this will typically be the case in practice. If we further assume that both products are always manufactured by the firm (as in assumption (A1) of Goyal and Netessine (2007)), then clearly the optimal investment in dedicated capacities will be positive and it follows from the above theorem that the *expected prices and production quantities do not depend on the cost of flexible capacity. Flexibility simply allows the firm to offer the same expected prices and quantities with a lower investment cost* (see Lus (2008) for numerical illustrations). Furthermore, the expected price for each product only depends on the mean price intercept and the unit cost of dedicated capacity, and increases with them. Thus, the optimal expected price for product *i* does not depend on the parameters a_i and *b* of the inverse demand functions, and hence on the level of product substitution. Note that the unconstrained optimal prices ($P_i = A_i/2$) for the monopoly model do not depend on these parameters. The average optimal prices for the capacity investment problem exhibit similar behavior, with an additional term proportional to the unit cost of capacity. The production quantity, however, is affected by all parameters except the cost of flexible capacity. The optimal expected production quantity increases with the mean price intercept of the product and decreases with that of the other product. Similarly, it decreases as the unit cost of dedicated capacity for the product increases while it increases with the unit capacity cost of the other product. Finally, we should point out that the expected prices and production quantities do not depend on the variability in demand or the correlation between the demand/price intercepts.

Chod and Rudi (2005, Proposition 8) also show that the expected output prices are unaffected by resource flexibility, but in a very different setting where the firm invests in a single technology, the costs of flexible and dedicated capacity are assumed identical and the solution is restricted to an approximate clearance strategy (capacity must be fully utilized and the non-negativity of the production quantities is ignored).

3.3 Computational Study

Our computational study shows that the impact of product substitutability on the optimal flexible capacity requirements depends heavily on the underlying assumptions on the distribution of the random product demand and price intercepts, and how they change as b increases. For this purpose, we study two extreme cases: in Section 3.3.2 (3.3.3) we assume that the correlation of the demand (price) intercepts remains constant as substitutability increases. Before that, in the next section, we characterize the relationship between the correlations of demand and price intercepts and discuss the merits of each of the two extreme cases to capture reality.

3.3.1 Relationship between Demand and Price Intercept Distributions

The following theorem characterizes the relationship between the correlations of demand and price intercepts.

Theorem 2: Assume that the price and demand intercepts of the linear demand functions, (A_1, A_2) and $(\varepsilon_1, \varepsilon_2)$, respectively, with $(\varepsilon_1, \varepsilon_2) = \left(\frac{a_2 A_1 - bA_2}{a_1 a_2 - b^2}, \frac{a_1 A_2 - bA_1}{a_1 a_2 - b^2} \right)$, are random variables.

The following relations hold between the demand and price intercepts:

1) If $|Corr(\varepsilon_1, \varepsilon_2)| = 1$ or $|Corr(A_1, A_2)| = 1$, we have $Corr(\varepsilon_1, \varepsilon_2) = Corr(A_1, A_2)$ for all levels of product substitutability.
2) As products become more substitutable, if the correlation between the demand intercepts is kept constant, the coefficient of variation for the two individual demand intercepts identical and also constant, and $|Corr(\varepsilon_1, \varepsilon_2)| \neq 1$, then the correlation of the price intercepts increases to 1.
3) As products become more substitutable, if the correlation between the price intercepts and the variances of the individual price intercepts are kept constant, and $|Corr(A_1, A_2)| \neq 1$, then the correlation of the demand intercepts decreases to -1.

See Appendix for the proof.

Note that for the case of constant correlation between the demand intercepts in 2), we require equal coefficients of variation of the two product demand intercepts. Extensive computational experiments show that the result holds in general, except when the coefficient of variation of one product is drastically larger than the other (in the order of ten times).

In the first part of the computational study we assume that the correlation between the demand intercepts is kept constant as the products become closer substitutes; this constant correlation can take any value from -1 to 1. This is a reasonable assumption since any level of correlation seems to be possible in practice between the demand intercepts for all levels of product substitutability. For example, consider the extreme case where the products are perfect substitutes and equally priced. In this case, the total demand of the two products, which we can refer to as the demand for the general commodity, depends only on that one price. In a particular setting, the correlation among the demand intercepts of the two products could be very negative if the demand curve for the general commodity is fairly well known but there is high uncertainty as to which of the two substitutable products consumers will choose. In a different setting, the two products might be identical in the eyes of the consumer and in that case the market potentials would be perfectly positively correlated. Furthermore, under the assumption of constant correlation of demand intercepts as products become more substitutable, the correlation between the price intercepts increases to 1 (Theorem 2). This makes sense at an intuitive level since one would expect the maximum possible prices for two substitutable products to become more similar as the degree of substitution increases and equal when the products are perfect substitutes.

The above argument also suggests that it may not be appropriate to fix the correlation of the price intercepts to study the impact of increasing *b*, at least for high levels of *b*. Theorem 2 shows that, in that case, as *b* increases the demand intercepts become more and more negatively correlated, which does not seem practical in general. For completeness we also study the case of fixed correlation of the price intercepts, see Section 3.3.3, in order to understand how the results in Goyal and Netessine (2007) extend to the case where investments in a mix of dedicated and flexible technology are possible.

In general, both demand and price intercept correlations may be affected by the product substitutability parameter. Such cases are beyond the scope of our study, since they require the specification of the intercept distributions as functions of *b*, for which we have no practical data/evidence.

3.3.2 Constant Correlation of Demand Intercepts

For the computational study, we consider 100 possible future demand scenarios, which are combinations of 10 scenarios obtained from discretizing a normal distribution with a given mean and standard deviation, as in Biller et al. (2006). We let $c_1 = c_2 = 4$, $c_f = 4.2$, $a_1 = a_2 = 0.02$, $E[A_1] = E[A_2] = 50$, and let $b \geq 0$ represent the level of substitution.

In the first case, we assume that the correlation of the demand intercepts ε_1 and ε_2 is fixed, and generate demand scenarios from a multivariate normal distribution $(\varepsilon_1, \varepsilon_2)$ with mean $E[\varepsilon_i] = \frac{a_j E[A_i] - bE[A_j]}{a_i a_j - b^2}$, coefficient of variation 10% i.e. $\sqrt{Var[\varepsilon_i]} = 0.10E[\varepsilon_i]$ for i=1,2, $i \neq j$, and the given correlation, $Corr(\varepsilon_1, \varepsilon_2) = \rho$. As noted in Biller et al. (2006), 10% variability of the demand intercepts in a linear demand model translates into a much higher demand coefficient of variation once you set the price to a realistic value.

Table 5 shows the optimal prices, production quantities, capacity investment levels and the resulting profits for the given example with $\rho = 0$. We see that as products become closer substitutes, the production levels and total capacity investment decrease, due to a reduction in total market size, and so do profits. Furthermore, investment in flexible capacity becomes less attractive as products become more substitutable because of the increased correlation between the maximum prices a customer is willing to pay for the products. As a result, not as much can be gained from shifting production to the higher premium product.

