From Morrison's Book
Understanding Rheology

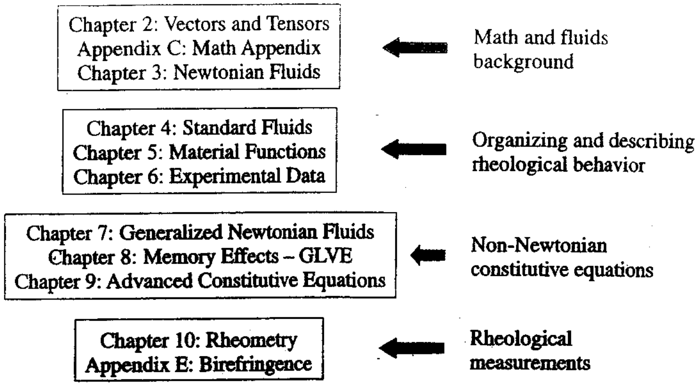
# Vector and Tensor Operations

We will focus our discussion of rheology on isothermal flows. The isothermal flow behavior of either conventional or complex fluids is determined by two physical laws, mass conservation and momentum conservation, plus the stress constitutive equation, a relationship that describes how a fluid responds to stress or to deformation. Mass conservation is a scalar equation, momentum conservation is a vector equation, and the constitutive equation is an equation of still higher mathematical complexity, a tensor equation.

To make fluid mechanics comprehensible to third-year chemical and mechanical engineering students, the vector and tensor nature of the subject is often given a light treatment in introductory courses. In these classes, emphasis lies with solving the vector momentum equation in the form of three scalar equations (conveniently tabulated in several coordinate systems). To relate the shear stress and the velocity gradient, the students incorporate the appropriate scalar component of the Newtonian constitutive equation, usually the 21-component, often called Newton's law of viscosity.

It is then straightforward to solve for the velocity and pressure fields and other quantities of interest. The scalar presentation of fluid mechanics works fine until, for example, shear-induced normal stresses are encountered or until one wishes to understand polymer die swell or memory effects. To describe such phenomena, more complex constitutive equations are required, and while these more complex constitutive equations may be expressed in scalar form, the scalar form will usually include six nontrivial equations. Furthermore, the forms of these six scalar equations will depend on the coordinate system in which the problem is written. This enormous increase in complexity can be understood and managed quite effectively if we employ the mathematical concept of a tensor. The tensor is thus a time-saving and simplifying device, and in studying rheology it is well worth the effort to learn tensor algebra. In fact, after taking the time to understand tensors, we will see that some aspects of Newtonian fluid mechanics become easier to understand and apply.

In this text we use tensor notation extensively. We assume no prior knowledge of tensors, however, and we begin in this chapter with a comprehensive review of scalars and vectors (Figure 2.1). This review is followed by the introduction of tensors and by the derivation of the conservation equations in vector/tensor format in Chapter 3. The review of Newtonian fluid mechanics in Chapter 3 provides an opportunity to work with tensors


Chapter 2: Vectors and Tensors
Appendix C: Math Appendix
Chapter 3: Newtonian Fluids

Math and fluids background

Chapter 4: Standard Fluids
Chapter 5: Material Functions
Chapter 6: Experimental Data

Organizing and describing rheological behavior

Chapter 7: Generalized Newtonian Fluids
Chapter 8: Memory Effects – GLVE
Chapter 9: Advanced Constitutive Equations

Non-Newtonian constitutive equations

Chapter 10: Rheometry
Appendix E: Birefringence

Rheological measurements

**Figure 2.1** Organizational structure of this book.

In Chapter 4 we define and discuss the characteristics of standard flows used in rheology, and in Chapter 5 we define the material functions that are used to describe non-Newtonian behavior. We also provide in Chapter 6 a summary of the observed rheological behavior for many fluids. The rest of the text is dedicated to understanding and applying several simple non-Newtonian constitutive equations.

As stated before, first we will establish a common vocabulary of mathematics and fluid mechanics on which to build an in-depth understanding of rheology. There are some tools to help the reader in the appendixes, including a detailed table of nomenclature and a glossary. We begin, then, with scalars, vectors, and tensors. Readers familiar with vector and tensor analysis and Newtonian fluid mechanics may wish to skip ahead to Chapter 4.

## 2.1 Scalars

Scalars are quantities that have magnitude only. Examples of scalars include mass, energy, density, volume, and the number of cars in a parking lot. When we do ordinary arithmetic, we are dealing with scalars. Scalars may be constants, such as $c$, the speed of light ($c = 3.0 \times 10^{10}$ cm/s), or scalars may be variables, such as your height $h(t)$ over the course of your lifetime, which is a function of time $t$, or the density of an ideal gas $\rho(T, P)$, which is a function of temperature $T$ and pressure $P$. The magnitude of a scalar has units associated with it since, for example, the numerical magnitude of your mass will be diffeent if it is expressed in kg or lbs.

Three scalars, for example, $\alpha$, $\beta$, and $\zeta$,[1] may be manipulated algebraically according to the following laws of scalar multiplication:

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| | | |
|---|---|---|
| Allowable algebraic operations of scalars with scalars | commutative law | $\alpha\beta = \beta\alpha$ |
| | associative law | $(\alpha\beta)\zeta = \alpha(\beta\zeta)$ |
| | distributive law | $\alpha(\beta + \zeta) = \alpha\beta + \alpha\zeta$ |

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Vectors are quantities that have both magnitude and direction. Examples of vectors that appear in fluid mechanics and rheology are velocity $\underline{v}$ and force $\underline{f}$. The velocity of a body is a vector because two properties are expressed: the speed at which the body is traveling (the magnitude of the velocity $|\underline{v}|$) and the direction in which the body is traveling. Likewise we can understand why force is a vector since to fully describe the force on, for example, a table (Figure 2.2), we must indicate both its magnitude $|\underline{f}|$ and the direction in which the force is applied. The same magnitude of force applied to the top of the table and to the side of the table will have different effects and must be treated differently. As with magnitudes of scalars, magnitudes of vectors have units associated with them.

In this text, most vectors will be distinguished from scalars by writing a single bar underneath vector quantities; vectors of unit length will be written without the underbar and with a caret (ˆ) over the symbol for the vector, as will be discussed. An important vector property is that both the magnitude and the direction of a vector are independent of the coordinate system in which the vector is written. We will return to this property shortly.

Since a vector has two properties associated with it, we can examine these two properties separately. The magnitude of a vector is scalar valued. The magnitude $a$ of a vector $\underline{a}$ is denoted as follows:

Vector magnitude
$$\boxed{|\underline{a}| = a} \tag{2.1}$$

The direction of a vector can be isolated by creating a new vector, such as $\hat{a}$, that points in the same direction as the original vector but has a magnitude of one. Called a *unit vector*, this is written as shown in Equation (2.2)

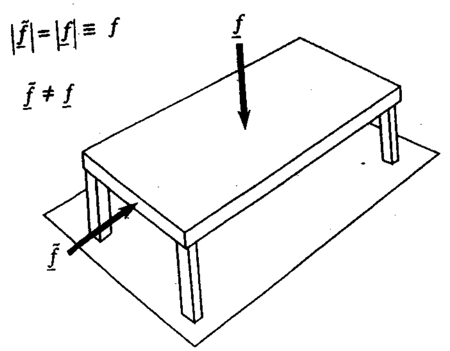


**Figure 2.2** Schematic representation of forces acting on a table. If the same magnitude of force $f$ is applied in different directions, the vectors describing those forces differ too $(\underline{f}, \tilde{f})$.

$$\hat{a} = \frac{\underline{a}}{a} \tag{2.2}$$

$$|\hat{a}| = 1 \tag{2.3}$$

As stated before, we will distinguish unit vectors from general vectors with a caret
A special vector is the zero vector $\underline{0}$, which has zero magnitude and whose direction is unspecified.

## 2.2.1 VECTOR RULES OF ALGEBRA

The rules of algebra for vectors are not the usual laws of scalar arithmetic, since when manipulating two vectors both the magnitude and the direction must be taken into account. The rules for the addition and subtraction of vectors are reviewed in Figure 2.3.

The operation of multiplication with vectors takes on several forms since vectors may be multiplied by scalars or by other vectors. Each type of multiplication has its own rules associated with it. When a scalar ($\alpha$) multiplies a vector ($\underline{a}$), it only affects the magnitude of the vector, leaving the direction unchanged,

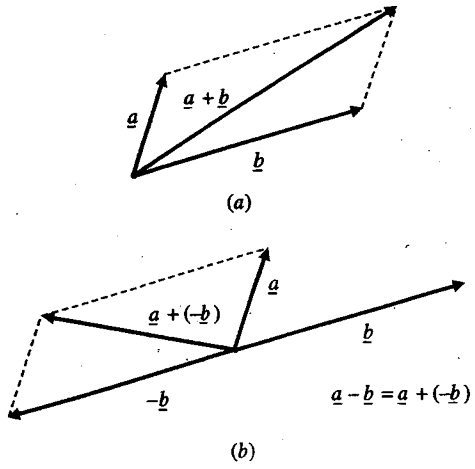
$$\underline{b} = \alpha\underline{a} \tag{2.4}$$

$$|\underline{b}| = |\alpha\underline{a}| = \alpha|\underline{a}| = \alpha a \tag{2.5}$$

$$\hat{b} = \frac{\underline{b}}{|\underline{b}|} = \frac{\alpha\underline{a}}{\alpha a} = \hat{a} \tag{2.6}$$

Since multiplication of a scalar with a vector only involves scalar quantities (the scalar $\alpha$ and the magnitude $a$), this type of multiplication has the same properties as scalar multiplication:

| | | |
|---|---|---|
| Allowable algebraic operations of scalars with vectors | commutative law | $\alpha\underline{a} = \underline{a}\alpha$ |
| | associative law | $(\alpha\underline{a})\beta = \alpha(\underline{a}\beta)$ |
| | distributive law | $\alpha(\underline{a} + \underline{b}) = \alpha\underline{a} + \alpha\underline{b}$ |



(a)

(b)

**Figure 2.3** Pictorial representation of the addition and subtraction of two vectors.

Two types of multiplication between vectors are the scalar product and the vector product. These are also called the *inner* (or *dot*) *product* and the *outer* (or *cross*) *product*. Their definitions are

$$\text{Scalar product} \quad \boxed{\underline{a} \cdot \underline{b} = ab \cos \psi} \tag{2.7}$$

$$\text{Vector product} \quad \boxed{\underline{a} \times \underline{b} = ab \sin \psi \, \hat{n}} \tag{2.8}$$

where $\hat{n}$ is a unit vector perpendicular to $\underline{a}$ and $\underline{b}$ subject to the right-hand rule (Figure 2.4), and $\psi$ is the angle between $\underline{a}$ and $\underline{b}$. When a vector $\underline{b}$ is dotted with a unit vector $\hat{a}$, the scalar product yields the projection of $\underline{b}$ in the direction of the unit vector $\hat{a}$:

$$\begin{array}{c}\text{Projection of } \underline{b} \\ \text{in direction } \hat{a}\end{array} \quad \boxed{\underline{b} \cdot \hat{a} = (b)(1) \cos \psi = b \cos \psi} \tag{2.9}$$

Also, when two vectors are perpendicular ($\psi = \pi/2$), the dot product is zero [$\cos (\pi/2) = 0$], and when two vectors are parallel ($\psi = 0$), the dot product is just the product of the magnitudes ($\cos 0 = 1$). The rules of algebra for the dot and cross products are:

$$\begin{array}{c}\text{Laws of algebra for} \\ \text{vector dot product}\end{array} \left\{ \begin{array}{ll} \text{commutative} & \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a} \\ \text{associative} & \text{not possible} \\ \text{distributive} & \underline{a} \cdot (\underline{c} + \underline{w}) = \underline{a} \cdot \underline{c} + \underline{a} \cdot \underline{w} \end{array} \right.$$

$$\begin{array}{c}\text{Laws of algebra for} \\ \text{vector cross product}\end{array} \left\{ \begin{array}{ll} \textit{not} \text{ commutative} & \underline{a} \times \underline{c} \neq \underline{c} \times \underline{a} \\ \text{associative} & (\underline{a} \times \underline{c}) \times \underline{w} = \underline{a} \times (\underline{c} \times \underline{w}) \\ \text{distributive} & \underline{a} \times (\underline{c} + \underline{w}) = \underline{a} \times \underline{c} + \underline{a} \times \underline{w} \end{array} \right.$$

Performing the dot product is a convenient way to calculate the magnitude of a vector, as shown in Equations (2.10) and (2.11)

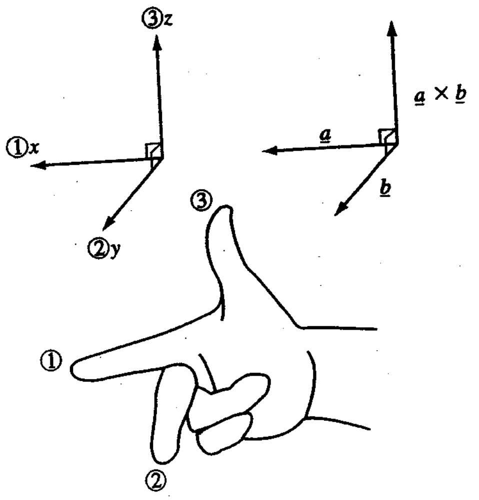


**Figure 2.4** Definition of a right-handed coordinate system.

$$\underline{a} \cdot \underline{a} = a^2 \tag{2.10}$$

and therefore

$$|\underline{a}| = \text{abs} \left(\sqrt{\underline{a} \cdot \underline{a}}\right) \tag{2.11}$$

where abs( ) denotes the absolute value of the quantity in parentheses. By convention the magnitude of a vector is taken to be positive. The graphical interpretations of the two types of vector products are shown in Figure 2.5.

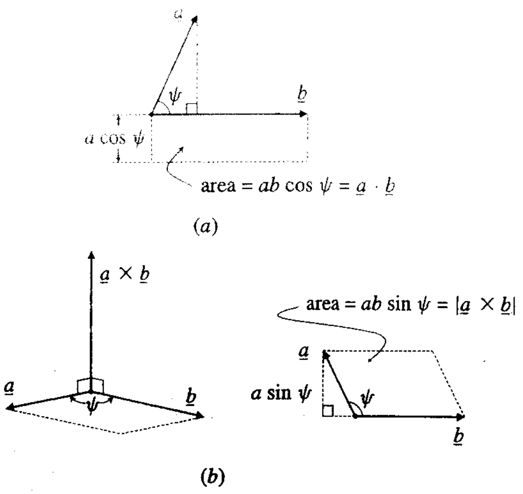


**Figure 2.5** Pictorial representation of the multiplication of two vectors. (*a*) Scalar product. (*b*) Vector product.

### 2.2.1.1 *Coordinate Systems*

When we introduced vectors we mentioned that an important vector property is that its magnitude and direction are independent of the coordinate system in which it is written. We now would like to elaborate on this concept, which will be important in understanding rheology.

Vectors, such as those that describe the forces on a body, exist independently of how we describe them mathematically. Imagine, for example, that you are leaning against a wall (Figure 2.6). Your hips are exerting a force on the wall. The vector direction in which you are exerting this force makes some angle $\psi$ with the wall. This force has a magnitude (related to your weight and the angle $\psi$), and it has a direction (related to how exactly you are positioned with respect to the wall). All of this is true despite the fact that we have yet to describe the vector force with any type of mathematical expression.

If we wish to do a calculation involving the force you are exerting on the wall, we must translate that real force into a mathematical expression that we can use in our calculations. What we would typically do is to choose some reference coordinate system, probably composed of three mutually perpendicular axes, and to write down the coefficients of the force vector in the chosen coordinate system. If we write down all the forces in a problem in the same coordinate system, we can then solve the problem.

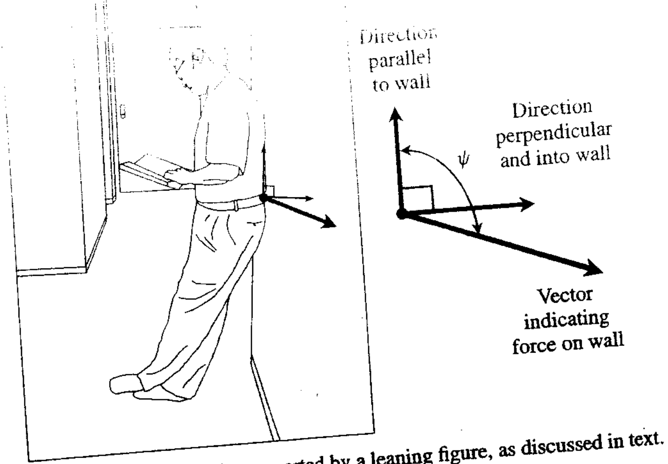


**Figure 2.6** Schematic of the vector force exerted by a leaning figure, as discussed in text.

Now we ask, what is a coordinate system? Can any vectors be chosen as the bases of the coordinate system? Must they be mutually perpendicular? Must they be unit vectors? We will answer these questions by laying down two rules for coordinate systems, also called coordinate bases (Table 2.1).

