BOUNDARY LAYER ANALYSIS WITH
NAVIER-STOKES EQUATION
IN 2D CHANNEL FLOW

Yunho Jang
Department of Mechanical and Industrial Engineering
Univ. of Massachusetts

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• Introduction

• Solution for Laminar and Turbulent Channel Flow

• Results

• Conclusion
Introduction

• The plane channel or duct flows
  – A canonical configuration for studying internal flows
  – Boundary layer analysis
  – The behavior of laminar and turbulence flow
  – Turbulence Model

• Numerical Method issue
  – Compare the Finite Element Method with other numerical methods
    (Finite Volume Method, Direct Numerical Simulation)
Solution for Laminar Channel Flow

- Laminar channel flow
  - Domain: \( h = 2 \text{m}, \ L = 100 \text{m} \)
  - Grid \( h \times L = 60 \times 200 \)
  - Mean velocity \( = 30 \text{m/s} \)
  - Kinematic viscosity \( = 0.01111 \text{kg/m} - \text{s} \), density \( = 1 \text{kg/m}^3 \)
    (incompressible)
  
  \[
  \text{Re}_\tau = \frac{u_\tau \delta}{\nu} = 90
  \]
  
  where \( u_\tau = \sqrt{\frac{\tau_w}{\rho}} \), and \( \delta \) is \( h/2 \)
  
  - Compare with Analytical solution for both Methods (FEM,FVM)
Solution for Laminar Channel Flow

- For steady state, two dimensional Navier-Stokes equation

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  \hspace{1cm} (2)

\[
u \frac{\partial u}{\partial x} + \rho \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]  \hspace{1cm} (3)

\[
u \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]  \hspace{1cm} (4)
Solution for Laminar Channel Flow

• Horizontal velocity

\[ u(y) = -u_{max} \left[ 1 - 4\left(\frac{y}{h}\right) \right] \]  
\[ u_{max} = \frac{h^2 \, dp}{8 \mu \, dx} \]  

• Shear stress

\[ \tau_{yx} = \mu \frac{du}{dy} = -8\mu u_{max} \frac{y}{h^2} \]

• Vorticity

\[ \omega_{yx} = -\frac{du}{dy} = 8u_{max} \frac{y}{h^2} \]
Solution for Turbulent Channel Flow

- Turbulent channel flow
  - Same domain and grid with laminar case
  - Mean velocity $= 18.4539\text{m/s}$
  - Kinematic viscosity $= 0.001695\text{kg/m} - \text{s}$, density $= 1\text{kg/m}^3$ (incompressible)

$$Re_{\tau} = \frac{u_{\tau}\delta}{\nu} = 590 \quad (9)$$

- Compare results from FEA with both Methods (FVM,DNS)
Solution for Turbulent Channel Flow

- $k - \epsilon$ model

\[ \mu_t = \rho C_\mu \frac{k^2}{\epsilon} \]  \hspace{1cm} (10)

- The standard wall function

\[ u^+ = \frac{1}{\kappa} \ln(Ey^+) \]  \hspace{1cm} (11)

where

\[ u^+ \equiv \frac{U_p C_\mu^{1/4} k_p^{1/2}}{\tau_w / \rho} \]  \hspace{1cm} (12)

\[ y^+ \equiv \frac{\rho C_\mu^{1/4} k_p^{1/2} y_p}{\mu} \]  \hspace{1cm} (13)
## Results of Laminar Channel Flow

**Table 1: Values from the laminar solutions**

<table>
<thead>
<tr>
<th>Method</th>
<th>$u_{max}$ (m/s)</th>
<th>$\tau_w$ (pascal)</th>
<th>$\omega_w$ ($s^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic solution</td>
<td>45</td>
<td>1.0</td>
<td>-90</td>
</tr>
<tr>
<td>FVM</td>
<td>44.9313</td>
<td>0.987</td>
<td>-88.8</td>
</tr>
<tr>
<td>FEA</td>
<td>44.248</td>
<td>0.935</td>
<td>-84.144</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>$C_f$</th>
<th>$C_{fo}$</th>
<th>$\delta^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic solution</td>
<td>0.00222</td>
<td>0.000987</td>
<td>0.333</td>
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<tr>
<td>FVM</td>
<td>0.00219</td>
<td>0.000974</td>
<td>0.356</td>
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<tr>
<td>FEA</td>
<td>0.00207</td>
<td>0.000923</td>
<td>0.338</td>
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</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>$\theta$</th>
<th>$Re_m$</th>
<th>$Re_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic solution</td>
<td>0.1333</td>
<td>5400</td>
<td>4050</td>
</tr>
<tr>
<td>FVM</td>
<td>0.1343</td>
<td>5400</td>
<td>4043</td>
</tr>
<tr>
<td>FEA</td>
<td>0.1338</td>
<td>5400</td>
<td>3982</td>
</tr>
</tbody>
</table>
Results of Laminar Channel Flow

- Velocity profiles in fully developed laminar channel flow at $X/L = 1$
Results of Laminar Channel Flow

• Shear stress profiles
Results of Laminar Channel Flow

• Vorticity profiles
Results of Turbulent Channel Flow

- Velocity profiles in fully developed turbulent channel flow at \( X/L = 1 \)
Results of Turbulent Channel Flow

- Shear stress profiles
Results of Turbulent Channel Flow

- Vorticity profiles
Results of Turbulent Channel Flow

- Mean velocity profiles
Conclusion

• Finite Element Method works well for laminar channel flow as well as Finite Volume Method.

• In turbulent flow, there are some errors from using turbulent modeling.

• The prediction of FEA for turbulent channel flow is believable.

• However, we need to develop more exact turbulent model for boundary layer flows.