Table 5: The effect of *b* on optimal decisions and profits; demand intercepts are independently distributed with coefficient of variation 10%

b	Expected Price	Expected Production	Dedicated Capacity	Flexible Capacity	Total Capacity	Profits	$Cor(A_1, A_2)$
0	27	1150	1044	221	2309	53,098	0
0.004	27	958	881	161	1923	44,184	0.38
0.008	27	821	767	116	1650	37,834	0.69
0.012	27	719	685	74	1444	33,082	0.88
0.016	27	639	632	20	1284	29,395	0.98
0.0196	27	581	583	0	1166	26,722	1.00

Numerical experiments to study the sensitivity of the investment decisions to demand variability and correlation (Table 6), and to the cost of a unit of dedicated capacity, *c*, and the relative cost of flexible capacity, c_f/c, (Lus (2008)) show the robustness of our conclusions and lead to the following results: (1) Consistent with the previous literature, the total capacity investment and the profits of the firm increase with demand variability and flexible capacity becomes more beneficial (Chod and Rudi (2005), Goyal and Netessine (2005, 2007)). (2) Similar to the results of Chod and Rudi (2005), while the total capacity investment increases with the correlation between the demand intercepts, flexible capacity becomes less valuable. (3) The firm's profits tend to decrease as the correlation of the demand intercepts increases. This is because flexible capacity cannot be used effectively to reduce investment costs. (4) For high levels of substitutability, however, profits increase with the correlation between the demand intercepts. This is because capacity flexibility has little value due to the high correlation of the price intercepts; hence, the firm mainly gains from the increased probability of higher realizations of both product demands associated with the increased correlation between them. (5) The study of the sensitivity to the cost ratio shows that increasing the cost of flexibility induces the firm to invest in higher total capacity but less flexible capacity in order to achieve the same average prices and production levels with lower profits (a clear consequence of Theorem 1).

Table 6: The effect of b on optimal decisions and profits for different levels of demand variability and correlation between demand intercepts (DC: Dedicated Capacity, FC: Flexible Capacity, TC: Total Capacity, $\rho = Corr(\varepsilon_1, \varepsilon_2)$, $\gamma = Corr(A_1, A_2)$)

		Coefficient of Variation for Demand Intercepts																
		5%				10%				15%				20%				
ρ	b	DC	FC	TC	Profits	DC	FC	TC	Profits	DC	FC	TC	Profits	DC	FC	TC	Profits	γ
-0.5	0	1095	109	2300	52,956	1017	267	2302	53,174	944	428	2315	53,573	874	588	2337	54,171	-0.50
	0.004	920	77	1917	44,109	860	199	1918	44,222	801	327	1929	44,439	744	460	1947	44,776	-0.14
	0.008	796	51	1643	37,796	749	146	1644	37,851	702	250	1653	37,968	658	352	1669	38,157	0.29
	0.012	706	26	1438	33,065	671	98	1439	33,087	634	178	1447	33,144	600	260	1460	33,246	0.68
	0.016	639	0	1278	29,389	621	38	1279	29,392	597	93	1286	29,413	574	150	1298	29,461	0.93
	0.0196	581	0	1162	26,717	581	0	1163	26,718	584	0	1169	26,727	590	0	1179	26,752	≈1.00
0	0	1106	89	2300	52,937	1044	221	2309	53,098	989	356	2335	53,424	946	484	2377	53,945	0
	0.004	928	60	1917	44,100	881	161	1924	44,184	839	268	1946	44,372	805	371	1980	44,688	0.38
	0.008	802	38	1643	37,792	767	116	1649	37,834	735	198	1668	37,945	709	279	1697	38,145	0.69
	0.012	711	16	1438	33,063	685	74	1443	33,082	663	134	1459	33,147	643	199	1485	33,275	0.88
	0.016	639	0	1278	29,389	632	20	1283	29,395	619	59	1297	29,432	610	100	1320	29,515	0.98
	0.0196	581	0	1162	26,717	583	0	1166	26,722	589	0	1179	26,749	599	0	1199	26,812	≈1.00
0.5	0	1120	59	2300	52,918	1084	148	2316	53,018	1056	244	2355	53,264	1038	337	2412	53,692	0.50
	0.004	939	38	1917	44,090	911	108	1930	44,146	893	177	1962	44,304	882	246	2010	44,593	0.74
	0.008	811	21	1643	37,788	792	71	1654	37,819	778	127	1682	37,925	770	182	1722	38,133	0.88
	0.012	716	5	1438	33,063	704	40	1447	33,080	697	79	1472	33,155	696	115	1507	33,311	0.96
	0.016	639	0	1278	29,389	643	0	1287	29,400	645	18	1309	29,455	650	41	1340	29,577	0.99
	0.0196	581	0	1162	26,717	585	0	1169	26,726	594	0	1188	26,774	608	0	1216	26,878	≈1.00

Table 7: The effect of b on optimal decisions and profits for different levels of demand variability and correlation between price intercepts (DC: Dedicated Capacity, FC: Flexible Capacity, TC: Total Capacity, $\rho = Corr(\varepsilon_1, \varepsilon_2)$, $\gamma = Corr(A_1, A_2)$)

		Coefficient of Variation for Price Intercepts																
		5%				10%				15%				20%				
γ	b	DC	FC	TC	Profits	DC	FC	TC	Profits	DC	FC	TC	Profits	DC	FC	TC	Profits	ρ
-0.5	0	1095	109	2300	52,956	1017	267	2302	53,174	944	428	2315	53,573	874	588	2337	54,171	-0.50
	0.004	890	136	1917	44,153	792	334	1918	44,425	696	536	1929	44,917	604	740	1948	45,644	-0.74
	0.008	730	182	1643	37,879	600	445	1644	38,240	468	718	1654	38,889	338	993	1670	39,839	-0.88
	0.012	582	273	1438	33,202	385	668	1439	33,743	NA	NA	NA	NA	NA	NA	NA	NA	-0.96
	0.016	366	546	1278	29,669	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	-0.99
0	0	1106	89	2300	52,937	1044	221	2309	53,098	989	356	2335	53,424	946	484	2377	53,945	0
	0.004	903	111	1917	44,130	824	277	1924	44,327	748	449	1946	44,713	684	613	1981	45,310	-0.38
	0.008	748	148	1643	37,848	639	370	1649	38,107	533	603	1668	38,599	435	829	1699	39,346	-0.69
	0.012	608	222	1438	33,156	443	558	1443	33,541	NA	NA	NA	NA	NA	NA	NA	NA	-0.88
	0.016	417	443	1278	29,576	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	-0.98
0.5	0	1120	59	2300	52,918	1084	148	2316	53,018	1056	244	2355	53,264	1038	337	2412	53,692	0.50
	0.004	921	74	1917	44,106	872	186	1930	44,223	827	309	1963	44,491	789	432	2011	44,939	0.14
	0.008	772	99	1643	37,816	702	250	1655	37,965	631	422	1683	38,284	566	591	1724	38,798	-0.29
	0.012	645	148	1438	33,107	534	381	1448	33,323	NA	NA	NA	NA	NA	NA	NA	NA	-0.68
	0.016	491	296	1278	29,479	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	-0.93