The coordinate system with which we are most familiar is the Cartesian coordinate system, usually called $\hat{i}$, $\hat{j}$, and $\hat{k}$, or alternatively $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ or $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ (Figure 2.7). In the Cartesian coordinate system the basis vectors are three mutually perpendicular unit vectors (orthonormal basis vectors), and $\hat{i} = \hat{e}_x = \hat{e}_1$ points along the $x$-axis, $\hat{j} = \hat{e}_y = \hat{e}_2$ points along the $y$-axis, and $\hat{k} = \hat{e}_z = \hat{e}_3$ points along the $z$-axis. These basis vectors are constant and therefore point in the same direction at every point in space. Although the Cartesian system is the most commonly used coordinate system, we see from the rules listed in Table 2.1 that a coordinate system need not be composed of either unit vectors or mutually perpendicular basis vectors. We will use bases in which the basis vectors are not mutually perpendicular when we consider advanced rheological constitutive equations in Chapter 9.

One requirement of all coordinate systems is that the basis vectors be noncoplanar, i.e., that the three vectors not all lie in the same plane. This requirement can be understood by

**TABLE 2.1**
**Rules for Coordinate Bases**

1. In three-dimensional space, any vector may be expressed as a linear combination of three nonzero, noncoplanar vectors, which we will call basis vectors.
2. The choice of coordinate system is arbitrary. We usually choose the coordinate system to make the problem easier to solve. The coordinate system serves as a reference system, providing both units for magnitude and reference directions for vectors and other quantities.

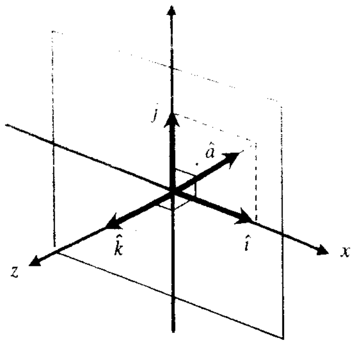


**Figure 2.7** Cartesian coordinate system ($x$, $y$, $z$) and Cartesian basis vectors ($\hat{i}$, $\hat{j}$, $\hat{k}$). The vector $\hat{a}$ is in the $xy$-plane, as discussed in text.

imagining that we choose $\hat{i}$ and $\hat{j}$ as two of our basis vectors, and then as the third basis vector we choose a vector $\hat{a} = (1/\sqrt{2})\hat{i} + (1/\sqrt{2})\hat{j}$, which is parallel to the sum of $\hat{i}$ and $\hat{j}$ (Figure 2.7). Note that $\hat{a}$ is a unit vector and that all three proposed basis vectors lie in the same plane. The problem arises when we try to express a vector such as $\hat{k}$ in our chosen coordinate system. Since $\hat{k}$ is perpendicular to all three of the vectors in our chosen coordinate system, there is no combination of $\hat{i}$, $\hat{j}$, and $\hat{a}$ that will produce $\hat{k}$.

Mathematically the requirement that the three basis vectors be noncoplanar is the same as saying that the three vectors must be linearly independent. The requirement of being linearly independent means that the linear combination of the three vectors, here $\underline{a}$, $\underline{b}$, and $\underline{c}$, can be made to be zero, that is, the vectors can be added together with scalar coefficients $\alpha$, $\beta$, and $\zeta$, such that

$$\alpha\underline{a} + \beta\underline{b} + \zeta\underline{c} = 0 \tag{2.12}$$

if and only if $\alpha = \beta = \zeta = 0$. If scalars $\alpha, \beta, \zeta$ can be found so that Equation (2.12) is satisfied but where one or more of these coefficients $(\alpha, \beta, \zeta)$ is nonzero, then $\underline{a}$, $\underline{b}$, and $\underline{c}$ are linearly dependent, coplanar, and may not form a set of basis vectors.

Once a set of appropriate basis vectors is chosen (such as $\underline{a}$, $\underline{b}$, and $\underline{c}$), we know that any vector may be expressed as a linear combination of these three vectors. This means that for an arbitrary vector $\underline{v}$ we can find three scalars $\tilde{v}_1$, $\tilde{v}_2$, and $\tilde{v}_3$, such that

$$\underline{v} = \tilde{v}_1\underline{a} + \tilde{v}_2\underline{b} + \tilde{v}_3\underline{c} \tag{2.13}$$

Note that for the chosen basis of $\underline{a}$, $\underline{b}$, $\underline{c}$, the scalar coefficients $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{v}_3$ are unique. Thus if we choose a different basis (oriented differently in space, or with different angles between the basis vectors, or composed of vectors with lengths different from those of the original basis vectors), different coefficients will be calculated. For example, for the orthonormal Cartesian-basis $\hat{e}_1 = \hat{i}$, $\hat{e}_2 = \hat{j}$, $\hat{e}_3 = \hat{k}$, $\underline{v}$ can be written as

$$\underline{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \tag{2.14}$$

and in general, $v_1 \neq \tilde{v}_1$, $v_2 \neq \tilde{v}_2$, and $v_3 \neq \tilde{v}_3$.

Write the vector that describes the force $\underline{f}$ on the string on the right in Figure 2.8a. Express
in the three different coordinate systems $\hat{e}_x, \hat{e}_y, \hat{e}_z$; $\hat{e}_{\bar{x}}, \hat{e}_{\bar{y}}, \hat{e}_{\bar{z}}$; and $\hat{e}_{\tilde{x}}, \hat{e}_{\tilde{y}}, \hat{e}_{\tilde{z}}$.

[redacted]

In the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system there are components of $\underline{f}$ in both the $x$- and $y$-directions. We can work these out in terms of the magnitude of the vector $\underline{f}$ using the usual trigonometric relations (Figure 2.9):

$$\underline{f} = f \cos \psi \, \hat{e}_x + f \sin \psi \, \hat{e}_y \tag{2.15}$$

The magnitude $f = |\underline{f}|$ can be found from a force balance between gravity affecting the mass $m$ and the upward component of $\underline{f}$ for each string,

$$2 f \sin \psi = mg \tag{2.16}$$

$$f = \frac{mg}{2 \sin \psi} \tag{2.17}$$

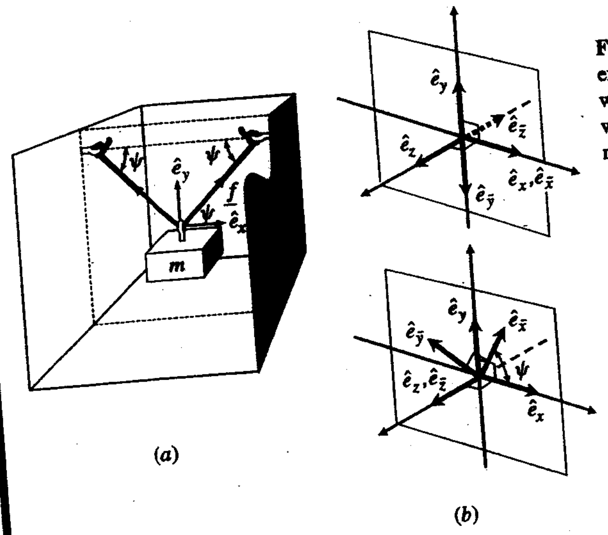


**Figure 2.8** (*a*) Schematic for example problem showing a weight hanging between two walls. (*b*) Coordinate systems referred to in example.

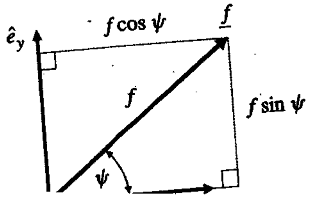


**Figure 2.9** Relationship between $\underline{f}$ and the unit vectors $\hat{e}_x$ and $\hat{e}_y$.

Therefore $\underline{f}$ may be written as

$$\underline{f} = \frac{mg}{2 \sin \psi} (\cos \psi \, \hat{e}_x + \sin \psi \, \hat{e}_y) \tag{2.18}$$

This same vector can be expressed in the $\hat{e}_{\bar{x}}, \hat{e}_{\bar{y}}, \hat{e}_{\bar{z}}$ and $\hat{e}_{\tilde{x}}, \hat{e}_{\tilde{y}}, \hat{e}_{\tilde{z}}$ coordinate systems by following the same procedure. Alternatively, we can write the new basis vectors in terms of the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ basis vectors and substitute into Equation (2.18). For example, $\hat{e}_x, \hat{e}_y$, and $\hat{e}_z$ are related to $\hat{e}_{\bar{x}}, \hat{e}_{\bar{y}}$, and $\hat{e}_{\bar{z}}$ as follows:

$$\hat{e}_x = \hat{e}_{\bar{x}} \qquad \hat{e}_y = -\hat{e}_{\bar{y}} \qquad \hat{e}_z = -\hat{e}_{\bar{z}} \tag{2.19}$$

Using either procedure, the results for the two alternate coordinate systems are

$$\underline{f} = \frac{mg}{2 \sin \psi} (\cos \psi \, \hat{e}_{\bar{x}} - \sin \psi \, \hat{e}_{\bar{y}}) \tag{2.20}$$

$$\underline{f} = \frac{mg}{2 \sin \psi} \, \hat{e}_{\tilde{x}} \tag{2.21}$$

EXAMPLE

**Express the vector force $\underline{f}$ from the previous example (Figure 2.8) in the two coordinate systems shown in Figure 2.10 (that is, with respect to the bases $\underline{a}$, $\underline{b}$, $\hat{e}_z$ and $\hat{e}_y, \hat{e}_{\bar{y}}, \hat{e}_z$). Note that $\underline{a}$ and $\underline{b}$ are not unit vectors, and $\hat{e}_y, \hat{e}_{\bar{y}}$, and $\hat{e}_z$ are not mutually perpendicular.**

[redacted]

**In the previous example we found that $\underline{f}$ could be written in the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system as**

$$\underline{f} = f \cos \psi \, \hat{e}_x + f \sin \psi \, \hat{e}_y \tag{2.22}$$

**where $f = mg/(2 \sin \psi)$. First we must express $\underline{f}$ using the following basis vectors, which are mutually orthogonal but not of unit length:**

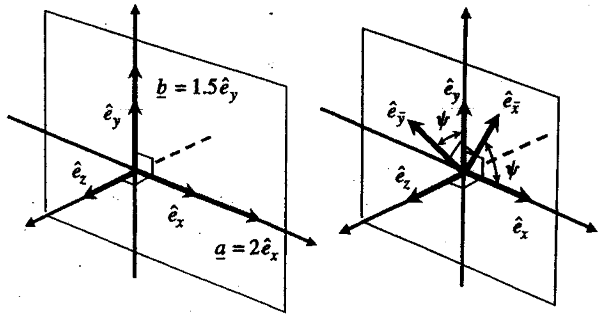


**Figure 2.10** Schematic of the two coordinate systems referred to in example problem.

[illegible]

Solving expressions (2.23) for $\hat{e}_x$ and $\hat{e}_y$ in terms of $\underline{a}$ and $\underline{b}$, we can substitute the results into Equation (2.22) and obtain the answer

$$\underline{f} = f\cos\psi\,\hat{e}_x + f\sin\psi\,\hat{e}_y \tag{2.24}$$

$$= \left(\frac{f\cos\psi}{2}\right)\underline{a} + \left(\frac{f\sin\psi}{1.5}\right)\underline{b} \tag{2.25}$$

$$= \frac{mg}{2\sin\psi}\left(\frac{\cos\psi}{2}\,\underline{a} + \frac{\sin\psi}{1.5}\,\underline{b}\right) \tag{2.26}$$

We can use the same technique to write $\underline{f}$ in terms of the second coordinate system, $\hat{e}_y, \hat{e}_{\bar{y}}, \hat{e}_z$ (see Figure 2.10), in which the basis vectors are all of unit length but not mutually orthogonal. First we must write $\hat{e}_{\bar{y}}$ in terms of $\hat{e}_x$ and $\hat{e}_y$. Then we will solve for $\hat{e}_x$ in terms of $\hat{e}_y$ and $\hat{e}_{\bar{y}}$, substitute the result into Equation (2.22), and simplify. Referring to Figure 2.10,

$$\hat{e}_{\bar{y}} = -\sin\psi\,\hat{e}_x + \cos\psi\,\hat{e}_y \tag{2.27}$$

Solving for $\hat{e}_x$ and substituting,

$$\hat{e}_x = -\frac{1}{\sin\psi}\left(\hat{e}_{\bar{y}} - \cos\psi\,\hat{e}_y\right) \tag{2.28}$$

$$\underline{f} = f\cos\psi\,\hat{e}_x + f\sin\psi\,\hat{e}_y \tag{2.29}$$

$$= -f\frac{\cos\psi}{\sin\psi}\,\hat{e}_{\bar{y}} + f\frac{\cos^2\psi}{\sin\psi}\,\hat{e}_y + f\sin\psi\,\hat{e}_y \tag{2.30}$$

$$= \frac{mg}{2\sin\psi}\left(-\cot\psi\,\hat{e}_{\bar{y}} + \sec\psi\,\hat{e}_y\right) \tag{2.31}$$

In the previous example, where we were working with orthonormal bases, it was easier to express vectors. The right angles in orthonormal bases allow us to relate vectors to their components directly, using trigonometric functions. When the basis vectors are not mutually orthogonal unit vectors (as in these last examples), we must do more work to get the final results. As we will see, vector multiplication is also much easier to carry out when the vectors are written with respect to orthonormal coordinate systems.

### 2.2.1.2 Vector Addition

We want to express vectors in a common coordinate system so that we can manipulate them. The advantage of expressing vectors in this way can be shown when we add two vectors.

Consider the addition of two vectors $\underline{v}$ and $\underline{u}$ to produce $\underline{w}$. We may write each of these vectors with respect to the Cartesian coordinate system $\hat{e}_i$ ($i = 1, 2, 3$) as follows:

$$\underline{u} = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3 \tag{2.32}$$

$$\underline{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \tag{2.33}$$

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$$\underline{w} = w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3 \tag{2.34}$$

Adding $\underline{u}$ and $\underline{v}$ and factoring out the basis vectors yields

$$\underline{w} = \underline{u} + \underline{v} = (u_1 + v_1)\hat{e}_1 + (u_2 + v_2)\hat{e}_2 + (u_3 + v_3)\hat{e}_3 \tag{2.35}$$

Thus by comparing Equations 2.35 and 2.34 we see that the coefficients of the vector $\underline{w}$ are just the sums of the coefficients of the two other vectors:

$$w_1 = u_1 + v_1 \tag{2.36}$$

$$w_2 = u_2 + v_2 \tag{2.37}$$

$$w_3 = u_3 + v_3 \tag{2.38}$$

If we know what set of basis vectors we are using, it is a bit easier not to write the basis vectors each time. Thus the Cartesian version of the vector $\underline{v}$ can be written in one of two ways—as written in Equation (2.33) or by writing just the coefficients, $v_1$, $v_2$, and $v_3$ and understanding that the Cartesian coordinate system is being used. A convenient way of writing these coefficients is in matrix form:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}_{123} \tag{2.39}$$

We write the subscript 123 on the matrix version of $\underline{v}$ to remind us what coordinate system was used to define $v_1$, $v_2$, and $v_3$. Since we are using the matrix notation only to hold the vector coefficients, it is arbitrary whether we write these vectors as column or row vectors.

MPLE

In the previous two examples we wrote the force on a string (see Figure 2.8) with respect to five different coordinate systems. Write each of these representations of the vector $\underline{f}$ in matrix notation.