3.3.3 Constant Correlation of Price Intercepts

We now assume that the correlation of the price intercepts A_1 and A_2 is fixed, and generate demand scenarios from a multivariate normal distribution (A_1, A_2) with mean $E[A_i] = 50$ and coefficient of variation 10%, i.e. $\sqrt{Var[A_i]} = 0.10E[A_i]$, for $i = 1,2$, and the given correlation, $Corr(A_1, A_2) = \gamma$. The values of the other parameters are taken as given in the previous section. In this case, we only consider *b* values for which all scenarios are feasible (i.e. $a_j A_{is} - bA_{js} > 0$ for all *s*=1,…,100, *i*=1,2, $i \neq j$) in order to analyze the sole effect of *b*, and report NA on the others. The results given in Table 7 show that the need for flexible capacity increases as products become more substitutable, in agreement with the results in Goyal and Netessine (2005,2007). We observe that as the parameter *b* increases the market potentials become more negatively correlated and the flexible capacity becomes more economically attractive, because it enables the firm to reap handsome profits by satisfying the most popular and profitable product.

Interestingly, the optimal total capacities reported in Table 6 and Table 7 are very similar for the same level of correlation between demand intercepts (Table 6) or price intercepts (Table 7). This is because for symmetric cases, such as the ones tested, the variance of total demand is identical in both cases and so is the amount of total capacity required to achieve the same expected production levels and expected prices. When the correlation of price intercepts is fixed, this is achieved by using capacity flexibility effectively since demand intercepts tend to be negatively correlated. When the correlation of demand intercepts is fixed, on the other hand, this is achieved almost entirely with dedicated capacity by shifting demand from one product to the other through pricing. Not surprisingly, higher profits are reaped in the case of fixing price intercept correlation as a result of effectively using a larger amount of flexible capacity to achieve those levels. In asymmetric cases, on the other hand, the optimal total capacity levels are generally different.

The results shown for substitutable products can easily be extended to include complements by considering the negative counterpart values of b. In that case, the need for flexibility diminishes as the products become stronger complements when we keep the correlation between price intercepts constant (which in this case makes sense since the prices of complementary products don't need to be related) and increases when that between demand intercepts is kept constant (which is questionable since one would expect perfect complements to have perfectly positively correlated demands). As a result, we can conclude that *the need for flexible capacity is generally reduced as the products are more closely related, either as substitutes or complements.* We should emphasize again, though, that the model has serious drawbacks when modeling complementarity since the production quantities and profits increase unboundedly as products become stronger complements.

Finally, we should point out that when β is used as a measure of product substitutability, then all prices, production quantities and the total capacity investment increase as products become more substitutable (see Biller et al. (2006) and Bish and Suwandechochai (2006)). Product prices become significantly higher than the unit capacity investment costs and additional capacity is acquired to ensure that the fat margins are not missed. The increase in total capacity is fueled by the higher premiums. The need for flexible capacity, however, depends on the underlying demand assumptions, as when using the b-measure. Again, when the correlation between the demand intercepts is kept constant, the optimal amount of flexible capacity decreases as the products become less differentiated; in this case, though, the investments in dedicated capacities grow. When the correlation between price intercepts is fixed, total capacity is increased through higher investments in flexible capacity while the amount of dedicated capacity is reduced as products become closer substitutes.

3.4 Optimal Decisions under Cournot Competition

In this section we analyze how the decisions of a manufacturing firm change when it has competitors contending in both markets. If there are other firms competing for the customers of the two substitutable products, then the market price for each product becomes a function of the total production quantities of the products, i.e. $P_i = A_i - a_i Q_i^T - bQ_j^T$, $i, j = 1,2$ and $i \neq j$,where Q_i^T denotes the sum of the production quantities of all the firms in the market for product *i*. When Cournot competition is considered, then each firm decides on the profit maximizing quantities and capacity investment levels given the production quantities of the rival firms. The optimization problem of a certain firm under Cournot competition can be formulated as follows:

$$\underset{\substack{Q_{1s},Q_{2s},s=1,\ldots,S\\K_1,K_2,K_f}}{Max} \left\{ \begin{array}{c} \sum_{s=1}^{S} q_s \left[Q_{1s}\left(A_{1s} - a_1\left(Q_{1s}+Q_{1s}^r\right) - b\left(Q_{2s}+Q_{2s}^r\right)\right) + Q_{2s}\left(A_{2s} - a_2\left(Q_{2s}+Q_{2s}^r\right) - b\left(Q_{1s}+Q_{1s}^r\right)\right)\right] \\ - \sum_{i=1,2,f} c_i K_i \end{array} \right\}$$

subject to the constraints (1)-(6). Note that in the objective function Q_{is} denotes the firm's production quantity for product *i* under scenario *s*, and Q_{is}^r denotes the total production of its competitors.

The following theorem states the relation between the optimal decisions of a firm for monopolistic and competitive settings.

Theorem 3: Assume that there are *n* identical firms competing in both markets. Let $E\left[\hat{P}_i^*\right]$ and $E\left[\hat{Q}_i^*\right]$ denote the optimal expected price and production quantity for product *i*, *i*=1,2, and

$\hat{K}_1^*$, $\hat{K}_2^*$ and $\hat{K}_f^*$ denote the optimal dedicated and flexible capacity investments, for each firm under Cournot competition. Let $E\left[P_i^*\right]$ and $E\left[Q_i^*\right]$ denote the optimal expected price and production quantity for product i, i=1,2, in a monopoly setting as given in Theorem1, and K_1^*, K_2^* and K_f^* denote the optimal capacity investments of the monopolist. The following relations hold between the optimal decisions under monopolistic and competitive settings:

1. $E\left[\hat{Q}_i^*\right]=\dfrac{2}{n+1}E\left[Q_i^*\right]$ for i=1,2
2. $E\left[\hat{P}_i^*\right]=\dfrac{2nE\left[P_i^*\right]-(n-1)E\left[A_i\right]}{n+1}$ for i=1,2
3. $\hat{K}_i^*=\dfrac{2}{n+1}K_i^*$ for i=1,2,f

See Appendix for the proof.

The Theorem shows that while the expected production quantity and capacity investment levels of a firm decrease by 100(n-1)/(n+1) % under competition, the system-wide production and the total capacity of the system increases by 100(n-1)/(n+1)%. The results in Goyal and Netessine (2005, 2007) for two competing firms lead to similar conclusions for the problem of investing in a single technology. Our results further demonstrate that the optimal ratio of flexible to dedicated capacity of the firm is the same for both monopoly and competition settings. Based on Theorem 3, one can also conclude that the results obtained on the impact of product substitutability, the correlation between the demand and price intercepts, and the demand variability on the optimal decisions of a monopolist also hold for an oligopolist under Cournot competition.