[redacted]

The five different representations of the vector $\underline{f}$ are

$$\underline{f} = \begin{pmatrix} \frac{mg\cot\psi}{2} \\ \frac{mg}{2} \\ 0 \end{pmatrix}_{xyz} \tag{2.40}$$

$$\underline{f} = \begin{pmatrix} \frac{mg\cot\psi}{2} \\ -\frac{mg}{2} \\ 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} \tag{2.41}$$

$$\underline{f} = \begin{pmatrix} \frac{mg}{2\sin\psi} \\ 0 \\ 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} \tag{2.42}$$

$$\underline{f} = \begin{pmatrix} \frac{mg \cot \psi}{2} \\ \frac{mg}{2} \\ 0 \end{pmatrix}_{xyz} \quad (2.43)$$

$$\underline{f} = \begin{pmatrix} \frac{mg}{2 \sin^2 \psi} \\ \frac{-mg \cos \psi}{2 \sin^2 \psi} \\ 0 \end{pmatrix}_{y\bar{y}z} \quad (2.44)$$

We see then that the coefficients associated with a vector will vary as the vector is expressed with respect to different coordinate systems. The magnitude and direction of the vector do not change, however. To completely describe a vector, both the identity of the basis vectors and the coefficients of the vector with respect to that basis are needed.

### 2.2.1.3 *Vector Dot Product*

Taking the dot product of two vectors is particularly easy when they are written with respect to the same orthonormal basis. For an example, we can take the dot product of the vectors $\underline{v}$ and $\underline{u}$:

$$\underline{v} \cdot \underline{u} = (v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3) \cdot (u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3) \quad (2.45)$$

Using the distributive and commutative rules for the dot product we get

$$\begin{aligned} \underline{v} \cdot \underline{u} = {} & v_1u_1\hat{e}_1 \cdot \hat{e}_1 + v_2u_1\hat{e}_2 \cdot \hat{e}_1 + v_3u_1\hat{e}_3 \cdot \hat{e}_1 + v_1u_2\hat{e}_1 \cdot \hat{e}_2 \\ & + v_2u_2\hat{e}_2 \cdot \hat{e}_2 + v_3u_2\hat{e}_3 \cdot \hat{e}_2 + v_1u_3\hat{e}_1 \cdot \hat{e}_3 + v_2u_3\hat{e}_2 \cdot \hat{e}_3 + v_3u_3\hat{e}_3 \cdot \hat{e}_3 \end{aligned} \quad (2.46)$$

Because the basis vectors are orthonormal, however, when two like basis vectors are multiplied (e.g., $\hat{e}_1 \cdot \hat{e}_1$) the answer is one ($\cos 0 = 1$), whereas when two unlike unit vectors are multiplied (e.g., $\hat{e}_1 \cdot \hat{e}_2$), the answer is zero ($\cos(\pi/2) = 0$). Thus expression (2.46) simplifies to

$$\underline{v} \cdot \underline{u} = v_1u_1 + v_2u_2 + v_3u_3 \quad (2.47)$$

This answer is a scalar, as required [see Equation (2.7)], that is, none of the basis vectors appears. We see that if we know two vectors with respect to the same orthonormal basis, we can easily calculate the dot product by multiplying the components term by term and adding them.

When we introduced the dot product of two vectors, we noted that the projection of a vector in a certain direction could be found by dotting the vector with a unit vector in the desired direction. For an orthonormal basis, the basis vectors themselves are unit vectors, and we can solve for the components of a vector with respect to the orthonormal basis by taking the following dot products:

$$v_1 = \underline{v} \cdot \hat{e}_1 \quad (2.48)$$

$$v_2 = \underline{v} \cdot \hat{e}_2 \quad (2.49)$$

$$v_3 = \underline{v} \cdot \hat{e}_3 \quad (2.50)$$

This may also be confirmed by dotting Equation (2.33) with each of the unit vectors in turn and remembering that we are taking the three basis vectors $\hat{e}_i$ $(i = 1, 2, 3)$ to be mutually perpendicular and of unit length. For example,

$$\hat{e}_1 \cdot \underline{v} = \hat{e}_1 \cdot (v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3) \quad (2.51)$$

$$= v_1 \quad (2.52)$$

**EXAMPLE**

Evaluate the dot product $\underline{f} \cdot \hat{e}_x$ for the vectors $\underline{f}$ and $\hat{e}_x$ shown in Figure 2.8. First use $\underline{f}$ expressed in the $x, y, z$ coordinate system. Then calculate the same dot product first with $\underline{f}$ expressed in the $\bar{x}, \bar{y}, \bar{z}$ system and then with $\underline{f}$ expressed in the $y, \bar{y}, z$ system.

[redacted]

In $x, y, z$-coordinates the dot product is straightforward to carry out since $x, y, z$ is an orthonormal coordinate system and $\hat{e}_x$ is one of the basis vectors of this system:

$$\underline{f} \cdot \hat{e}_x = (f_x\hat{e}_x + f_y\hat{e}_y + f_z\hat{e}_z) \cdot \hat{e}_x \quad (2.53)$$

$$= f_x = \frac{mg \cot \psi}{2} \quad (2.54)$$

where $f_x$ was obtained from Equation (2.40). Alternatively, we can write out the calculation using column-vector notation:

$$\underline{f} \cdot \hat{e}_x = \begin{pmatrix} \frac{mg \cot \psi}{2} \\ \frac{mg}{2} \\ 0 \end{pmatrix}_{xyz} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{xyz} \quad (2.55)$$

$$= \frac{mg \cot \psi}{2} \quad (2.56)$$

To carry out this calculation in $\bar{x}, \bar{y}, \bar{z}$ coordinates, we must first determine the coefficients of $\underline{f}$ and $\hat{e}_x$ in that system. Once the $\bar{x}, \bar{y}, \bar{z}$ coefficients are obtained, the procedure matches that employed before since $\bar{x}, \bar{y}, \bar{z}$ coordinates are orthonormal. The result is, of course, the same,

$$\underline{f} \cdot \hat{e}_x = (f_{\bar{x}}\hat{e}_{\bar{x}} + f_{\bar{y}}\hat{e}_{\bar{y}} + f_{\bar{z}}\hat{e}_{\bar{z}}) \cdot \hat{e}_x \quad (2.57)$$

$$= \begin{pmatrix} \frac{mg}{2 \sin \psi} \\ 0 \\ 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} \cdot \begin{pmatrix} \cos \psi \\ -\sin \psi \\ 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} \quad (2.58)$$

$$= \frac{mg \cot \psi}{2} \quad (2.59)$$

The basis vectors of the $y, \bar{y}, z$ system are not mutually orthogonal. This complicates any calculation carried out in this coordinate system. First we write $\underline{f}$ in the $y, \bar{y}, z$

coordinate system, and then we carry out the dot product using the distributive rule of algebra.

$$\underline{f} = f_y\hat{e}_y + f_{\bar{y}}\hat{e}_{\bar{y}} + f_z\hat{e}_z \tag{2.60}$$

$$\underline{f}\cdot\hat{e}_x = (f_y\hat{e}_y + f_{\bar{y}}\hat{e}_{\bar{y}} + f_z\hat{e}_z)\cdot\hat{e}_x \tag{2.61}$$

$$= f_y(\hat{e}_y\cdot\hat{e}_x) + f_{\bar{y}}(\hat{e}_{\bar{y}}\cdot\hat{e}_x) + f_z(\hat{e}_z\cdot\hat{e}_x) \tag{2.62}$$

Since $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ are mutually perpendicular, the first and last terms vanish. We are left with one dot product:

$$\underline{f}\cdot\hat{e}_x = f_{\bar{y}}(\hat{e}_{\bar{y}}\cdot\hat{e}_x) \tag{2.63}$$

where $f_{\bar{y}}$ is the $\bar{y}$ coefficient of $\underline{f}$ when that vector is written in the $y, \bar{y}, z$ coordinate system [see Equation (2.44)],

$$f_{\bar{y}} = -\frac{mg\cos\psi}{2\sin^2\psi} \tag{2.64}$$

We can evaluate the vector dot product on the right side of Equation (2.63) by recalling the definition of dot product (Equation (2.7)]:

$$\hat{e}_{\bar{y}}\cdot\hat{e}_x = |\hat{e}_{\bar{y}}||\hat{e}_x|\cos\begin{pmatrix}\text{angle between}\\ \text{two vectors}\end{pmatrix} \tag{2.65}$$

$$= (1)(1)\cos\left(\frac{\pi}{2}+\psi\right) = -\sin\psi \tag{2.66}$$

where we have used Figure 2.10 and a trigonometric identity to evaluate the cosine function. Substituting this result into Equation (2.63), we obtain the correct answer:

$$\underline{f}\cdot\hat{e}_x = f_{\bar{y}}(\hat{e}_{\bar{y}}\cdot\hat{e}_x) \tag{2.67}$$

$$= \frac{mg\cos\psi}{2\sin\psi} \tag{2.68}$$

This last example, which dealt with evaluating a dot product for a vector written with respect to a nonorthonormal basis, was simplified by the orthogonality of $\hat{e}_x$ with two of the basis vectors ($\hat{e}_y$ and $\hat{e}_z$). In the general case of calculating a dot product with vectors written with respect to nonorthonormal bases, there would be six independent dot products among the basis vectors to evaluate geometrically or trigonometrically [see Equation (2.46); recall that the dot product is commutative]. This is why we prefer to carry out vector calculations by writing the vectors with respect to orthonormal bases.

### 2.2.1.4 *Vector Cross Product*

Cross products may also be written simply using an orthonormal basis [239]. For the basis vectors, any cross products of like vectors (e.g., $\hat{e}_1\times\hat{e}_1$) vanish since $\sin\psi = 0$ [Equation (2.8)]. For unlike vectors, following the right-hand rule (Figure 2.7) we can see that

$$\hat{e}_1\times\hat{e}_2 = \hat{e}_3 \tag{2.69}$$

$$\hat{e}_2\times\hat{e}_3 = \hat{e}_1 \tag{2.70}$$

$$\hat{e}_3\times\hat{e}_1 = \hat{e}_2 \tag{2.71}$$

$$\hat{e}_3\times\hat{e}_2 = -\hat{e}_1 \tag{2.72}$$

$$\hat{e}_2\times\hat{e}_1 = -\hat{e}_3 \tag{2.73}$$

$$\hat{e}_1\times\hat{e}_3 = -\hat{e}_2 \tag{2.74}$$

Notice that $\hat{e}_i\times\hat{e}_j = +\hat{e}_k$ when $ijk$ are permutations of 123 that are produced by removing the last digit and placing it in front. This is called a *cyclic* permutation. Likewise $\hat{e}_i\times\hat{e}_j = -\hat{e}_k$ when $ijk$ are cyclic permutations of 321.

For the cross product of arbitrary vectors we write each vector in terms of an orthonormal basis and carry out the individual cross products. Remember that the cross product of parallel vectors is zero.

$$\underline{v}\times\underline{u} = (v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3)\times(u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3) \tag{2.75}$$

$$\begin{aligned} &= v_1u_1\hat{e}_1\times\hat{e}_1 + v_1u_2\hat{e}_1\times\hat{e}_2 + v_1u_3\hat{e}_1\times\hat{e}_3 + v_2u_1\hat{e}_2\times\hat{e}_1\\ &\quad + v_2u_2\hat{e}_2\times\hat{e}_2 + v_2u_3\hat{e}_2\times\hat{e}_3 + v_3u_1\hat{e}_3\times\hat{e}_1 + v_3u_2\hat{e}_3\times\hat{e}_2\\ &\quad + v_3u_3\hat{e}_3\times\hat{e}_3 \end{aligned} \tag{2.76}$$

$$= \hat{e}_1(v_2u_3 - v_3u_2) - \hat{e}_2(v_1u_3 - v_3u_1) + \hat{e}_3(v_1u_2 - v_2u_1) \tag{2.77}$$

This operation can be summarized using the matrix operation of taking a determinant, denoted by enclosing a square array in vertical lines and defined for a 3 × 3 matrix:

$$\det|Z| = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13}\\ Z_{21} & Z_{22} & Z_{23}\\ Z_{31} & Z_{32} & Z_{33}\end{vmatrix} \tag{2.78}$$

$$\begin{aligned} &= Z_{11}(Z_{22}Z_{33} - Z_{23}Z_{32}) - Z_{12}(Z_{21}Z_{33} - Z_{23}Z_{31})\\ &\quad + Z_{13}(Z_{21}Z_{32} - Z_{22}Z_{31}) \end{aligned} \tag{2.79}$$

To carry out the cross product of two arbitrary vectors, we construct a 3 × 3 matrix using the basis vectors and the coefficients of the vectors as follows:

$$\underline{v}\times\underline{u} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{vmatrix} \tag{2.80}$$

$$= \hat{e}_1(v_2u_3 - v_3u_2) - \hat{e}_2(v_1u_3 - v_3u_1) + \hat{e}_3(v_1u_2 - v_2u_1) \tag{2.81}$$

$$= \begin{pmatrix}(v_2u_3 - v_3u_2)\\ (v_3u_1 - v_1u_3)\\ (v_1u_2 - v_2u_1)\end{pmatrix}_{123} \tag{2.82}$$

which is the same result as before. Again the subscript 123 denotes that this 3 × 1 array holds the coefficients of the vector $\underline{v}\times\underline{u}$ with respect to the basis vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$.

When using this kind of mathematics in rheology we will often be generating even more complex expressions than those shown. To take advantage of the order in this cha of letters and subscripts, we use a short-hand called *Einstein* notation

## 2.2.2 VECTOR EINSTEIN NOTATION

Einstein notation, also called *summation convention*, is a way of writing the effects of operations on vectors in a compact and easier-to-read format. To use Einstein notation most effectively, the vectors must be written with respect to orthonormal basis vectors. We begin with the basic expression for a vector written with respect to an orthonormal basis:

$$\underline{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \tag{2.83}$$

This can be written more compactly as

$$\underline{v} = \sum_{i=1}^{3} v_i\hat{e}_i \tag{2.84}$$

A further simplification can be made by leaving out the summation sign and understanding that when an index ($i$ in the example) is repeated, a summation from 1 to 3 over that index is understood,

$$\underline{v} = v_i\hat{e}_i \tag{2.85}$$

The power of Einstein notation is harnessed when expressing the results of the multiplication of vectors and, later, tensors. Consider the dot product of two vectors $\underline{v}$ and $\underline{u}$, which was carried out in detail in Equation (2.46). This becomes

$$\underline{v} \cdot \underline{u} = v_i\hat{e}_i \cdot u_j\hat{e}_j = v_i u_j\ \hat{e}_i \cdot \hat{e}_j \tag{2.86}$$

Remember that the summation signs over indices $i$ and $j$ are understood since these are repeated. If we expanded Equation (2.86) by reinstating the summations, the result would be

$$\underline{v} \cdot \underline{u} = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i u_j\ \hat{e}_i \cdot \hat{e}_j \tag{2.87}$$

Also note that we used different indices for the two vectors. This is important since there are two summations in this expression—one for each vector. If we use the same index (e.g., $i$) for both summations, we incorrectly reduce the number of summations to one.

Now, to carry out the dot product it is helpful to use the Kronecker delta $\delta_{ij}$:

**Kronecker delta**

$$\boxed{\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}} \tag{2.88}$$

This quantity expresses exactly the result of $\hat{e}_i \cdot \hat{e}_j$, where $\hat{e}_i$ and $\hat{e}_j$ are any of the three orthonormal basis vectors,

$$\cdot\ \hat{e}_j = \delta_{ij} \tag{2.89}$$

Note that the subscripts $i$ and $j$ can take on any of the values 1, 2, or 3. When the Kronecker delta appears in an expression, it tells us that the two indices associated with the delta are now redundant as a result of some action. In the example outlined [Equation (2.89)], the indices are made redundant as a result of dot multiplication of orthonormal basis vectors. To simplify an Einstein expression in which the Kronecker delta appears, we replace the two subscript indices with a single index and drop the delta:

$$\underline{v} \cdot \underline{u} = v_i\hat{e}_i \cdot u_j\hat{e}_j \tag{2.90}$$

$$= v_i u_j\ \hat{e}_i \cdot \hat{e}_j \tag{2.91}$$

$$= v_i u_j\ \delta_{ij} \tag{2.92}$$

$$= v_i u_i \tag{2.93}$$

Note that the choice of the index $i$ for the final scalar result of this operation is completely arbitrary. We could just as well have written $v_j u_j$ or $v_m u_m$. Remember that the indices serve to remind us that a summation is required, and they identify which terms change as the summation is performed; the specific letter used as the index is arbitrary. The final result in Equation (2.93) is the same result we derived in Equation (2.47):

$$\underline{v} \cdot \underline{u} = v_i u_i \tag{2.94}$$

$$= \sum_{i=1}^{3} v_i u_i \tag{2.95}$$

$$= v_1 u_1 + v_2 u_2 + v_3 u_3 \tag{2.96}$$

The vector cross product ($\underline{v} \times \underline{u}$) can also be expressed in Einstein notation. To do this we must use a new expression, the epsilon permutation symbol $\varepsilon_{ijk}$:

**Epsilon permutation symbol**

$$\boxed{\varepsilon_{ijk} \equiv \begin{cases} 1 & ijk = 123,\ 231,\ \text{or } 312 \\ -1 & ijk = 321,\ 213,\ \text{or } 132 \\ 0 & i = j,\ j = k,\ \text{or } k = i \end{cases}} \tag{2.97}$$

The combinations of indices that give +1 are called even permutations of 123, and the combinations that give −1 are called odd permutations of 123. Using this function, the cross product can be written in Einstein notation as

$$\underline{v} \times \underline{u} = v_p\hat{e}_p \times u_s\hat{e}_s \tag{2.98}$$

$$= v_p u_s\ \hat{e}_p \times \hat{e}_s \tag{2.99}$$

$$= v_p u_s \varepsilon_{psj}\hat{e}_j \tag{2.100}$$

Remember that summing of repeated indices ($p$, $s$, and $j$ in this case) is assumed; thus there are three summations understood in Equation (2.100). The reader can verify that this result is the same as that obtained by the determinant method described in Section 2.2.1.4. As with all aspects of Einstein notation, this method of expressing the dot and cross products is limited to orthonormal bases.[2]

## .3 Tensors

Now that we have reviewed scalar and vector operations, we move on to the more complex quantities called *tensors*. A tensor is a mathematical entity related to vectors, but it is not easy to represent a tensor graphically. A tensor is better explained first by mathematical description and then by showing what it does.

A tensor is an ordered pair of coordinate directions. It is also called the *indeterminate vector product*. The simplest tensor is called a *dyad* or *dyadic product*, and it is written as two vectors side by side,

$$\text{Tensor} \qquad \boxed{\underline{\underline{A}} = \underline{a}\,\underline{b}} \tag{2.101}$$

As you see, in our notation tensors will appear with two underbars. This indicates that they are of higher complexity, or order, than vectors. The tensors we are discussing are *second order* tensors, and we will have more to say about tensor order in Section 2.3.4. While scalars and vectors are physical entities (magnitude, magnitude and direction), tensors are *operators* (magnitude and two or more directions). We will discuss this in more detail after we familiarize ourselves with the algebraic rules for second-order tensors.

### 2.3.1 TENSOR RULES OF ALGEBRA

To understand tensors,[3] we must first know their rules of algebra. The indeterminate vecto product that forms a tensor is not commutative, although it is associative and distributive

$$\begin{array}{c} \text{Laws of algebra} \\ \text{for indeterminate} \\ \text{vector product} \end{array} \left\{ \begin{array}{ll} \textit{not}\text{ commutative} & \underline{a}\,\underline{b} \neq \underline{b}\,\underline{a} \\ \text{associative} & (\underline{a}\,\underline{b})\underline{c} = \underline{a}(\underline{b}\,\underline{c}) \\ \text{distributive} & \underline{a}(\underline{b}+\underline{c}) = \underline{a}\,\underline{b} + \underline{a}\,\underline{c} \\ & (\underline{a}+\underline{b})(\underline{c}+\underline{d}) = \underline{a}\,\underline{c} + \underline{a}\,\underline{d} + \underline{b}\,\underline{c} + \underline{b}\,\underline{d} \end{array} \right.$$

Scalar multiplication of a tensor follows the same rules as scalar multiplication of a vecto such as $\alpha\underline{a}\,\underline{b} = \underline{a}(\alpha\underline{b}) = (\alpha\underline{a})\underline{b}$.

#### 2.3.1.1 Tensor Addition

Adding and subtracting tensors is also possible, but unlike the case of adding vectors, it not possible to graphically illustrate these. As we will show, the easiest way to carry o the addition of two tensors is to write them with respect to a common basis and to coll terms, as we did when adding two vectors.

Recall that any vector may be expressed as the linear combination of any three basis vectors. If we express two vectors this way and then take the indeterminate vector product to make a tensor (following the rules of algebra outlined earlier), we see that to express a tensor in the most general form we need a linear combination of nine pairs of coordinate basis vectors:

$$\underline{\underline{A}} = \underline{u}\,\underline{v} \tag{2.102}$$

$$= (u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3)(v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3) \tag{2.103}$$

$$\begin{aligned} &= u_1v_1\hat{e}_1\hat{e}_1 + u_1v_2\hat{e}_1\hat{e}_2 + u_1v_3\hat{e}_1\hat{e}_3 + u_2v_1\hat{e}_2\hat{e}_1 \\ &\quad + u_2v_2\hat{e}_2\hat{e}_2 + u_2v_3\hat{e}_2\hat{e}_3 + u_3v_1\hat{e}_3\hat{e}_1 + u_3v_2\hat{e}_3\hat{e}_2 + u_3v_3\hat{e}_3\hat{e}_3 \end{aligned} \tag{2.104}$$

where we have used an orthonormal basis for convenience. Since the indeterminate vector product is not commutative, the coefficients of the terms $\hat{e}_1\hat{e}_2$ and $\hat{e}_2\hat{e}_1$ cannot be combined and must remain distinct. Thus, for any chosen coordinate system we can always express a tensor as a linear combination of nine ordered pairs of basis vectors, as outlined before and shown here:

$$\begin{aligned} \underline{\underline{A}} = {} & A_{11}\hat{e}_1\hat{e}_1 + A_{12}\hat{e}_1\hat{e}_2 + A_{13}\hat{e}_1\hat{e}_3 + A_{21}\hat{e}_2\hat{e}_1 + A_{22}\hat{e}_2\hat{e}_2 \\ & + A_{23}\hat{e}_2\hat{e}_3 + A_{31}\hat{e}_3\hat{e}_1 + A_{32}\hat{e}_3\hat{e}_2 + A_{33}\hat{e}_3\hat{e}_3 \end{aligned} \tag{2.105}$$

To add two tensors we write each out with respect to the same basis vectors and add, grouping like terms:

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} \tag{2.106}$$

$$\begin{aligned} &= A_{11}\hat{e}_1\hat{e}_1 + A_{12}\hat{e}_1\hat{e}_2 + A_{13}\hat{e}_1\hat{e}_3 + A_{21}\hat{e}_2\hat{e}_1 + A_{22}\hat{e}_2\hat{e}_2 \\ &\quad + A_{23}\hat{e}_2\hat{e}_3 + A_{31}\hat{e}_3\hat{e}_1 + A_{32}\hat{e}_3\hat{e}_2 + A_{33}\hat{e}_3\hat{e}_3 \\ &\quad + B_{11}\hat{e}_1\hat{e}_1 + B_{12}\hat{e}_1\hat{e}_2 + B_{13}\hat{e}_1\hat{e}_3 + B_{21}\hat{e}_2\hat{e}_1 + B_{22}\hat{e}_2\hat{e}_2 \\ &\quad + B_{23}\hat{e}_2\hat{e}_3 + B_{31}\hat{e}_3\hat{e}_1 + B_{32}\hat{e}_3\hat{e}_2 + B_{33}\hat{e}_3\hat{e}_3 \end{aligned} \tag{2.107}$$

$$\begin{aligned} &= (A_{11} + B_{11})\hat{e}_1\hat{e}_1 + (A_{12} + B_{12})\hat{e}_1\hat{e}_2 + (A_{13} + B_{13})\hat{e}_1\hat{e}_3 \\ &\quad + (A_{21} + B_{21})\hat{e}_2\hat{e}_1 + (A_{22} + B_{22})\hat{e}_2\hat{e}_2 + (A_{23} + B_{23})\hat{e}_2\hat{e}_3 \\ &\quad + (A_{31} + B_{31})\hat{e}_3\hat{e}_1 + (A_{32} + B_{32})\hat{e}_3\hat{e}_2 + (A_{33} + B_{33})\hat{e}_3\hat{e}_3 \end{aligned} \tag{2.108}$$

As was the case when adding vectors, when two tensors are expressed with respect to the same coordinate system, they may be added by simply adding the appropriate coefficients together.

Nine coordinates are unwieldy, and it is common to write the nine coefficients of a tensor in matrix form:

$$\underline{\underline{A}} = \underline{u}\,\underline{v} \tag{2.109}$$

$$\begin{aligned} &= u_1v_1\hat{e}_1\hat{e}_1 + u_1v_2\hat{e}_1\hat{e}_2 + u_1v_3\hat{e}_1\hat{e}_3 + u_2v_1\hat{e}_2\hat{e}_1 \\ &\quad + u_2v_2\hat{e}_2\hat{e}_2 + u_2v_3\hat{e}_2\hat{e}_3 + u_3v_1\hat{e}_3\hat{e}_1 + u_3v_2\hat{e}_3\hat{e}_2 \\ &\quad + u_3v_3\hat{e}_3\hat{e}_3 \end{aligned} \tag{2.110}$$

---

[2] See Chapter 9 for a discussion of nonorthonormal bases.

[3] Throughout the text we will use the term tensor to mean second-order tensor.

$$\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}_{123} \qquad (2.111)$$

More generally:

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{123} \qquad (2.112)$$

where the subscripts of each coefficient indicate which two basis vectors are associated with the scalar coefficient and in what order. Because tensors are made of ordered pairs, $A_{12}$ will not generally be equal to $A_{21}$, and so on. Further, in the convention used here and followed throughout the text, the first index indicates the row in which the coefficient is placed, and the second index indicates the column.

### 2.3.1.2 *Tensor Dot Product*

There exists a dot product between two dyads. This is carried out by dotting the two vectors that are closest together:

Tensor dot product

$$\boxed{\begin{aligned} \underline{a}\,\underline{b}\cdot\underline{c}\,\underline{d} &= \underline{a}(\underline{b}\cdot\underline{c})\,\underline{d} \\ &= (\underline{b}\cdot\underline{c})\underline{a}\,\underline{d} \end{aligned}} \qquad (2.113)$$

Since scalar multiplication is commutative and $(\underline{b}\cdot\underline{c})$ is a scalar, we can move this quantity around to the front, as shown in Equation (2.113). We can also see from the example that the dot product of two tensors is a tensor, but the overall magnitude of the resulting tensor differs from the magnitude of either of the original tensors (since the magnitude now involves $\underline{b}\cdot\underline{c}$), and only certain vector directions ($\underline{a}$ and $\underline{d}$ in the example) are preserved [the directions of $\underline{b}$ and $\underline{c}$ do not appear in Equation (2.113)].

Similarly we may dot a vector with a tensor:

$$\underline{a}\cdot\underline{b}\,\underline{c} = (\underline{a}\cdot\underline{b})\underline{c} \equiv \underline{w} \qquad (2.114)$$

The result $\underline{w}$ is a vector pointing in a direction that was part of the original tensor (parallel to $\underline{c}$), but the magnitude of $\underline{w}$ differs from the magnitudes of any of the original vectors ($\underline{a}$, $\underline{b}$, and $\underline{c}$ above). Neither the dot product of two tensors nor the dot product of a vector with a tensor is commutative. Both are associative and distributive, however,

$$\text{Laws of algebra for tensor dot product}\left\{\begin{array}{ll} \textit{not}\text{ commutative} & \underline{a}\,\underline{b}\cdot\underline{c}\,\underline{d} \neq \underline{c}\,\underline{d}\cdot\underline{a}\,\underline{b} \\ & \underline{\underline{A}}\cdot\underline{\underline{B}} \neq \underline{\underline{B}}\cdot\underline{\underline{A}} \\ \text{associative} & (\underline{a}\,\underline{b}\cdot\underline{c}\,\underline{d})\cdot\underline{f}\,\underline{g} = \underline{a}\,\underline{b}\cdot(\underline{c}\,\underline{d}\cdot\underline{f}\,\underline{g}) \\ & (\underline{\underline{A}}\cdot\underline{\underline{B}})\cdot\underline{\underline{C}} = \underline{\underline{A}}\cdot(\underline{\underline{B}}\cdot\underline{\underline{C}}) \\ \text{distributive} & \underline{a}\,\underline{b}\cdot(\underline{c}\,\underline{m}+\underline{n}\,\underline{w}) = (\underline{a}\,\underline{b}\cdot\underline{c}\,\underline{m}) + (\underline{a}\,\underline{b}\cdot\underline{n}\,\underline{w}) \\ & \underline{\underline{A}}\cdot(\underline{\underline{D}}+\underline{\underline{M}}) = \underline{\underline{A}}\cdot\underline{\underline{D}} + \underline{\underline{A}}\cdot\underline{\underline{M}} \end{array}\right.$$

$$\text{Laws of algebra for vector dot product with a tensor}\left\{\begin{array}{ll} \textit{not}\text{ commutative} & \underline{b}\cdot\underline{c}\,\underline{d} \neq \underline{c}\,\underline{d}\cdot\underline{b} \\ & \underline{b}\cdot\underline{\underline{M}} \neq \underline{\underline{M}}\cdot\underline{b} \\ \text{associative} & \underline{a}\cdot(\underline{c}\,\underline{d}\cdot\underline{w}) = (\underline{a}\cdot\underline{c}\,\underline{d})\cdot\underline{w} \\ & \underline{a}\cdot(\underline{\underline{B}}\cdot\underline{w}) = (\underline{a}\cdot\underline{\underline{B}})\cdot\underline{w} \\ \text{distributive} & \underline{d}\cdot(\underline{c}\,\underline{m}+\underline{n}\,\underline{w}) = (\underline{d}\cdot\underline{c}\,\underline{m}) + (\underline{d}\cdot\underline{n}\,\underline{w}) \\ & \underline{d}\cdot(\underline{\underline{A}}+\underline{\underline{C}}) = \underline{d}\cdot\underline{\underline{A}} + \underline{d}\cdot\underline{\underline{C}} \end{array}\right.$$

In the previous section we started to use matrix notation for writing tensor and vector components. It may seem like shaky ground to begin to use matrix notation for these new entities called tensors. After all, in linear algebra courses, properties and techniques associated with matrices are taught, and we have yet to show whether these properties and techniques are appropriate for matrices composed of vector and tensor coefficients.

To address this concern, consider the multiplication of a vector $\underline{v}$ and a tensor $\underline{\underline{A}}$. To calculate the new vector that results from this multiplication, we first write out the two quantities in terms of their coefficients with respect to an orthonormal basis, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$:

$$\begin{aligned} \underline{v}\cdot\underline{\underline{A}} = (v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3)\cdot(&A_{11}\hat{e}_1\hat{e}_1 + A_{12}\hat{e}_1\hat{e}_2 + A_{13}\hat{e}_1\hat{e}_3 + \\ &A_{21}\hat{e}_2\hat{e}_1 + A_{22}\hat{e}_2\hat{e}_2 + A_{23}\hat{e}_2\hat{e}_3 + A_{31}\hat{e}_3\hat{e}_1 + A_{32}\hat{e}_3\hat{e}_2 + A_{33}\hat{e}_3\hat{e}_3) \end{aligned} \qquad (2.115)$$

Now we use the distributive rule of the dot product to multiply. Recall that every time the indices of the orthonormal basis vectors match, their dot product is one (e.g., $\hat{e}_1\cdot\hat{e}_1 = 1$); when the indices of two dotting basis vectors differ, their dot product is zero (e.g., $\hat{e}_1\cdot\hat{e}_2 = 0$). This allows us to simplify this complex expression. Recall that we will dot the unit vector from $\underline{v}$ with the first (leftmost) unit vectors in the tensor dyads.

$$\begin{aligned} \underline{v}\cdot\underline{\underline{A}} &= v_1(A_{11}\hat{e}_1 + A_{12}\hat{e}_2 + A_{13}\hat{e}_3) + v_2(A_{21}\hat{e}_1 + A_{22}\hat{e}_2 + A_{23}\hat{e}_3) \\ &\quad + v_3(A_{31}\hat{e}_1 + A_{32}\hat{e}_2 + A_{33}\hat{e}_3) && (2.116) \\ &= (v_1A_{11} + v_2A_{21} + v_3A_{31})\hat{e}_1 + (v_1A_{12} + v_2A_{22} + v_3A_{32})\hat{e}_2 \\ &\quad + (v_1A_{13} + v_2A_{23} + v_3A_{33})\hat{e}_3 && (2.117) \end{aligned}$$

This final expression follows the rules of matrix multiplication if we write $\underline{v}$ and $\underline{\underline{A}}$ as follows:

$$\begin{aligned} \underline{w} &= \underline{v}\cdot\underline{\underline{A}} && (2.118) \\ &= (v_1 \quad v_2 \quad v_3)_{123}\cdot\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{123} && (2.119) \\ &= (w_1 \quad w_2 \quad w_3)_{123} && (2.120) \end{aligned}$$

where

$$w_1 = v_1A_{11} + v_2A_{21} + v_3A_{31} \qquad (2.121)$$

$$u_2 = v_1A_{12} + v_2A_{22} + v_3A_{32} \qquad (2.122)$$

$$u_3 = v_1A_{13} + v_2A_{23} + v_3A_{33} \qquad (2.123)$$

Thus, using matrix algebra to carry out the dot product on components of vectors with tensors written with respect to orthonormal bases is correct. Similarly, we can show that the dot product of two tensors also follows the rules of matrix multiplication.