4. Conclusions

In conclusion, we show that if linear demand functions are of the form $Q_i = \varepsilon_i - \alpha_i P_i + \beta P_j$, studying the effect of product substitutability/complementarity by varying the parameter β leads to highly unrealistic results. There is, however, an alternative parameterization of the linear demand functions that provides reasonable inputs to strategic decision models. The key is to choose the right dependence structure among the linear coefficients, as in the case where $\varepsilon_i = \left(a_j A_i - bA_j\right) / \left(a_i a_j - b^2\right)$, $\alpha_i = a_j / \left(a_i a_j - b^2\right)$, $\beta_i = b / \left(a_i a_j - b^2\right)$. Using these relations between the coefficients of the linear demand functions, we show that the study of the parameter b as the measure of product substitutability/complementarity yields results in line with our intuition.

Using parameter b as the measure of product substitutability, we find that the total capacity investment decreases with product substitutability, as a result of the reduction in market size. The optimal amount of flexible capacity, however, may decrease or increase depending on the underlying assumptions on the distribution of the demand and price potentials and how their correlations change as the products become closer substitutes. This can be explained as follows. Recall that the value of capacity flexibility lays in the ability to better balance supply and demand (Fine and Freund (1990)) and/or shift production to the more profitable product (Van Mieghem (1998)). As the level of product substitutability increases, the price potentials (intercepts) for the two products will tend to be highly correlated, reducing the benefit associated with shifting production to the higher margin product. At the same time, demand for the substitutes can be easily managed through pricing, which reduces the need for flexible capacity to balance supply and demand. Consequently, the flexible capacity requirements decrease with product substitutability.

In contrast, if the correlation between the price potentials of the two products is assumed constant, as in Goyal and Netessine (2007), then the correlation of the demand potentials decreases and converges to -1 as the products become closer substitutes, making the capacity flexibility increasingly more effective and thus resulting in increased flexibility investments. This theory unifies and explains the contradicting results found in the literature regarding whether or not the value of flexibility increases with product substitutability.

In addition, our analysis shows that when a monopolist invests in a mix of dedicated and flexible capacity, the expected optimal prices and production quantities do not depend on the cost of flexible capacity. Flexibility simply allows the firm to offer the same expected prices and quantities with a lower investment cost. Furthermore, only the expected demand/price potentials, and not their variance or correlation, affect the optimal expected prices and production quantities. Interestingly, the level of substitution does not affect the expected prices of the two products, but only their expected production levels. These results can be extended to symmetric firms competing in the markets for both substitutable products.

Numerous extensions to our study deserve further attention. It would be interesting to understand how the results change when firms are not risk neutral and when using more complex, accurate models of demand. In particular, new demand models need to be developed to capture the effect of product complementarity, and validated with market research studies. In addition, the relationship between the correlation of the demand and price potentials of the two products and the level of product substitutability in practice needs to be tested and better understood.

Another important extension would be to incorporate product substitutability considerations into the design of product-plant manufacturing networks where a number of plants produces multiple products and the firm needs to decide the amounts of dedicated and flexible capacities to invest in, as well as the specific products that will share each of the flexible resources (see e.g. Jordan and Graves (1995), Lus and Muriel (2008a)). Finally, firms should understand the impact of not appropriately modeling product substitution on their capacity and flexibility investment decisions (see Lus and Muriel (2008b)).

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Appendix

Example 1 (Unconstrained Price/Quantity Decision Models): The purpose of this example is to illustrate the large magnitude of the effects discussed in Section 2. We use the deterministic version of the representative scenario described in Section 3.

Table A1: Optimal prices, production and profits for substitutable products (β–measure)

MODELS	β	P_1	P_2	Q_1	Q_2	Π_1	Π_2
Monopoly	0	25	25	1250	1250	31,250	31,250
	10	31.25	31.25	1250	1250	39,063	39,063
	20	41.67	41.67	1250	1250	52,083	52,083
	30	62.50	62.50	1250	1250	78,125	78,125
	40	125	125	1250	1250	156,250	156,250
	49	1250	1250	1250	1250	1,562,500	1,562,500
Bertrand	0	25	25	1250	1250	31,250	31,250
	10	27.78	27.78	1389	1389	38,580	38,580
	20	31.25	31.25	1563	1563	48,828	48,828
	30	35.71	35.71	1786	1786	63,776	63,776
	40	41.67	41.67	2083	2083	86,806	86,806
	49	49.02	49.02	2451	2451	120,146	120,146
Price Stackelberg	0	25	25	1250	1250	31,250	31,250
	10	28.06	27.81	1375	1390	38,584	38,659
	20	32.61	31.52	1500	1576	48,913	49,681
	30	39.63	36.89	1625	1845	64,405	68,045
	40	51.47	45.59	1750	2279	90,074	103,914
	49	71.66	60.11	1863	3006	133,471	180,687
Cournot	0	25	25	1250	1250	31,250	31,250
	10	28.41	28.41	1364	1364	38,740	38,740
	20	34.72	34.72	1458	1458	50,637	50,637
	30	48.08	48.08	1538	1538	73,964	73,964
	40	89.29	89.29	1607	1607	143,495	143,495
	49	838.93	838.93	1661	1661	1,393,518	1,393,518
Quantity Stackelberg	0	25	25	1250	1250	31,250	31,250
	10	28.13	28.38	1378	1362	38,744	38,661
	20	33.33	34.42	1522	1446	50,725	49,760
	30	43.75	46.49	1707	1488	74,695	69,174
	40	75.00	80.88	1985	1456	148,897	117,755
	49	637.50	649.05	2428	1285	1,548,065	834,101

Table A2: Optimal prices, production and profits for substitutable products (*b*–measure)