### 2.3.1.3 *Tensor Scalar Product*

There is a scalar product of two tensors, which is defined as follows:

$$\text{Tensor scalar product} \qquad \boxed{\underline{a}\,\underline{b}:\underline{c}\,\underline{d} = (\underline{b}\cdot\underline{c})(\underline{a}\cdot\underline{d})} \qquad (2.124)$$

This amounts to dotting the two closest vectors (the inner pair) and then dotting the two remaining vectors (the outer pair),

$$\underbrace{\underline{a}\,\overbrace{\underline{b}\cdot\underline{c}}^{\underline{b}\cdot\underline{c}}\,\underline{d}}_{\underline{a}\cdot\underline{d}}$$

The rules of algebra for the scalar product of two tensors are summarized as follows:

$$\begin{array}{c}\text{Laws of algebra}\\ \text{for tensor}\\ \text{scalar product}\end{array}\left\{\begin{array}{ll}\text{commutative} & \underline{a}\,\underline{w}:\underline{n}\,\underline{d} = \underline{n}\,\underline{d}:\underline{a}\,\underline{w}\\ \text{associative} & \text{not possible}\\ \text{distributive} & \underline{b}\,\underline{a}:(\underline{m}\,\underline{n}+\underline{w}\,\underline{d}) = \underline{b}\,\underline{a}:\underline{m}\,\underline{n}+\underline{b}\,\underline{a}:\underline{w}\,\underline{d}\end{array}\right.$$

## 2.3.2 TENSOR EINSTEIN NOTATION

The Einstein summation convention can be used to simplify the notation that goes along with tensor multiplication. A tensor in the summation notation requires a double sum:

$$\underline{\underline{A}} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\hat{e}_i\hat{e}_j \qquad (2.125)$$

In Einstein notation this becomes

$$\underline{\underline{A}} = A_{ij}\hat{e}_i\hat{e}_j \qquad (2.126)$$

Using $i$ and $j$ as the dummy indices is completely arbitrary. Each of the following expressions is equivalent:

$$\underline{\underline{A}} = A_{ij}\hat{e}_i\hat{e}_j = A_{mp}\hat{e}_m\hat{e}_p = A_{rs}\hat{e}_r\hat{e}_s \qquad (2.127)$$

What is important in these expressions is that the first index on the symbol $A$ matches the index on the first unit vector; the second index on the symbol $A$ matches that on the second unit vector. An example of a different tensor, related to $\underline{\underline{A}}$, is the transpose of $\underline{\underline{A}}$, written as $\underline{\underline{A}}^T$. This is a tensor that has the same coefficients as $\underline{\underline{A}}$, but they are associated with differe

ordered pairs of basis vectors. In coefficient matrix notation, $\underline{\underline{A}}^T$ is the mirror image of $\underline{\underline{A}}$ across the main diagonal; that is, to obtain the matrix of coefficients of $\underline{\underline{A}}^T$, interchange the rows and the columns of $\underline{\underline{A}}$.

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}_{123} \qquad (2.128)$$

$$\underline{\underline{A}}^T = \begin{pmatrix} A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{pmatrix}_{123} \qquad (2.129)$$

In the Einstein expression for $\underline{\underline{A}}^T$, the first index of $A_{pk}$ is associated with the *second* basis vector in the basis vector dyad:

$$\underline{\underline{A}} = A_{pk}\hat{e}_p\hat{e}_k \qquad (2.130)$$

$$\underline{\underline{A}}^T = A_{pk}\hat{e}_k\hat{e}_p \qquad (2.131)$$

Again, since the letters used to indicate the implicit summations (in this case $p$ and $k$) are arbitrary, $\underline{\underline{A}}^T$ can be written any number of ways. In particular, note the difference between the first two examples:

$$\underline{\underline{A}}^T = A_{ji}\hat{e}_i\hat{e}_j = A_{ij}\hat{e}_j\hat{e}_i = A_{sr}\hat{e}_r\hat{e}_s \qquad (2.132)$$

Since tensors are written in terms of vectors, the methods outlined earlier for vector multiplication with the Einstein convention works just as well for the multiplication of two tensors or the multiplication of a vector with a tensor. When preparing to multiply tensors in Einstein notation, all different letters (e.g., $i$, $j$, $p$, or $k$) must be used for indexing the implicit summations.

$$\underline{\underline{A}}\cdot\underline{\underline{B}} = A_{ij}\hat{e}_i\hat{e}_j \cdot B_{pk}\hat{e}_p\hat{e}_k \qquad (2.133)$$

$$= A_{ij}B_{pk}\;\hat{e}_i\hat{e}_j\cdot\hat{e}_p\hat{e}_k \qquad (2.134)$$

$$= A_{ij}B_{pk}\;\hat{e}_i\delta_{jp}\hat{e}_k \qquad (2.135)$$

$$= A_{ip}B_{pk}\;\hat{e}_i\hat{e}_k \qquad (2.136)$$

The Kronecker delta in Equation (2.135) tells us that one index, $j$ or $p$, is redundant. We therefore replace all of the $j$'s with $p$'s (we could have replaced $p$'s with $j$'s or both $p$ and $j$ with a third letter) to arrive at Equation (2.136).

In the final result there are three summations, two of which involve the unit vectors, and one that does not. To clarify the answer obtained, we can carry out the summation that does not involve the unit vectors, the summation over $p$:

$$A_{ip}B_{pk}\hat{e}_i\hat{e}_k = \sum_{i=1}^{3}\sum_{p=1}^{3}\sum_{k=1}^{3} A_{ip}B_{pk}\hat{e}_i\hat{e}_k \qquad (2.137)$$

$$= \sum_{i=1}^{3}\sum_{k=1}^{3}(A_{i1}B_{1k} + A_{i2}B_{2k} + A_{i3}B_{3k})\hat{e}_i\hat{e}_k \qquad (2.138)$$

Now the result of the multiplication of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ looks more like a usual tensor. Each coefficient term $(A_{i1}B_{1k} + A_{i2}B_{2k} + A_{i3}B_{3k})$ contains only two unknown subscripts that are associated with the basis dyad $(\hat{e}_i\hat{e}_k)$. This coefficient expression is more complicated than a basic tensor in that the coefficients each consist of the sum of three scalars, but we can see that the tensor is still composed of a double sum of scalar coefficients multiplying nine unique pairs of basis vectors.

Multiplication of a vector with a tensor can be carried out no matter whether the vector comes first $(\underline{v} \cdot \underline{\underline{A}})$ or the tensor comes first $(\underline{\underline{A}} \cdot \underline{v})$, but in general the answers in these two cases differ:

$$\underline{v} \cdot \underline{\underline{A}} = v_i\hat{e}_i \cdot A_{rs}\hat{e}_r\hat{e}_s \qquad (2.139)$$

$$= v_i A_{rs}\hat{e}_i \cdot \hat{e}_r\hat{e}_s \qquad (2.140)$$

$$= v_i A_{rs}\delta_{ir}\hat{e}_s \qquad (2.141)$$

$$= v_r A_{rs}\hat{e}_s \qquad (2.142)$$

$$\underline{\underline{A}} \cdot \underline{v} = A_{mp}\hat{e}_m\hat{e}_p \cdot v_j\hat{e}_j \qquad (2.143)$$

$$= A_{mp}v_j\hat{e}_m\hat{e}_p \cdot \hat{e}_j \qquad (2.144)$$

$$= A_{mp}v_j\hat{e}_m\delta_{pj} \qquad (2.145)$$

$$= A_{mp}v_p\hat{e}_m \qquad (2.146)$$

$$= v_p A_{mp}\hat{e}_m \qquad (2.147)$$

To convince yourself that these two answers differ, note that the index that is on the surviving unit vector is not in the same place on the tensor coefficient $A_{ij}$ in the two final expressions.

What is the 2-component of $\underline{\underline{A}} \cdot \underline{v}$?

[redacted]

From the example in the text we know that $\underline{\underline{A}} \cdot \underline{v} = v_p A_{mp}\hat{e}_m$. Expanding that into summation and then into vector matrix notation, we obtain

$$\underline{\underline{A}} \cdot \underline{v} = \sum_{p=1}^{3}\sum_{m=1}^{3} v_p A_{mp}\hat{e}_m \qquad (2.148)$$

$$= \sum_{p=1}^{3} v_p A_{1p}\hat{e}_1 + \sum_{p=1}^{3} v_p A_{2p}\hat{e}_2 + \sum_{p=1}^{3} v_p A_{3p}\hat{e}_3 \qquad (2.149)$$

$$= \begin{pmatrix} \sum_{p=1}^{3} v_p A_{1p} \\ \sum_{p=1}^{3} v_p A_{2p} \\ \sum_{p=1}^{3} v_p A_{3p} \end{pmatrix}_{123} \qquad (2.150)$$

$$= \begin{pmatrix} v_1 A_{11} + v_2 A_{12} + v_3 A_{13} \\ v_1 A_{21} + v_2 A_{22} + v_3 A_{23} \\ v_1 A_{31} + v_2 A_{32} + v_3 A_{33} \end{pmatrix}_{123} \qquad (2.151)$$

The 2-component of $\underline{\underline{A}} \cdot \underline{v}$ is the coefficient of $\hat{e}_2$ in Equation (2.151), that is, $v_1 A_{21} + v_2 A_{22} + v_3 A_{23}$. Equation (2.151) is a vector (note the + signs between the terms). The three scalar components of this vector are long and spread out; do not confuse such vectors with tensors.

### 2.3.3 LINEAR VECTOR FUNCTIONS

We have been concerned, thus far, with familiarizing ourselves with tensors and with tensor algebra. We saved the discussion of what tensors are for until now. In rheology we use tensors because they are a convenient way to express linear vector functions. Linear vector functions arise naturally in the equations describing physical quantities such as linear and angular momentum, light traversing a medium, and stress in a body. We will now show how a tensor expresses a linear vector function through a simple calculation.

We are familiar with functions such as $y = f(x)$. This is a scalar function because it takes a scalar variable $x$ and transforms it into a scalar variable $y$. A vector function, for example, $f(\underline{b})$, behaves analogously, transforming a vector $\underline{b}$ to another vector $\underline{a}$,

$$\underline{a} = f(\underline{b}) \qquad (2.152)$$

A function, as described, is a general transformation. We can further qualify the type of function we are talking about by describing its mathematical properties. An important type of function is a linear function. A function is linear if for all vectors $\underline{a}$, $\underline{b}$ and scalars $\alpha$, the following properties hold:

Definition of
a linear function

$$\boxed{\begin{aligned} f(\underline{a} + \underline{b}) &= f(\underline{a}) + f(\underline{b}) \\ f(\alpha\underline{a}) &= \alpha f(\underline{a}) \end{aligned}} \qquad (2.153)$$

To show that a tensor embodies the properties of a linear vector function, consider the function $f(\underline{b})$ and expand the vector $\underline{b}$ with respect to an orthonormal basis,

$$\underline{a} = f(\underline{b}) \qquad (2.154)$$

$$= f(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3) \qquad (2.155)$$

Since $f$ is a linear function, we can expand Equation (2.155) as follows:

$$\underline{a} = f(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3) \qquad (2.156)$$

$$\ldots + b_3 f(\hat{e}_3) \qquad (2.157)$$

We know that when $f$ operates on a vector a new vector is produced. Thus each of the expressions $f(\hat{e}_1)$, $f(\hat{e}_2)$, and $f(\hat{e}_3)$ is a vector. We do not know what these vectors are, but we can call them $\underline{v}$, $\underline{u}$, and $\underline{w}$.

$$\underline{v} \equiv f(\hat{e}_1) \qquad (2.158)$$

$$\underline{u} \equiv f(\hat{e}_2) \qquad (2.159)$$

$$\underline{w} \equiv f(\hat{e}_3) \qquad (2.160)$$

Thus,

$$\underline{a} = b_1\underline{v} + b_2\underline{u} + b_3\underline{w} \qquad (2.161)$$

$$= \underline{v}b_1 + \underline{u}b_2 + \underline{w}b_3 \qquad (2.162)$$

where we have used the commutative rule of scalar multiplication in writing the second expression.

The three scalars $b_1$, $b_2$, and $b_3$ are just the coefficients of $\underline{b}$ with respect to the orthonormal basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and thus we can write them as [see Equations (2.48)–(2.50)]

$$b_1 = \hat{e}_1 \cdot \underline{b} \qquad (2.163)$$

$$b_2 = \hat{e}_2 \cdot \underline{b} \qquad (2.164)$$

$$b_3 = \hat{e}_3 \cdot \underline{b} \qquad (2.165)$$

Substituting these expressions into Equation (2.162) gives

$$\underline{a} = \underline{v}\,\hat{e}_1 \cdot \underline{b} + \underline{u}\,\hat{e}_2 \cdot \underline{b} + \underline{w}\,\hat{e}_3 \cdot \underline{b} \qquad (2.166)$$

Factoring out $\underline{b}$ by the distributive rule of the dot product gives us

$$\underline{a} = (\underline{v}\,\hat{e}_1 + \underline{u}\,\hat{e}_2 + \underline{w}\,\hat{e}_3) \cdot \underline{b} \qquad (2.167)$$

The expression in parentheses is the sum of three dyadic products and thus is a tensor. If we call this sum $\underline{\underline{M}}$, then our final result is

$$\underline{\underline{M}} \equiv \underline{v}\,\hat{e}_1 + \underline{u}\,\hat{e}_2 + \underline{w}\,\hat{e}_3 \qquad (2.168)$$

$$\underline{a} = f(\underline{b}) = \underline{\underline{M}} \cdot \underline{b} \qquad (2.169)$$

We see that the linear vector function $f$ acting on the vector $\underline{b}$ is the equivalent of dotting a tensor $\underline{\underline{M}}$ with $\underline{b}$. We will often use tensors in this manner, taking advantage of Einstein notation to simplify the calculations.

## 2.3.4 ASSOCIATED DEFINITIONS

As we study rheology we will have use for some specialized definitions that relate to tensors. We have already encountered the transpose of a tensor. Some other definitions are summarized next.

*Identity tensor I*

$$\underline{\underline{I}} \equiv \hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 + \hat{e}_3\hat{e}_3 \qquad (2.170)$$

$$= \hat{e}_i\hat{e}_i \qquad (2.171)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123} \qquad (2.172)$$

This tensor has the same properties as the identity matrix that it resembles. For example, $\underline{\underline{I}} \cdot \underline{v} = \underline{v} \cdot \underline{\underline{I}} = \underline{v}$ and $\underline{\underline{I}} \cdot \underline{\underline{B}} = \underline{\underline{B}} \cdot \underline{\underline{I}} = \underline{\underline{B}}$, where $\underline{v}$ is any vector and $\underline{\underline{B}}$ is any tensor. $\underline{\underline{I}}$ is written as in Equation (2.172) for any orthonormal basis. Any tensor proportional to $\underline{\underline{I}}$ is an isotropic tensor. The linear vector function represented by an isotropic tensor has the same effect in all directions.

*Zero tensor* $\underline{\underline{0}}$*:* The zero tensor is a quantity of tensor order that has all coefficients equal to zero in any coordinate system. It is a linear vector function that transforms any vector to the zero vector

$$\underline{\underline{0}} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.173)$$

*Magnitude of a tensor* $|\underline{\underline{A}}|$*:*

$$|\underline{\underline{A}}| \equiv +\sqrt{\frac{\underline{\underline{A}} : \underline{\underline{A}}}{2}} \qquad (2.174)$$

The magnitude of a tensor is a scalar that is associated with a tensor. The value of the magnitude does not depend on the coordinate system in which the tensor is written.

*Symmetric and antisymmetric tensors:* A tensor is said to be symmetric if

$$\underline{\underline{A}} = \underline{\underline{A}}^T \qquad (2.175)$$

In Einstein notation this means that $A_{sm} = A_{ms}$. An example of the matrix of coefficients of a symmetric tensor is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}_{123} \qquad (2.176)$$

A tensor is said to be antisymmetric if

$$\underline{\underline{A}} = -\underline{\underline{A}}^T \qquad (2.177)$$

In Einstein notation this means that $A_{sm} = -A_{ms}$. An example of the matrix of coefficients of an antisymmetric tensor is

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}_{123} \tag{2.178}$$

The diagonal elements of an antisymmetric tensor are always zeros.

*Invariants of a tensor:* Tensors of the type we have been discussing so far have three scalar quantities associated with them that are independent of the coordinate system. These are called the invariants of the tensor. Combinations of the three invariants are also invariant to change in a coordinate system, and therefore how the three invariants are defined is not unique (see Appendix C.6). The definitions of the tensor invariants that we will use are shown here for a tensor $\underline{\underline{B}}$ [26]. These definitions in terms of tensor coefficients are only valid when the tensor is written in an orthonormal coordinate system. Tensor invariants are

$$I_{\underline{\underline{B}}} \equiv \sum_{i=1}^{3} B_{ii} \tag{2.179}$$

$$II_{\underline{\underline{B}}} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}B_{ji} = \underline{\underline{B}} : \underline{\underline{B}} \tag{2.180}$$

$$III_{\underline{\underline{B}}} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} B_{ij}B_{jk}B_{ki} \tag{2.181}$$

The magnitude $|\underline{\underline{B}}|$ of a tensor (defined previously) is equal to $+\sqrt{II_{\underline{\underline{B}}}/2}$.

***Trace of a tensor:*** The trace of a tensor, written trace($\underline{\underline{A}}$), is the sum of the diagonal elements,

$$\underline{\underline{A}} = A_{pj}\hat{e}_p\hat{e}_j \tag{2.182}$$

$$= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{123} \tag{2.183}$$

$$\text{trace}(\underline{\underline{A}}) = A_{mm} \tag{2.184}$$

$$= A_{11} + A_{22} + A_{33} \tag{2.185}$$

The first invariant, defined by Equation (2.179), is the trace of the tensor written with respe to orthonormal coordinates. The second and third invariants may also be written as trace

$$II_{\underline{\underline{B}}} = \text{trace}(\underline{\underline{B}} \cdot \underline{\underline{B}}) \qquad (2.18$$

$$III_{\underline{\underline{B}}} = \text{trace}(\underline{\underline{B}} \cdot \underline{\underline{B}} \cdot \underline{\underline{B}}) \qquad (2.18$$

*Order of a tensor:* The types of tensors we have been dealing with so far are called *seco order* tensors. Second-order tensors are formed by the indeterminate vector product of t vectors. Higher order tensors may be formed by taking the indeterminate vector product more than two tensors,

third-order tensor $\underline{v}\ \underline{u}\ \underline{w}$

fourth-order tensor $\underline{v}\ \underline{u}\ \underline{w}\ \underline{b}$

In addition, a vector may be considered to be a first-order tensor, and a scalar may l considered to be a zero-order tensor. The number of components in three-dimensional spa required to express a tensor depends on the order $\nu$, as summarized in Table 2.2. The ord of a mathematical quantity is important to know when performing algebraic manipulation Since scalars have magnitude only, while vectors denote magnitude and direction, scala cannot equal vectors. Likewise, vectors cannot equal tensors, which are of higher orde When writing an equation, the rule is that each term must be of the same order. Example of scalar, vector, and tensor equations used in engineering and physics are

$$\text{Scalar equation} \quad Q = mC_p(T_1 - T_2) \quad \begin{cases} Q = \text{ heat transferred} \\ m = \text{ mass} \\ C_p = \text{ heat capacity} \\ T_1, T_2 = \text{temperatures} \end{cases}$$

$$\text{Scalar equation} \quad f_1 = \hat{e}_1 \cdot \underline{f} \quad \begin{cases} \underline{f} = \text{ force vector} \\ f_1 = \text{ scalar component of } \underline{f} \\ \hat{e}_1 = \text{ unit vector} \end{cases}$$

$$\text{Vector equation} \quad \underline{f} = m\underline{a} \quad \begin{cases} \underline{f} = \text{ force vector} \\ m = \text{ mass} \\ \underline{a} = \text{ acceleration vector} \end{cases}$$

$$\begin{matrix}\text{Vector equation} \\ \text{(see Appendix E)}\end{matrix} \quad \underline{D} = \underline{\underline{\epsilon}} \cdot \underline{E} \quad \begin{cases} \underline{D} = \text{ electric displacement vector} \\ \underline{\underline{\epsilon}} = \text{ dielectric tensor} \\ \underline{E} = \text{ electric field vector} \end{cases}$$

$$\begin{matrix}\text{Tensor equation} \\ \text{(see Chapter 3)}\end{matrix} \quad \underline{\underline{\tau}} = -\mu\underline{\underline{\dot{\gamma}}} \quad \begin{cases} \underline{\underline{\tau}} = \text{ stress tensor} \\ \mu = \text{ Newtonian viscosity} \\ \underline{\underline{\dot{\gamma}}} = \text{ rate-of-deformation tensor} \end{cases}$$

The net effect of vector–tensor operations on the order of an expression is summarized in Table 2.3 [28].

***Inverse of a tensor $\underline{\underline{A}}^{-1}$:*** The inverse $\underline{\underline{A}}^{-1}$ of a tensor $\underline{\underline{A}}$ is a tensor that when dot multiplied by $\underline{\underline{A}}$ gives the identity tensor $\underline{\underline{I}}$,

$$\underline{\underline{A}} \cdot \underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} = \underline{\underline{I}} \tag{2.188}$$

TABLE 2.2
Summary of the Orders of Vector and Tensor Quantities and Their Properties

| Order $\nu$ | Name | Number of Associated Directions | Number of Components | Examples |
|---|---|---|---|---|
| 0 | scalar | 0 | $3^0$ | Mass, energy, temperature |
| 1 | vector | 1 | $3^1$ | Velocity, force, electric field |
| 2 | 2nd-order tensor | 2 | $3^2$ | Stress, deformation |
| 3 | 3rd-order tensor | 3 | $3^3$ | Gradient of stress |
| $\nu$ | $\nu$th-order tensor | $\nu$ | $3^\nu$ | |

TABLE 2.3
Summary of the Effect of Various Operations on the Order of an Expression

| Operation | Order of Result | Example |
|---|---|---|
| No symbol | $\sum_{\text{orders}}$ | $\alpha\underline{\underline{B}}$, order = 2 |
| $\times$ | $\sum_{\text{orders}} -1$ | $\underline{w} \times \underline{\underline{C}}$, order = 2 |
| $\cdot$ | $\sum_{\text{orders}} -2$ | $\underline{u} \cdot \underline{\underline{A}}$, order = 1 |
| $:$ | $\sum_{\text{orders}} -4$ | $\underline{\underline{B}} : \underline{\underline{C}}$, order = 0 |

*Notes:* $\sum_{\text{orders}}$ is the summation of the orders of the quantities in the expression.
*Source:* After [28].

An inverse does not exist for a tensor whose determinant is zero. It is straightforward to show that the determinant of $\underline{\underline{A}}$ [written det$|\underline{\underline{A}}|$ and defined in Equation (2.79)] is related to the tensor invariants of $\underline{\underline{A}}$ as follows:

$$\det|\underline{\underline{A}}| = \frac{1}{6}\left(I_{\underline{\underline{A}}}^2 - 3I_{\underline{\underline{A}}}II_{\underline{\underline{A}}} + 2III_{\underline{\underline{A}}}\right) \tag{2.189}$$

The determinant of a tensor is invariant to any coordinate transformation.

## fferential Operations with Vectors and Tensors

The rheologically important equations of conservation of mass and momentum are differential equations, and thus we must learn how to differentiate vector and tensor quantities. In vector and tensor notation, differentiation in physical space (three dimensions) is handled by the vector differential operator $\nabla$, called *del* or *nabla*. In this section we will cover the operation of $\nabla$ on scalars, vectors, and tensors.

To calculate the derivative of a vector or tensor we must first express the quantity with respect to a basis. Differentiation is then carried out by having a differential operator, for example, $\partial/\partial y$, act on each term, including the basis vectors. For example, if the chosen basis is the arbitrary basis $\underline{\tilde{e}}_1, \underline{\tilde{e}}_2, \underline{\tilde{e}}_3$ (not necessarily orthonormal or constant in space), we can express a vector $\underline{v}$ as

$$\underline{v} = \tilde{v}_1\underline{\tilde{e}}_1 + \tilde{v}_2\underline{\tilde{e}}_2 + \tilde{v}_3\underline{\tilde{e}}_3 \tag{2.190}$$

The derivative of $\underline{v}$ is thus

$$[illegible] \tag{2.191}$$

$$= \frac{\partial}{\partial y}(\tilde{v}_1\underline{\tilde{e}}_1) + \frac{\partial}{\partial y}(\tilde{v}_2\underline{\tilde{e}}_2) + \frac{\partial}{\partial y}(\tilde{v}_3\underline{\tilde{e}}_3) \tag{2.192}$$

$$= \tilde{v}_1\frac{\partial\underline{\tilde{e}}_1}{\partial y} + \underline{\tilde{e}}_1\frac{\partial\tilde{v}_1}{\partial y} + \tilde{v}_2\frac{\partial\underline{\tilde{e}}_2}{\partial y} + \underline{\tilde{e}}_2\frac{\partial\tilde{v}_2}{\partial y} + \tilde{v}_3\frac{\partial\underline{\tilde{e}}_3}{\partial y} + \underline{\tilde{e}}_3\frac{\partial\tilde{v}_3}{\partial y} \tag{2.193}$$

Note that we used the product rule of differentiation in obtaining Equation (2.193). This complex situation is simplified if for the basis vectors $\underline{\tilde{e}}_i$ we choose to use the Cartesian coordinate system $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$. In Cartesian coordinates, the basis vectors are constant in length and fixed in direction, and with this choice the terms in Equation (2.193) involving differentiation of the basis vectors $\underline{\tilde{e}}_i$ are zero; thus half of the terms disappear.

Since vector and tensor quantities are independent of the coordinate system, any vector or tensor quantity derived in Cartesian coordinates is valid when properly expressed in any other coordinate system. Thus, when deriving general expressions, it is most convenient to represent vectors and tensors in Cartesian coordinates. Limiting ourselves to spatially homogeneous (the directions of the unit vectors do not vary with position), orthonormal basis vectors (the Cartesian system) allows us to use Einstein notation for differential operations, as we shall see. This is a distinct advantage. There are times when coordinate systems other than the spatially homogeneous Cartesian system are useful, and we will discuss two such coordinate systems (cylindrical and spherical) in the next section. In addition, there are times when nonorthonormal bases are preferred to orthonormal systems. This will be discussed in Chapter 9. Remember that the choice of coordinate system is simply one of convenience, since vector and tensor expressions are independent of the coordinate system.

In Cartesian coordinates ($x = x_1$, $y = x_2$, $z = x_3$) the spatial differentiation operator $\nabla$ is defined as

$$\nabla \equiv \hat{e}_1\frac{\partial}{\partial x_1} + \hat{e}_2\frac{\partial}{\partial x_2} + \hat{e}_3\frac{\partial}{\partial x_3} \tag{2.194}$$

In Einstein notation this becomes

$$\nabla = \hat{e}_i\frac{\partial}{\partial x_i} \tag{2.195}$$

$\nabla$ is a vector operator, not a vector. This means that it has the same order as a vector, but it cannot stand alone. We cannot sketch it on a set of axes, and it does not have a magnitude in the usual sense. Also, although $\nabla$ is of vector order, convention omits the underbar from this symbol.

Since $\nabla$ is an operator, it must operate on something. $\nabla$ may operate on scalars, vectors, or tensors of any order. When $\nabla$ operates on a scalar, it produces a vector,

$$\nabla\alpha = \left(\hat{e}_1\frac{\partial}{\partial x_1} + \hat{e}_2\frac{\partial}{\partial x_2} + \hat{e}_3\frac{\partial}{\partial x_3}\right)\alpha \tag{2.196}$$

$$= \hat{e}_1\frac{\partial\alpha}{\partial x_1} + \hat{e}_2\frac{\partial\alpha}{\partial x_2} + \hat{e}_3\frac{\partial\alpha}{\partial x_3} \tag{2.197}$$

$$= \hat{e}_i \frac{\partial \alpha}{\partial x_i} \qquad (2.198)$$

$$= \begin{pmatrix} \frac{\partial \alpha}{\partial x_1} \\ \frac{\partial \alpha}{\partial x_2} \\ \frac{\partial \alpha}{\partial x_3} \end{pmatrix}_{123} \qquad (2.199)$$

The vector it produces, $\nabla\alpha$, is called the *gradient* of the scalar quantity $\alpha$.

We pause here to clarify two terms we have used, scalars and constants. Scalars are quantities that are of order zero. They convey magnitude only. They may be variables, however, such as the distance $x(t)$ between two moving objects or the temperature $T(x, y, z)$ at various positions in a room with a fireplace. Multiplication by scalars follows the rules outlined earlier, namely, it is commutative, associative, and distributive. When combined with a $\nabla$ operator, however, the position of a scalar is quite important. If the position of a scalar variable is moved with respect to the $\nabla$ operator, the meaning of the expression has changed. We can summarize some of this by pointing out the following rules with respect to $\nabla$ operating on scalars $\alpha$ and $\zeta$:

$$\begin{array}{c} \text{Laws of algebra} \\ \text{for } \nabla \text{ operating} \\ \text{on scalars} \end{array} \left\{ \begin{array}{ll} \textit{not}\text{ commutative} & \nabla\alpha \neq \alpha\nabla \\ \textit{not}\text{ associative} & \nabla(\zeta\alpha) \neq (\nabla\zeta)\alpha \\ \text{distributive} & \nabla(\zeta + \alpha) = \nabla\zeta + \nabla\alpha \end{array} \right.$$

The first limitation, that $\nabla$ is not commutative, relates to the fact that $\nabla$ is an operator: $\nabla\alpha$ is a vector whereas $\alpha\nabla$ is an operator, and they cannot be equal. The second limitation reflects the rule that the differentiation operator $(\partial/\partial x)$ acts on all quantities to its right until a plus, minus, equals sign, or bracket ((), {}, []) is reached. Thus, expressions of the type $\partial(\zeta\alpha)/\partial x$ must be expanded using the usual product rule of differentiation, and $\nabla$ is not associative,

$$\frac{\partial(\zeta\alpha)}{\partial x} = \zeta\frac{\partial\alpha}{\partial x} + \alpha\frac{\partial\zeta}{\partial x} \qquad (2.200)$$

The term "constant" is sometimes confused with the word "scalar." Constant is a word that describes a quantity that does not change. Scalars may be constant [as in the speed of light $c$ (= $3 \times 10^8$ m/s) or the number of cars sold last year worldwide], vectors may be constant (as in the Cartesian coordinate basis vectors $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$), and tensors may be constant (as in the isotropic pressure 2 m below the surface of the ocean). The issue of constancy only comes up now because we are dealing with the change operator $\nabla$. Constants may be positioned arbitrarily with respect to a differential operator since they do not change.