MODELS	b	P_1	P_2	Q_1	Q_2	Π_1	Π_2
Monopoly	0	25	25	1250	1250	31,250	31,250
	0.004	25	25	1042	1042	26,042	26,042
	0.008	25	25	893	893	22,321	22,321
	0.012	25	25	781	781	19,531	19,531
	0.016	25	25	694	694	17,361	17,361
	0.0196	25	25	631	631	15,783	15,783
Bertrand	0	25	25	1250	1250	31,250	31,250
	0.004	22.22	22.22	1157	1157	25,720	25,720
	0.008	18.75	18.75	1116	1116	20,926	20,926
	0.012	14.29	14.29	1116	1116	15,944	15,944
	0.016	8.33	8.33	1157	1157	9,645	9,645
	0.0196	0.98	0.98	1238	1238	1,214	1,214
Price Stackelberg	0	25	25	1250	1250	31,250	31,250
	0.004	22.45	22.24	1146	1159	25,723	25,773
	0.008	19.57	18.91	1071	1126	20,963	21,292
	0.012	15.85	14.76	1016	1153	16,101	17,011
	0.016	10.29	9.12	972	1266	10,008	11,546
	0.0196	1.43	1.20	941	1518	1,348	1,825
Cournot	0	25	25	1250	1250	31,250	31,250
	0.004	22.73	22.73	1136	1136	25,826	25,826
	0.008	20.83	20.83	1042	1042	21,701	21,701
	0.012	19.23	19.23	962	962	18,491	18,491
	0.016	17.86	17.86	893	893	15,944	15,944
	0.0196	16.78	16.78	839	839	14,076	14,076
Quantity Stackelberg	0	25	25	1250	1250	31,250	31,250
	0.004	22.50	22.70	1148	1135	25,829	25,774
	0.008	20.00	20.65	1087	1033	21,739	21,326
	0.012	17.50	18.60	1067	930	18,674	17,293
	0.016	15.00	16.18	1103	809	16,544	13,084
	0.0196	12.75	12.98	1226	649	15,637	8,425

Table A3: Optimal prices, production and profits for complementary products (β –measure)

MODELS	$-\beta$	P_1	P_2	Q_1	Q_2	Π_1	Π_2
Monopoly	0	25	25	1250	1250	31,250	31,250
	10	20.83	20.83	1250	1250	26,042	26,042
	20	17.86	17.86	1250	1250	22,321	22,321
	30	15.63	15.63	1250	1250	19,531	19,531
	40	13.89	13.89	1250	1250	17,361	17,361
	49	12.63	12.63	1250	1250	15,783	15,783
Bertrand	0	25	25	1250	1250	31,250	31,250
	10	22.73	22.73	1136	1136	25,826	25,826
	20	20.83	20.83	1042	1042	21,701	21,701
	30	19.23	19.23	962	962	18,491	18,491
	40	17.86	17.86	893	893	15,944	15,944
	49	16.78	16.78	839	839	14,076	14,076
Price Stackelberg	0	25	25	1250	1250	31,250	31,250
	10	22.96	22.70	1125	1135	25,829	25,774
	20	21.74	20.65	1000	1033	21,739	21,326
	30	21.34	18.60	875	930	18,674	17,293
	40	22.06	16.18	750	809	16,544	13,084
	49	24.53	12.98	638	649	15,637	8,425
Cournot	0	25	25	1250	1250	31,250	31,250
	10	23.15	23.15	1111	1111	25,720	25,720
	20	22.32	22.32	938	938	20,926	20,926
	30	22.32	22.32	714	714	15,944	15,944
	40	23.15	23.15	417	417	9,645	9,645
	49	24.76	24.76	49	49	1,214	1,214
Quantity Stackelberg	0	25	25	1250	1250	31,250	31,250
	10	22.92	23.17	1122	1112	25,723	25,773
	20	21.43	22.52	978	946	20,963	21,292
	30	20.31	23.06	793	738	16,101	17,011
	40	19.44	25.33	515	456	10,008	11,546
	49	18.81	30.36	72	60	1,348	1,825

Table A4: Optimal prices, production and profits for complementary products (b–measure)

MODELS	b	P_1	P_2	Q_1	Q_2	Π_1	Π_2
Monopoly	0	25	25	1250	1250	31,250	31,250
	0.004	25	25	1563	1563	39,063	39,063
	0.008	25	25	2083	2083	52,083	52,083
	0.012	25	25	3125	3125	78,125	78,125
	0.016	25	25	6250	6250	156,250	156,250
	0.0196	25	25	62500	62500	1,562,500	1,562,500
Bertrand	0	25	25	1250	1250	31,250	31,250
	0.004	27.27	27.27	1420	1420	38,740	38,740
	0.008	29.17	29.17	1736	1736	50,637	50,637
	0.012	30.77	30.77	2404	2404	73,964	73,964
	0.016	32.14	32.14	4464	4464	143,495	143,495
	0.0196	33.22	33.22	41946	41946	1,393,518	1,393,518
Price Stackelberg	0	25	25	1250	1250	31,250	31,250
	0.004	27.55	27.24	1406	1419	38,744	38,661
	0.008	30.43	28.91	1667	1721	50,725	49,760
	0.012	34.15	29.76	2188	2325	74,695	69,174
	0.016	39.71	29.12	3750	4044	148,897	117,755
	0.0196	48.57	25.70	31875	32452	1,548,065	834,101
Cournot	0	25	25	1250	1250	31,250	31,250
	0.004	27.78	27.78	1389	1389	38,580	38,580
	0.008	31.25	31.25	1563	1563	48,828	48,828
	0.012	35.71	35.71	1786	1786	63,776	63,776
	0.016	41.67	41.67	2083	2083	86,806	86,806
	0.0196	49.02	49.02	2451	2451	120,146	120,146
Quantity Stackelberg	0	25	25	1250	1250	31,250	31,250
	0.004	27.50	27.81	1403	1390	38,584	38,659
	0.008	30.00	31.52	1630	1576	48,913	49,681
	0.012	32.50	36.89	1982	1845	64,405	68,045
	0.016	35.00	45.59	2574	2279	90,074	103,914
	0.0196	37.25	60.11	3583	3006	133,471	180,687

Proof of Theorem 1:

We will study the model without the non-negativity constraints for the prices and production quantities, and show that the optimal solutions to these variables are always positive.

$$Max \quad \sum_{s=1}^{S} q_s \left[Q_{1s} \cdot \left(A_{1s} - a_1 Q_{1s} - b Q_{2s} \right) + Q_{2s} \cdot \left(A_{2s} - a_2 Q_{2s} - b Q_{1s} \right) \right] - \sum_{i=1,2,f} c_i K_i$$

subject to

$$Q_{1s} \le K_1 + K_f \qquad \forall s \qquad (u_{1s})$$

$$Q_{2s} \le K_2 + K_f \qquad \forall s \qquad (u_{2s})$$

$$Q_{1s} + Q_{2s} \le K_1 + K_2 + K_f \qquad \forall s \qquad (v_s)$$

$$K_1, K_2, K_f \ge 0 \qquad (t_1, t_2, t_f)$$

The objective function is concave; hence the first order KKT conditions (1)-(11) given below are sufficient for optimality.