Another thing to notice about the $\nabla$ operator is that it increases the order of the expression on which it acts. We saw that when $\nabla$ operates on a scalar, a vector results. We will now see that when $\nabla$ operates on a vector, it yields a second-order tensor, and when $\nabla$ operates on a second-order tensor, a third-order tensor results. Note that since the Cartesian basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are constant (do not vary with $x_1, x_2, x_3$), it does not matt where they are written with respect to the differentiation operator $\partial/\partial x_j$ in the Einste summation convention. However, the order of the unit vectors in the final expression [ shown in Equation (2.203)] and in the original expansion into Einstein notation must matc

$$\nabla \underline{w} = \hat{e}_p \frac{\partial}{\partial x_p} (w_k \hat{e}_k) = \hat{e}_p \frac{\partial (w_k \hat{e}_k)}{\partial x_p} \qquad (2.201)$$

$$= \hat{e}_p \hat{e}_k \frac{\partial w_k}{\partial x_p} \qquad (2.202)$$

$$= \frac{\partial w_k}{\partial x_p} \hat{e}_p \hat{e}_k \qquad (2.203)$$

$$= \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_2}{\partial x_1} & \frac{\partial w_3}{\partial x_1} \\ \frac{\partial w_1}{\partial x_2} & \frac{\partial w_2}{\partial x_2} & \frac{\partial w_3}{\partial x_2} \\ \frac{\partial w_1}{\partial x_3} & \frac{\partial w_2}{\partial x_3} & \frac{\partial w_3}{\partial x_3} \end{pmatrix}_{123} \qquad (2.204)$$

$$\nabla \underline{\underline{B}} = \hat{e}_i \frac{\partial}{\partial x_i} (B_{rs} \hat{e}_r \hat{e}_s) = \hat{e}_i \frac{\partial (B_{rs} \hat{e}_r \hat{e}_s)}{\partial x_i} \qquad (2.205)$$

$$= \hat{e}_i \hat{e}_r \hat{e}_s \frac{\partial B_{rs}}{\partial x_i} \qquad (2.206)$$

$$= \frac{\partial B_{rs}}{\partial x_i} \hat{e}_i \hat{e}_r \hat{e}_s \qquad (2.207)$$

There are 27 components ($3^3$) associated with the third-order tensor $\nabla\underline{\underline{B}}$, and we will not list them here. This is left as an exercise for the interested reader. $\nabla\underline{w}$ is called the gradient of the vector $\underline{w}$,[4] and $\nabla\underline{\underline{B}}$ is called the gradient of the tensor $\underline{\underline{B}}$.[5]

The rules of algebra for $\nabla$ operating on nonconstant scalars ($\nabla$ is not commutative, not associative, but is distributive) also hold for nonconstant vectors and tensors as outlined:

$$\begin{array}{c} \text{Laws of algebra for} \\ \nabla \text{ operating on} \\ \text{vectors and tensors} \end{array} \left\{ \begin{array}{ll} \textit{not}\text{ commutative} & \nabla\underline{w} \neq \underline{w}\nabla \\ & \nabla\underline{\underline{B}} \neq \underline{\underline{B}}\nabla \\ \textit{not}\text{ associative} & \nabla(\underline{a}\,\underline{b}) \neq (\nabla\underline{a})\underline{b} \\ & \nabla(\underline{a} \cdot \underline{b}) \neq (\nabla\underline{a}) \cdot \underline{b} \\ & \nabla(\underline{a} \times \underline{b}) \neq (\nabla\underline{a}) \times \underline{b} \\ & \nabla(\underline{\underline{B}}\,\underline{\underline{C}}) \neq (\nabla\underline{\underline{B}})\underline{\underline{C}} \\ & \nabla(\underline{\underline{B}} \cdot \underline{\underline{C}}) \neq (\nabla\underline{\underline{B}}) \cdot \underline{\underline{C}} \\ \text{distributive} & \nabla(\underline{w} + \underline{b}) = \nabla\underline{w} + \nabla\underline{b} \\ & \nabla(\underline{\underline{B}} + \underline{\underline{C}}) = \nabla\underline{\underline{B}} + \nabla\underline{\underline{C}} \end{array} \right.$$

---

[4] Note that in the convention we follow [138, 26] the unit vector that accompanies $\nabla$ is the first unit vector in the tensor $\nabla\underline{w}$, and the unit vector from the vector being operated upon is the second unit vector. Thus $\nabla\underline{w} = (\partial w_j/\partial x_i)\hat{e}_i\hat{e}_j$. The opposite convention is also in wide use, namely, $\tilde{\nabla}\underline{w} = (\partial w_i/\partial x_j)\hat{e}_i\hat{e}_j$ [162, 9, 205, 166, 61, 238, 179]. When reading other texts it is important to check which convention is in use (see Tables D.1 and D.2).

[5] The fact that $\nabla\underline{w}$ and $\nabla\underline{\underline{B}}$ are tensors, that is, frame-invariant quantities that express linear vector (or tensor) functions, should not simply be assumed. By examining these quantities under the action of a change in basis, however, both of these gradients as well as the gradients of all higher order tensors can be shown to be tensors [7].

A second type of differential operation is performed when ∇ is dot-multiplied with a vector or a tensor. This operator, the divergence ∇·, lowers by one the order of the entity o which it operates. Since the order of a scalar is already zero, one cannot take the divergence of a scalar. The following operations are defined.

*Divergence of a vector:*

$$\nabla \cdot \underline{w} = \frac{\partial}{\partial x_i} \hat{e}_i \cdot w_m \hat{e}_m \quad (2.208)$$

$$= \hat{e}_i \cdot \hat{e}_m \frac{\partial w_m}{\partial x_i} \quad (2.209)$$

$$= \delta_{im} \frac{\partial w_m}{\partial x_i} \quad (2.210)$$

$$= \frac{\partial w_m}{\partial x_m} \quad (2.211)$$

$$= \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} \quad (2.212)$$

The result is a scalar [no unit vectors present in Equation (2.211)].

*Divergence of a tensor:*

$$\nabla \cdot \underline{\underline{B}} = \frac{\partial}{\partial x_p} \hat{e}_p \cdot B_{mn} \hat{e}_m \hat{e}_n \quad (2.213)$$

$$= \hat{e}_p \cdot \hat{e}_m \hat{e}_n \frac{\partial B_{mn}}{\partial x_p} \quad (2.214)$$

$$= \delta_{pm} \hat{e}_n \frac{\partial B_{mn}}{\partial x_p} \quad (2.215)$$

$$= \frac{\partial B_{pn}}{\partial x_p} \hat{e}_n \quad (2.216)$$

$$= \begin{pmatrix} \frac{\partial B_{p1}}{\partial x_p} \\ \frac{\partial B_{p2}}{\partial x_p} \\ \frac{\partial B_{p3}}{\partial x_p} \end{pmatrix}_{123} = \begin{pmatrix} \sum_{p=1}^{3} \frac{\partial B_{p1}}{\partial x_p} \\ \sum_{p=1}^{3} \frac{\partial B_{p2}}{\partial x_p} \\ \sum_{p=1}^{3} \frac{\partial B_{p3}}{\partial x_p} \end{pmatrix}_{123} \quad (2.217)$$

$$= \begin{pmatrix} \frac{\partial B_{11}}{\partial x_1} + \frac{\partial B_{21}}{\partial x_2} + \frac{\partial B_{31}}{\partial x_3} \\ \frac{\partial B_{12}}{\partial x_1} + \frac{\partial B_{22}}{\partial x_2} + \frac{\partial B_{32}}{\partial x_3} \\ \frac{\partial B_{13}}{\partial x_1} + \frac{\partial B_{23}}{\partial x_2} + \frac{\partial B_{33}}{\partial x_3} \end{pmatrix}_{123} \quad (2.218)$$

The result is a vector [one unit vector is present in Equation (2.216)]. The rules of alge for the operation of the divergence ∇· on vectors and tensors can be deduced by wri the expression of interest in Einstein notation and following the rules of algebra for operation of the differentiation operator $\partial/\partial x_p$ on scalars and vectors.

One final differential operation is the Laplacian, ∇ · ∇ or ∇². This operation leav order of its object unchanged, and thus we may take the Laplacian of scalars, vectors, tensors as shown next.

*Laplacian of a scalar:*

$$\nabla \cdot \nabla \alpha = \frac{\partial}{\partial x_k} \hat{e}_k \cdot \frac{\partial}{\partial x_m} \hat{e}_m \alpha \quad (2.21$$

$$= \hat{e}_k \cdot \hat{e}_m \frac{\partial}{\partial x_k} \frac{\partial \alpha}{\partial x_m} = \delta_{km} \frac{\partial}{\partial x_k} \frac{\partial \alpha}{\partial x_m} \quad (2.22$$

$$= \frac{\partial}{\partial x_k} \frac{\partial \alpha}{\partial x_k} = \frac{\partial^2 \alpha}{\partial x_k^2} \quad (2.22$$

$$= \frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} + \frac{\partial^2 \alpha}{\partial x_3^2} \quad (2.22$$

The result is a scalar. Although $k$ appears only once in Equation (2.221), it is a repeate subscript with respect to Einstein notation since we have used the usual shorthand notatic $\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} = \left( \frac{\partial^2}{\partial x_k^2} \right)$ for twice differentiating with respect to $x_k$.

*Laplacian of a vector:*

$$\nabla \cdot \nabla \underline{w} = \frac{\partial}{\partial x_k} \hat{e}_k \cdot \frac{\partial}{\partial x_m} \hat{e}_m \, w_j \hat{e}_j \quad (2.22$$

$$= \hat{e}_k \cdot \hat{e}_m \, \hat{e}_j \frac{\partial}{\partial x_k} \frac{\partial w_j}{\partial x_m} = \delta_{km} \hat{e}_j \frac{\partial}{\partial x_k} \frac{\partial w_j}{\partial x_m} \quad (2.22$$

$$= \frac{\partial}{\partial x_k} \frac{\partial w_j}{\partial x_k} \hat{e}_j = \frac{\partial^2 w_j}{\partial x_k^2} \hat{e}_j \quad (2.22$$

$$= \begin{pmatrix} \frac{\partial^2 w_1}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2^2} + \frac{\partial^2 w_1}{\partial x_3^2} \\ \frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_2}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_3^2} \\ \frac{\partial^2 w_3}{\partial x_1^2} + \frac{\partial^2 w_3}{\partial x_2^2} + \frac{\partial^2 w_3}{\partial x_3^2} \end{pmatrix}_{123} \quad (2.22$$

The result is a vector. The same procedure may be followed to determine the expression fc the Laplacian of a tensor.

**Correctly identifying the quantities on which ∇ operates is an important issue, and th rules are worth repeating. The differentiation operator $\partial/\partial x_i$ acts on all quantities to it right until a plus, minus, equals sign, or bracket ((), {}, []) is reached. To show how thi property affects terms in a vector/tensor expression, we now give an example.**

What is $\nabla \cdot \underline{a}\,\underline{b}$?

[redacted]

We begin with Einstein notation:

Here, as always, we use different indices for the various implied summations.

Since the coefficients of both $\underline{a}$ and $\underline{b}$ are to the right of the $\nabla$ operator, they are both acted upon by its differentiation action. The Cartesian unit vectors are also affected, but these are constant.

$$\nabla \cdot \underline{a}\,\underline{b} = \frac{\partial}{\partial x_m}\hat{e}_m \cdot (a_p\hat{e}_p\, b_n\hat{e}_n) \tag{2.228}$$

$$= \hat{e}_m \cdot \hat{e}_p\, \hat{e}_n \frac{\partial(a_p\, b_n)}{\partial x_m} = \delta_{mp}\, \hat{e}_n \frac{\partial(a_p\, b_n)}{\partial x_m} \tag{2.229}$$

$$= \hat{e}_n \frac{\partial(a_m\, b_n)}{\partial x_m} \tag{2.230}$$

To further expand this expression, we use the product rule of differentiation on the quantity in parentheses,

$$\nabla \cdot \underline{a}\,\underline{b} = \hat{e}_n \frac{\partial(a_m\, b_n)}{\partial x_m} \tag{2.231}$$

$$= \hat{e}_n \left( a_m \frac{\partial b_n}{\partial x_m} + b_n \frac{\partial a_m}{\partial x_m} \right) \tag{2.232}$$

This is as far as this expression may be expanded. One can write these two terms in vector (also called *Gibbs*) notation,

$$\nabla \cdot \underline{a}\,\underline{b} = a_m \frac{\partial b_n}{\partial x_m}\hat{e}_n + \frac{\partial a_m}{\partial x_m}\, b_n\hat{e}_n \tag{2.233}$$

$$= \underline{a} \cdot \nabla \underline{b} + (\nabla \cdot \underline{a})\underline{b} \tag{2.234}$$

The equivalence of the last two equations may be verified by working backward from Equation (2.234). If the differentiation of the product had not been carried out correctly, the first term on the right-hand side would have been omitted.

## .5 Curvilinear Coordinates

Until now we have used (almost exclusively) the Cartesian coordinate system to expres vectors and tensors with respect to scalar coordinates. Since vector and tensor quantities ar independent of the coordinate system, we have chosen to use the Cartesian system, which i orthonormal and constant in space, to derive vector/tensor relations. This choice allows us use Einstein notation to keep track of vector/tensor operations. The Cartesian system is al a natural choice when solving flow problems if the flow boundaries are straight lines, th is, if the boundaries coincide with coordinate surfaces (e.g., at $x = B$, $v = V$). When t boundaries are curved, however, as, for example, in flow in a pipe or flow around a falli

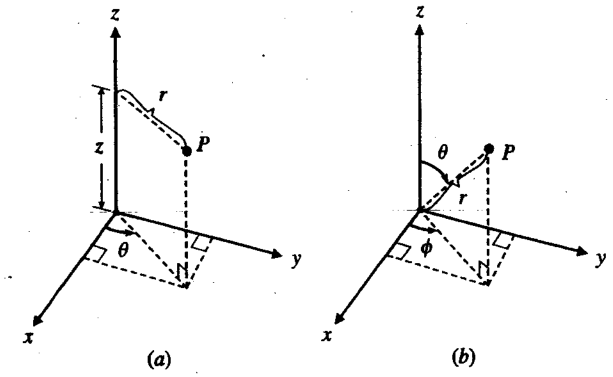
sphere, it is mathematically awkward to use the Cartesian system to solve problem cylindrical and spherical symmetry we will choose to use coordinate systems that sh these symmetries (see Figures 2.11 and 2.12).

The cylindrical and spherical coordinate systems, jointly called curvilinear coordin systems, allow for considerable simplification of the analysis of problems for which boundaries are coordinate surfaces of these systems, that is, problems for which boundaries are cylindrical or spherical (see Chapter 3 for worked-out examples). T disadvantage of these systems, however, is that for both the cylindrical and the spheri coordinate systems the basis vectors vary with position, as we will demonstrate in the ne section. Hence, the differential operator $\nabla$ is more complicated in curvilinear systems, a care must be taken to use the correct form of vector and tensor quantities involving Also, Einstein notation is inconvenient to use in curvilinear systems when $\nabla$ is involv since $\nabla$ must be made to operate on the (spatially varying) basis vectors as well on the vector and tensor coefficients. We will discuss both of these concerns in t next section.

### 2.5.1 CYLINDRICAL COORDINATE SYSTEM

**The coordinates and unit vectors that are used for the cylindrical and spherical coordin systems are shown in Figures 2.11 and 2.12. In cylindrical coordinates (Figure 2.12*a*), t three basis vectors are $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$. The vector $\hat{e}_z$ is the same as the vector of the sa name in the Cartesian coordinate system. The vector $\hat{e}_r$ is a vector that is perpendicular the $z$-axis and makes an angle $\theta$ with the positive $x$-axis of the Cartesian system. The t vector, $\hat{e}_\theta$, is perpendicular to $\hat{e}_r$, resides in an $xy$-plane of the Cartesian system, and poi in the direction counterclockwise to the $x$-axis, that is, in the direction of increasing $\theta$. N that this is an orthonormal basis system.**

**In cylindrical coordinates the vector $\hat{e}_z$ remains constant in direction and in magnitu no matter what point in space is being considered. Both $\hat{e}_r$ and $\hat{e}_\theta$ vary with positi however. To convince yourself of this, consider two points in the $xy$-plane, (1, 1, 0) and (− −1, 0) (Figure 2.13). We can write the vectors $\hat{e}_r$ and $\hat{e}_\theta$ with respect to Cartesian coordina**



(*a*) (*b*)

**Figure 2.11** Geometries of (*a*) cylindrical and (*b*) spherical coordinate systems.

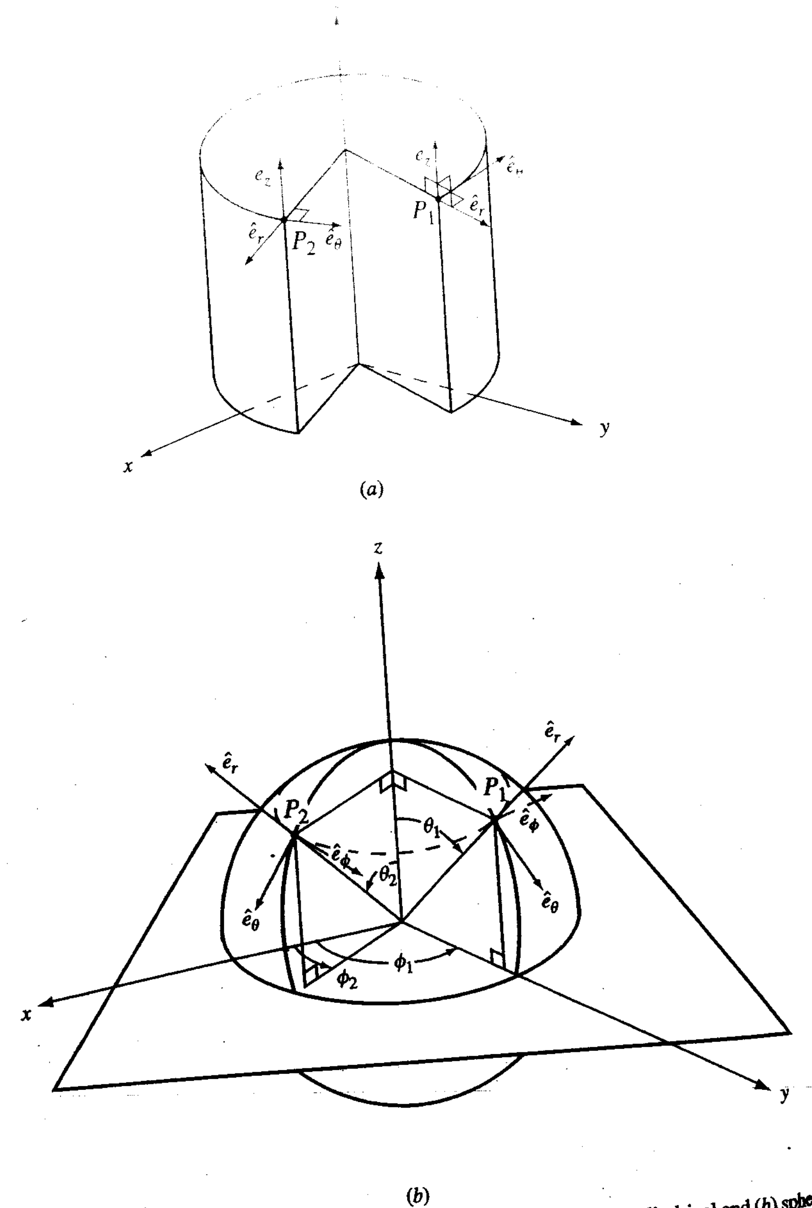


**Figure 2.12** Pictorial representation of the basis vectors associated with (*a*) cylindrical and (*b*) spherical coordinate systems.