$$q_s A_{1s} - 2a_1 q_s Q_{1s} - 2bq_s Q_{2s} = u_{1s} + v_s \qquad \forall s \qquad (7)$$

$$q_s A_{2s} - 2a_2 q_s Q_{2s} - 2bq_s Q_{1s} = u_{2s} + v_s \qquad \forall s \qquad (8)$$

$$c_1 = \sum_{s=1}^{S} (u_{1s} + v_s) + t_1 \qquad (9)$$

$$c_2 = \sum_{s=1}^{S} (u_{2s} + v_s) + t_2 \qquad (10)$$

$$c_f = \sum_{s=1}^{S} (u_{1s} + u_{2s} + v_s) + t_f \qquad (11)$$

$$(Q_{1s} - K_1 - K_f) u_{1s} = 0 \qquad \forall s \qquad (12)$$

$$(Q_{2s} - K_2 - K_f) u_{2s} = 0 \qquad \forall s \qquad (13)$$

$$(Q_{1s} + Q_{2s} - K_1 - K_2 - K_f) v_s = 0 \qquad \forall s \qquad (14)$$

$$t_1 K_1 = 0, \quad t_2 K_2 = 0, \quad t_f K_f = 0 \qquad (15,16,17)$$

From equations (7) and (8), the optimal prices and production quantities for product *i* for scenario *s* can be derived as

$$P_{is} = A_{is} - a_i Q_{is} - bQ_{js} = \frac{q_s A_{is} + u_{is} + v_s}{2q_s} \qquad \text{for } i, j = 1,2, \ i \neq j$$

$$Q_{is} = \frac{q_s\left(a_j A_{is} - bA_{js}\right) - a_j u_{is} + bu_{js} - \left(a_j - b\right)v_s}{2q_s\left(a_i a_j - b^2\right)} \qquad \text{for } i, j = 1,2, \ i \neq j$$

Observe that $P_{is} \geq 0$ for all *i*=1,2 and *s*=1,…,*S*. We will now show that Q_{is} is also positive for any given *i* and *s*. Assume that $Q_{is} < K_i$, then clearly $Q_{1s} + Q_{2s} < K_1 + K_2 + K_3$. So we have $u_{is} = v_s = 0$ and $Q_{is} = \dfrac{q_s\left(a_j A_{is} - bA_{js}\right) + bu_{js}}{2q_s\left(a_i a_j - b^2\right)} \geq 0$.

The expected price for product *i* can be calculated as follows:

$$E\left[P_i\right] = \sum_{s=1}^{S} q_s P_{is} = \sum_{s=1}^{S} q_s \frac{q_s A_{is} + u_{is} + v_s}{2q_s} = \frac{\sum_{i=1}^{S} q_s A_{is} + \sum_{i=1}^{S}\left(u_{is} + v_s\right)}{2}$$

By using equations (9) and (10) we get

$$E\left[P_i\right] = \frac{E\left[A_i\right] + \left(c_i - t_i\right)}{2} \text{ for } i=1,2$$

The expected production quantity for product *i* can be calculated similarly as

$$E\left[Q_i\right] = \sum_{s=1}^{S} q_s Q_{is} = \sum_{s=1}^{S} q_s \frac{q_s\left(a_j A_{is} - bA_{js}\right) - a_j u_{is} + bu_{js} - \left(a_j - b\right)v_s}{2q_s\left(a_i a_j - b^2\right)}$$

$$= \frac{\sum_{i=1}^{S} q_s\left(a_j A_{is} - bA_{js}\right) - a_j \sum_{i=1}^{S}\left(u_{is} + v_s\right) + b\sum_{i=1}^{S}\left(u_{js} + v_s\right)}{2\left(a_i a_j - b^2\right)}$$

By using equations (9) and (10) we get

$$E\left[Q_i\right] = \frac{a_j E\left[A_i\right] - bE\left[A_j\right] - a_j\left(c_i - t_i\right) + b\left(c_j - t_j\right)}{2\left(a_i a_j - b^2\right)} \text{ for } i,j=1,2 \text{ and } i \neq j$$

1) If $K_1^*, K_2^* > 0$ then $t_1 = t_2 = 0$ and the expected price and production quantity for product *i* becomes equal to $E[P_i] = \dfrac{E[A_i]+c_i}{2}$ and $E[Q_i] = \dfrac{a_j E[A_i] - bE[A_j] - a_j c_i + bc_j}{2(a_i a_j - b^2)}$ for *i*, *j*=1,2. The expected production quantities can easily be obtained using expected prices since they are linearly related.

Assume that $K_1^* = 0$ ($K_2^* = 0$) and $K_f^* > 0$. Then, at optimality, the second (first) set of constraints, with associated multipliers u_{2s} (u_{1s}), is redundant. Hence the expected prices and production quantities for this case can be easily obtained by letting $u_{2s} = 0$ ($u_{1s} = 0$) for all $s = 1, \ldots, S$, which omits the second (first) set of constraints from the model, and $t_f = 0$ since $K_f^* > 0$. This leads to the expression $c_f = c_1 - t_1$ ($c_f = c_2 - t_2$) using equations (9)-(11).

2) If the firm invests in a single product only, say product *i*, then clearly $K_i^* > 0$, $K_j^* = K_f^* = 0$ and $E[Q_j] = 0$ ($i \neq j$). Since $K_i^* > 0$ we have $t_i = 0$ and $E[P_i] = \dfrac{E[A_i]+c_i}{2}$. The expected production quantity for product *i* can easily be obtained from the inverse demand function of product *i*. □

<u>Proof of Theorem 2:</u> 1) This can be directly observed from the relations between the correlation of the demand intercepts and the one of the price intercepts given in parts 2 and 3 below.

2) Based on the relation between the price and demand intercepts, that is, $A_i = a_i \varepsilon_i + b\varepsilon_j$ for *i*,*j*=1,2 and $i \neq j$, the correlation between the price intercepts can be calculated as follows:

$$Corr(A_1, A_2) = \frac{Cov(A_1, A_2)}{\sigma_{A_1}\sigma_{A_2}} = \frac{Cov(a_1\varepsilon_1 + b\varepsilon_2, a_2\varepsilon_2 + b\varepsilon_1)}{\sigma_{a_1\varepsilon_1 + b\varepsilon_2}\sigma_{a_2\varepsilon_2 + b\varepsilon_1}}$$

$$= \frac{a_1 b Cov(\varepsilon_1, \varepsilon_1) + a_2 b Cov(\varepsilon_2, \varepsilon_2) + (a_1 a_2 + b^2) Cov(\varepsilon_1, \varepsilon_2)}{\sqrt{a_1^2 Var\varepsilon_1 + b^2 Var\varepsilon_2 + 2a_1 b Cov(\varepsilon_1, \varepsilon_2)} \sqrt{a_2^2 Var\varepsilon_2 + b^2 Var\varepsilon_1 + 2a_2 b Cov(\varepsilon_1, \varepsilon_2)}}$$
$$= \frac{a_1 b Var\varepsilon_1 + a_2 b Var\varepsilon_2 + (a_1 a_2 + b^2) Cov(\varepsilon_1, \varepsilon_2)}{\sqrt{a_1^2 Var\varepsilon_1 + b^2 Var\varepsilon_2 + 2a_1 b Cov(\varepsilon_1, \varepsilon_2)} \sqrt{a_2^2 Var\varepsilon_2 + b^2 Var\varepsilon_1 + 2a_2 b Cov(\varepsilon_1, \varepsilon_2)}}$$

Note that $\sigma_{\varepsilon_i} = CV \cdot \mu_{\varepsilon_i}$ and $Cov(\varepsilon_1, \varepsilon_2) = \rho \sigma_{\varepsilon_1} \sigma_{\varepsilon_2}$ where μ_{ε_i} and σ_{ε_i} represent the mean and standard deviation of the marginal distribution of ε_i, respectively, and CV and ρ represent the coefficient of variation of individual demand intercepts and correlation coefficient between the demand intercepts, respectively. Observe that $\mu_{\varepsilon_i} = \dfrac{a_j \mu_{A_i} - b\mu_{A_j}}{a_i a_j - b^2}$ for $i, j = 1, 2,\ i \neq j$. Letting $E = (a_2 \mu_{A_1} - b\mu_{A_2}) > 0$ and $F = (a_1 \mu_{A_2} - b\mu_{A_1}) > 0$, we have

$$Corr(A_1, A_2) = \frac{a_1 b E^2 + a_2 b F^2 + \rho (a_1 a_2 + b^2) EF}{\sqrt{a_1^2 E^2 + b^2 F^2 + 2a_1 b \rho EF} \sqrt{a_2^2 F^2 + b^2 E^2 + 2a_2 b \rho EF}}$$

The partial derivation of $Corr(A_1, A_2)$ with respect to the parameter *b* can be calculated as

$$\frac{\partial Corr(A_1, A_2)}{\partial b} = K \frac{b(a_1 a_2 - b^2)^2 (a_2 \mu_{A_1}^2 - a_1 \mu_{A_2}^2)^2 + EF\left[(a_1 a_2 + b^2)(a_1 E^2 + a_2 F^2) + 4a_1 a_2 b \rho EF\right]}{(a_1^2 E^2 + b^2 F^2 + 2a_1 b \rho EF)^{3/2} (a_2^2 F^2 + b^2 E^2 + 2a_2 b \rho EF)^{3/2}}$$

where $K = (a_1 a_2 - b^2)(1 - \rho^2) EF > 0$.

Observe that

$$(a_1 a_2 + b^2)(a_1 E^2 + a_2 F^2) + 4a_1 a_2 b \rho EF \geq (a_1 a_2 + b^2)(a_1 E^2 + a_2 F^2) - 4a_1 a_2 b EF \quad \text{since } \rho > -1 \text{ and } b > 0$$
$$= a_2 (a_1 E - bF)^2 + a_1 (a_2 F - bE)^2 \geq 0$$

$\Rightarrow \dfrac{\partial Corr(A_1, A_2)}{\partial b} > 0$, hence the correlation between the price intercepts increases with the product substitutability parameter *b*. Furthermore, in the case of perfect substitutes, that is, when $a_1 = a_2 = b$, $Corr(A_1, A_2) = 1$.

3) Similarly,

$$Corr(\varepsilon_1,\varepsilon_2)=\frac{Cov(\varepsilon_1,\varepsilon_2)}{\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}}=\frac{Cov\left(\frac{a_2A_1-bA_2}{a_1a_2-b^2},\frac{a_1A_2-bA_1}{a_1a_2-b^2}\right)}{\sigma_{\frac{a_2A_1-bA_2}{a_1a_2-b^2}}\sigma_{\frac{a_1A_2-bA_1}{a_1a_2-b^2}}}=\frac{Cov(a_2A_1-bA_2,a_1A_2-bA_1)}{\sigma_{a_2A_1-bA_2}\sigma_{a_1A_2-bA_1}}$$

$$=\frac{-a_2bVarA_1-a_1bVarA_2+(a_1a_2+b^2)Cov(A_1,A_2)}{\sqrt{a_2^2VarA_1+b^2VarA_2-2a_2bCov(A_1,A_2)}\sqrt{a_1^2VarA_2+b^2VarA_1-2a_1bCov(A_1,A_2)}}$$

$$\frac{\partial Corr(\varepsilon_1,\varepsilon_2)}{\partial b}=\frac{-(a_1a_2-b^2)^3(a_1Var\varepsilon_1+a_2Var\varepsilon_2)\left(VarA_1\cdot VarA_2-(Cov(A_1,A_2))^2\right)}{\left(a_2^2VarA_1+b^2VarA_2-2a_2bCov(A_1,A_2)\right)^{3/2}\left(a_1^2VarA_2+b^2VarA_1-2a_1bCov(A_1,A_2)\right)^{3/2}}\leq 0$$

Thus, the correlation between the demand intercepts decreases with *b*. In this case, when the products are perfect substitutes we have that $Corr(\varepsilon_1,\varepsilon_2)=-1$ □

Proof of Theorem 3: Since the *n* firms are identical, their optimal decisions should also be identical. Using this fact, we can solve the capacity investment problem for one of the firms under the assumption that all others will behave in exactly the same way.

The KKT conditions for the model with the above objective function are the same, except for equations (7) and (8) which become

$$q_sA_{1s}-2a_1q_sQ_{1s}-a_1q_sQ_{1s}^r-bq_sQ_{2s}^r-2bq_sQ_{2s}^1=u_{1s}+v_s \qquad \forall s \qquad (7')$$

$$q_sA_{2s}-2a_2q_sQ_{2s}-a_2q_sQ_{2s}^r-bq_sQ_{1s}^r-2bq_sQ_{1s}^1=u_{1s}+v_s \qquad \forall s \qquad (8')$$

Under the assumption of symmetric firms, the production quantities of the firms for each product under each scenario will be equivalent at optimality. Thus, we have $Q_{is}^r=(n-1)Q_{is}$ for i=1,2, and equations ($7'$) and ($8'$) reduce to

$$q_sA_{1s}-(n+1)a_1q_sQ_{1s}-(n+1)bq_sQ_{2s}=u_{1s}+v_s \qquad \forall s \qquad (7'')$$

$$q_sA_{2s}-(n+1)a_2q_sQ_{2s}-(n+1)bq_sQ_{1s}=u_{1s}+v_s \qquad \forall s \qquad (8'')$$

Observe that if $\left(Q_{1s},Q_{2s},u_{1s},u_{2s},v_s\right)=\left(Q_{1s}^*,Q_{2s}^*,u_{1s}^*,u_{2s}^*,v_s^*\right)$ for all $s=1,\ldots,S$ and $\left(K_1,K_2,K_f,t_1,t_2,t_f\right)=\left(K_1^*,K_2^*,K_f^*,t_1^*,t_2^*,t_f^*\right)$ satisfy the KKT conditions (7)-(17) of the model for the monopoly setting, then $\left(Q_{1s},Q_{2s},u_{1s},u_{2s},v_s\right)=\left(\frac{2Q_{1s}^*}{n+1},\frac{2Q_{2s}^*}{n+1},u_{1s}^*,u_{2s}^*,v_s^*\right)$ for all $s=1,\ldots,S$ and $\left(K_1,K_2,K_f,t_1,t_2,t_f\right)=\left(\frac{2K_1^*}{n+1},\frac{2K_2^*}{n+1},\frac{2K_f^*}{n+1},t_1^*,t_2^*,t_f^*\right)$ satisfy the KKT conditions ($7''$), ($8''$), (9)-(17) of the model for the competitive setting, and the results follow.

Comparison with McGuire and Staelin (1983)

McGuire and Staelin (1983) propose a different parameterization of linear demand functions, defined as follows:

$$Q_1=\mu\delta\left[1-\frac{\gamma}{1-\theta}P_1+\frac{\gamma\theta}{1-\theta}P_2\right]$$

$$Q_2=\left(1-\mu\right)\delta\left[1+\frac{\gamma\theta}{1-\theta}P_1-\frac{\gamma}{1-\theta}P_2\right]$$

where $0\le\mu\le 1$, $0\le\theta<1$, $\gamma>0$ and $\delta>0$; μ and θ are relative product preference and substitutability parameters, respectively; δ is a scale factor which is equal to industry demand when both prices are zero. For these demand functions, observe that the customers become more sensitive to changes in prices when products become more substitutable (similar to the effect of b) and total market size is not affected by product substitutability (similar to the effect of β).

To insure that the industry demand cannot increase with increases in either price, the parameters μ and θ should satisfy the following relation

$$\frac{\theta}{1+\theta}\le\mu\le\frac{1}{1+\theta}.$$

Table A5: Optimal prices, production and profits (McGuire&Staelin)

MODELS	θ	P_1	P_2	Q_1	Q_2	Π_1	Π_2
Monopoly	0	25	25	1250	1250	31,250	31,250
	0.2	25	25	1250	1250	31,250	31,250
	0.4	25	25	1250	1250	31,250	31,250
	0.6	25	25	1250	1250	31,250	31,250
	0.8	25	25	1250	1250	31,250	31,250
	0.99	25	25	1250	1250	31,250	31,250
Bertrand	0	25	25	1250	1250	31,250	31,250
	0.2	22.22	22.22	1389	1389	30,864	30,864
	0.4	18.75	18.75	1563	1563	29,297	29,297
	0.6	14.29	14.29	1786	1786	25,510	25,510
	0.8	8.33	8.33	2083	2083	17,361	17,361
	0.99	0.50	0.50	2475	2475	1,225	1,225
Price Stackelberg	0	25	25	1250	1250	31,250	31,250
	0.2	22.45	22.24	1375	1390	30,867	30,927
	0.4	19.57	18.91	1500	1576	29,348	29,809
	0.6	15.85	14.76	1625	1845	25,762	27,218
	0.8	10.29	9.12	1750	2279	18,015	20,783
	0.99	0.73	0.61	1869	3064	1,370	1,878
Cournot	0	25	25	1250	1250	31,250	31,250
	0.2	22.73	22.73	1364	1364	30,992	30,992
	0.4	20.83	20.83	1458	1458	30,382	30,382
	0.6	19.23	19.23	1538	1538	29,586	29,586
	0.8	17.86	17.86	1607	1607	28,699	28,699
	0.99	16.72	16.72	1664	1664	27,824	27,824
Quantity Stackelberg	0	25	25	1250	1250	31,250	31,250
	0.2	22.50	22.70	1378	1362	30,995	30,929
	0.4	20.00	20.65	1522	1446	30,435	29,856
	0.6	17.50	18.60	1707	1488	29,878	27,670
	0.8	15.00	16.18	1985	1456	29,779	23,551
	0.99	12.63	12.75	2463	1268	31,100	16,163

The optimal prices, production quantities and profits for the price and quantity decision models we considered can be easily obtained in closed form by doing the appropriate substitutions in Tables 1 and 3. Table A5 presents the optimal prices, production levels and profits for a simple example under the demand functions defined by McGuire and Staelin (1983) with $\mu = 1/2$, $\delta = 5000$ and $\gamma = 0.02$, for different levels of product substitutability. The results for the monopoly case show that an increase in product substitutability does not affect the optimal prices and production levels, and hence the profits of the firm. These results are not surprising since the total system demand for both products remains the same ($Q_1 + Q_2 = \delta\left[2 - \gamma P_1 - \gamma P_2\right]/2$ for $\mu = 1/2$) as products become closer substitutes. For the competitive setting, the prices decrease with product substitutability while the total demand significantly increases. These results seem to be plausible when the market size for both products does not vary as products become more substitutable, which is questionable in many practical settings.

Now we study the optimal capacity investments under the above parameterization of linear demand models. Similar to our study in Section 3, we assume randomness in demand intercepts, which in this case requires considering both μ and δ as random variables. Observe that changes in μ or δ affect the rates of change of quantities with respect to prices, but not own-price and cross-price elasticity. We consider 100 possible scenarios for (μ, δ), i.e. 100 scenarios for the demand intercepts, that correspond to the scenarios generated for demand intercepts when *b*=0 in the numerical example in Section 3, to allow for comparison; i.e., for each scenario $(\varepsilon_{1s}, \varepsilon_{2s})$ considered, the pair (μ_s, δ_s) is derived from the equations $\varepsilon_{1s} = \mu_s \delta_s$ and $\varepsilon_{2s} = (1 - \mu_s)\delta_s$. Here, we only consider the case where demand intercepts are independently distributed, i.e. $Corr(\varepsilon_1, \varepsilon_2) = 0$, and study four levels of product substitutability for which all scenarios are feasible (for higher values of θ some scenarios become infeasible because of the relation between μ and θ given above). The stochastic programming model given in Section 3 can be modified for this case by taking the observed demand functions for products 1 and 2 for scenario s as $Q_{1s} = \mu_s \delta_s \left[1 - \frac{\gamma}{1-\theta} P_{1s} + \frac{\gamma\theta}{1-\theta} P_{2s}\right]$ and $Q_{2s} = (1 - \mu_s)\delta_s \left[1 + \frac{\gamma\theta}{1-\theta} P_{1s} - \frac{\gamma}{1-\theta} P_{2s}\right]$, respectively.

Table A6: Results for Demand Functions defined by McGuire and Staelin ($\gamma = 0.02$)

θ	Expected Price	Expected Production	Dedicated Capacity	Flexible Capacity	Total Capacity	Profits
0	27	1145	1051	197	2298	52,651
0.2	27	1145	1033	233	2298	52,649
0.4	27	1146	1001	297	2299	52,663
0.6	27	1147	936	430	2302	52,706

From the results given in Table A6 we see that for all levels of product substitutability the expected product prices, production levels, total capacity and profits are not significantly different. Note that for this parameterization of linear demand functions the decrease in total demand as a result of a unit increase in the price of the low demand product (the product with a lower demand intercept) is lower for high values of θ. Hence the firm becomes more inclined to shift production from the low demand product to the high demand product, which makes flexible capacity more favorable as the products become closer substitutes. These conclusions are shown to be robust in a larger numerical study; see Lus (2008).