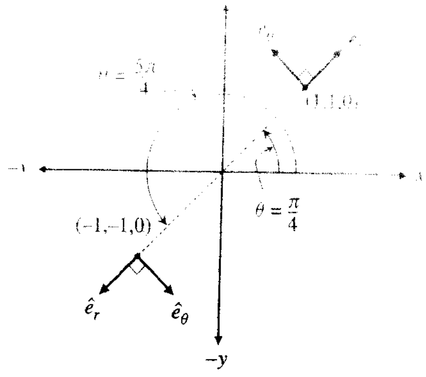


**Figure 2.13** Variation of the basis vectors $\hat{e}_r$ and $\hat{e}_\theta$ in cylindrical coordinates.

for each point. The vector $\hat{e}_r$ is a unit vector pointing in the direction of increasing $r$, that is, in the direction of a line from the origin to the point of interest. The unit vector $\hat{e}_\theta$ points from the point of interest in the direction of increasing $\theta$. For the point (1,1,0), $\hat{e}_r$ and $\hat{e}_\theta$ are

$$\hat{e}_r\big|_{(110)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}_{xyz} \tag{2.235}$$

$$\hat{e}_\theta\big|_{(110)} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}_{xyz} \tag{2.236}$$

where the subscript $xyz$ emphasizes that the vectors are written in the Cartesian, $x$, $y$, $z$, coordinate system. For (−1,−1,0), $\hat{e}_r$ and $\hat{e}_\theta$ are

$$\hat{e}_r\big|_{(-1,-1,0)} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}_{xyz} \tag{2.237}$$

$$\hat{e}_\theta\big|_{(-1,-1,0)} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}_{xyz} \tag{2.238}$$

The vectors $\hat{e}_r$ and $\hat{e}_\theta$ clearly differ at the two points. For an arbitrary point at coordinates $x, y, z$ or $r, \theta, z$, the cylindrical basis vectors are related to the Cartesian basis vectors as follows:

$$\hat{e}_r = \cos\theta\ \hat{e}_x + \sin\theta\ \hat{e}_y \tag{2.239}$$

$$(2.240)$$

$$\hat{e}_\theta = -\sin\theta \ldots \qquad (2.241)$$

$$\hat{e}_z = \hat{e}_z$$

The cylindrical coordinate variables $r$, $\theta$, and $z$ are related to the Cartesian coordinate variables $x$, $y$, and $z$ as follows:

$$(2.242)$$

$$x = r\cos\theta \qquad (2.243)$$

$$y = r\sin\theta \qquad (2.244)$$

$$z = z$$

The fact that the cylindrical basis vectors vary with position impacts the use of the $\nabla$ operator in this coordinate system. The $\nabla$ operator in cylindrical coordinates may be derived from the Cartesian expression by making use of the chain rule ([239] and Chapter 9) and the relations between $x$, $y$, and $z$ and $r$, $\theta$, and $z$. The result is

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z} \qquad (2.245)$$

We will now operate $\nabla$ on a vector $\underline{v}$, writing both $\nabla$ and the vector $\underline{v}$ in cylindrical coordinates and following the rules of algebra outlined earlier in this chapter,

$$(2.246)$$

$$\nabla\underline{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z) \qquad (2.247)$$

$$= \hat{e}_r\frac{\partial}{\partial r}(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z) + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z)$$

$$+ \hat{e}_z\frac{\partial}{\partial z}(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z)$$

$$= \hat{e}_r\left[\frac{\partial(v_r\hat{e}_r)}{\partial r} + \frac{\partial(v_\theta\hat{e}_\theta)}{\partial r} + \frac{\partial(v_z\hat{e}_z)}{\partial r}\right] + \hat{e}_\theta\frac{1}{r}\left[\frac{\partial(v_r\hat{e}_r)}{\partial\theta} + \frac{\partial(v_\theta\hat{e}_\theta)}{\partial\theta} + \frac{\partial(v_z\hat{e}_z)}{\partial\theta}\right] \qquad (2.248)$$

$$+ \hat{e}_z\left[\frac{\partial(v_r\hat{e}_r)}{\partial z} + \frac{\partial(v_\theta\hat{e}_\theta)}{\partial z} + \frac{\partial(v_z\hat{e}_z)}{\partial z}\right]$$

Each of the derivatives operates on a product of two quantities that vary in space, the coefficient of $\underline{v}$ and the basis vector. When we operated in the Cartesian coordinate system, since the basis vectors are not a function of position, they could be removed from the differentiation. For the ~~cylindrical (and spherical)~~ coordinate systems, which have variable unit vectors, this is not possible. One correct way to calculate $\nabla\underline{v}$ in cylindrical coordinates is to write $r, \theta, z, \hat{e}_r, \hat{e}_\theta$, and $\hat{e}_z$ with respect to the constant Cartesian system and then carry out the appropriate differentiations. The results are shown in Table C.7 in Appendix C.2.

## 2.5.2 SPHERICAL COORDINATE SYSTEM

In the spherical coordinate system, all three basis vectors vary with position (Figure 2.12 The three unit vectors are $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$. The vector $\hat{e}_r$ emits radially from the origin tow The vector $\hat{e}_\theta$, which lies in the plane formed by the point of interest

the Cartesian $z$-direction, is perpendicular to $\hat{e}_r$, and points in the direction that rotates away from the positive $z$-axis. The vector $\hat{e}_\phi$ lies in a Cartesian $xy$-plane, is perpendicular to the projection of $\hat{e}_r$ in the Cartesian $xy$-plane, and points counterclockwise from the $x$-axis. Note that the definitions of $\hat{e}_r$ and $\hat{e}_\theta$ in cylindrical and spherical coordinates differ.

The spherical coordinate variables $r$, $\theta$, and $\phi$ and basis vectors $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ are related to their Cartesian counterparts as follows:

$$x = r\sin\theta\cos\phi \qquad (2.249)$$

$$y = r\sin\theta\sin\phi \qquad (2.250)$$

$$z = r\cos\theta \qquad (2.251)$$

$$\hat{e}_r = (\sin\theta\cos\phi)\hat{e}_x + (\sin\theta\sin\phi)\hat{e}_y + (\cos\theta)\hat{e}_z \qquad (2.252)$$

$$\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z \qquad (2.253)$$

$$\hat{e}_\phi = (-\sin\phi)\hat{e}_x + (\cos\phi)\hat{e}_y \qquad (2.254)$$

The $\nabla$ operator for spherical coordinates is also calculated from the chain rule (see Chapter 9),

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad (2.255)$$

The extra difficulty caused by using the curvilinear coordinates is offset by the mathematical simplifications that result when cylindrically or spherically symmetric flow problems are expressed in these coordinate systems. To simplify the use of these coordinate systems, the effects of the $\nabla$ operator on scalars, vectors, and tensors in the cylindrical and spherical coordinate systems are summarized in Tables C.7 and C.8 in Appendix C.2. We will often refer to these tables when we solve Newtonian and non-Newtonian flow problems in curvilinear coordinates.

## ctor and Tensor Integral Theorems

Upon completion of this chapter on mathematics review, we will move on to define and to derive the governing equations of Newtonian and non-Newtonian fluid mechanics. We will need some theorems and formulas from vector mathematics, and these are presented here without proof. The reader is directed to textbooks on advanced mathematics for a more detailed discussion of these subjects [100].

### 2.6.1 GAUSS–OSTROGRADSKII DIVERGENCE THEOREM

The Gauss–Ostrogradskii divergence theorem[6] relates the change of a vector property $\underline{b}$, taking place in a closed volume $V$, with the flux of that property through the surface $S$ that encloses $V$ [100, 7] (Figure 2.14). The theorem is shown in Equation (2.256)

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[6] Also known as Green's theorem or simply as the divergence theorem; see Aris [7].

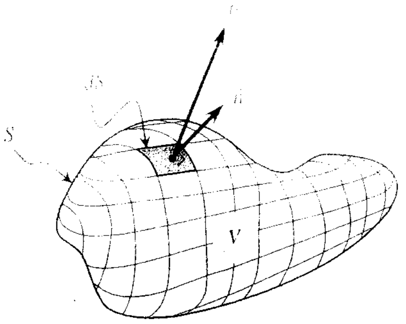

**Figure 2.14** Arbitrary volume V enclosed by a surface area S. Each particular piece of the surface is characterized by the direction of its unit normal n̂. b represents any vector property associated with dS.

Gauss–Ostrogradskii divergence theorem

$$\boxed{\int_V \nabla \cdot \underline{b}\, dV = \int_S \hat{n} \cdot \underline{b}\, dS} \qquad (2.256)$$

where $\hat{n}$ is the outwardly pointing unit normal of the differential surface element $dS$ (Figure 2.14). The volume $V$ is not necessarily constant in time. Use of the Gauss–Ostrogradskii divergence theorem allows us to convert an integral over a volume into a surface integral (or vice versa) without loss of information. This is handy when it is more intuitive to write part of an expression as a surface integral but all other terms of the equation as volume integrals.

## 2.6.2 LEIBNITZ FORMULA

The Leibnitz formula interprets for us the effect of differentiating an integral [100]. Most simple integrals encountered by beginning students involve integrals over fixed limits. For example, a quantity $J(x, t)$ is defined as the following integral:

$$J(x,t) = \int_\alpha^\beta f(x,t)\, dx \qquad (2.257)$$

where $\alpha$ and $\beta$ are constants, and $f$ is a function of $x$ and $t$. When it is desired to take th derivative of $J$, the procedure is straightforward:

$$\frac{dJ(x,t)}{dt} = \frac{d}{dt}\left[\int_\alpha^\beta f(x,t)\, dx\right] \qquad (2.258$$

$$= \int_\alpha^\beta \frac{\partial f(x,t)}{\partial t}\, dx \qquad (2.25$$

This is actually a simplified version of the Leibnitz formula, which tells us how to carry this differentiation if the limits are not constant but rather are functions of $t$. The Leib formula is given.

$$J(x,t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx \qquad (2.260)$$

$$\frac{dJ(x,t)}{dt} = \frac{d}{dt}\left[\int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx\right] \qquad (2.261)$$

Leibnitz formula (single integral)

$$\boxed{\frac{dJ(x,t)}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\, dx + f(\beta,t)\frac{d\beta}{dt} - f(\alpha,t)\frac{d\alpha}{dt}} \qquad (2.262)$$

There are two versions of the Leibnitz formula that we will use, the preceding one for differentiating single integrals, and a second version for differentiating in three dimensions.[7] For a time-varying volume $V(t)$, enclosed by the moving surface $S(t)$, $J$ is defined as

$$J(x,y,z,t) = \int_{V(t)} f(x,y,z,t)\, dV \qquad (2.263)$$

The derivative of $J$ is given by the three-dimensional Leibnitz formula:

Leibnitz formula (volume integral)

$$\boxed{\frac{dJ}{dt} = \int_{V(t)} \frac{\partial f}{\partial t}\, dV + \int_{S(t)} f\,(\underline{v}_{\text{surface}} \cdot \hat{n})\, dS} \qquad (2.264)$$

The meaning of $\hat{n}$ is the same as in the Gauss–Ostrogradskii divergence theorem (see Figure 2.14), and $\underline{v}_{\text{surface}}$ is the velocity of the surface element $dS$ (the surface is moving). If the volume is fixed in space, the second term goes to zero because $\underline{v}_{\text{surface}}$ is zero.

## 2.6.3 SUBSTANTIAL DERIVATIVE

In fluid mechanics and rheology, we often deal with properties that vary in space and also change with time. Thus we must consider the differentials of multivariable functions.

Consider such a multivariable function $f(x_1, x_2, x_3, t)$ associated with a particle of fluid, where $x_1$, $x_2$, and $x_3$ are the three spatial coordinates and $t$ is time. This might be, for example, the density of flowing material as a function of time and position. We know that the differential of $f$ is given by

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 \qquad (2.265)$$

If we divide this expression through by $dt$ we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} \qquad (2.266)$$

The terms $dx_i/dt$ are just the velocity components of the particle $v_i$:

---

[7] Also known as Reynolds' transport theorem [7,148].

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1} \ldots$$

Equation (2.267) may be written in Einstein notation and in vector (Gibbs) notation as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} v_i \qquad (2.268)$$

Substantial derivative

$$\boxed{\frac{df}{dt} = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f} \qquad (2.269)$$

This expression is called the *substantial derivative* and is often written as $Df/Dt$. It indicates the rate of change of the function $f$ as observed from a particle of fluid moving with velocity $\underline{v}$.

The mathematical techniques discussed in this chapter will be used extensively throughout the text. In the next chapter we will apply them to deriving conservation equations for mass and momentum.

[redacted]

t is the magnitude of $\underline{a} = \hat{i} + \hat{j} = \hat{e}_1 + \hat{e}_2$?

w that when a vector is dotted with an arbitrary vector, the scalar product yields the projection he vector in the direction of the unit vector.

hat is the unit vector parallel to $\underline{v} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}_{xyz}$?

hat vector goes between the points (1, 0, 3) and (0, 1)?

Vhat is a unit vector perpendicular to $\underline{v} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}_{xyz}$?

Show that $\underline{u} = a\hat{i} + b\hat{j} + c\hat{k}$ is perpendicular to the plane $ax + by + cz = \alpha$.

For a general vector $\underline{v}$,

$$\underline{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$$

show that

$$v_1 = \underline{v} \cdot \hat{e}_1$$

$$v_2 = \underline{v} \cdot \hat{e}_2$$

**2.8** Do the following vectors form a basis? If yes, write the vector $\underline{w} = 2\hat{e}_x + 3\hat{e}_y + \hat{e}_z$ in the basis. Prove your answers.

(a) $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}_{xyz}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}_{xyz}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_{xyz}$

(b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{xyz}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}_{xyz}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_{xyz}$

**2.9** (a) Show that the vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ form a basis:

$$\underline{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{xyz}, \quad \underline{v} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}_{xyz}, \quad \underline{w} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}_{xyz}$$

(b) Write the vector $\underline{t}$ with respect to the basis $\underline{u}$, $\underline{v}$, $\underline{w}$:

$$\underline{t} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}_{xyz}$$

**2.10** What is the difference between a 3 by 3 matrix an a 2nd-order tensor?

**2.11** In Cartesian coordinates $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ the coefficie of a tensor $\underline{\underline{A}}$ are given by

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

What are the coefficients of $\underline{\underline{A}}$ written with respect to the basis vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ given below?

$$\underline{a} = \frac{1}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2$$

$$\underline{b} = \frac{1}{2}\hat{e}_1 - \frac{1}{2}\hat{e}_2$$

$$\underline{c} = \hat{e}_3$$

**2.12** How are $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$ related?

**2.13** How can we simplify $(\underline{\underline{B}}^T)^T$? Use Einstein notation.

**2.14** Express $\underline{\underline{A}} \cdot \underline{\underline{B}}$ in Einstein notation.

Expand $(\underline{a}\,\underline{b})^T$ using Einstein notation.

What is the component of $(\underline{v} \cdot \underline{\underline{A}} \cdot \underline{b}\,\underline{c})$ in the 2-direction?

Using Einstein notation, show that $\underline{\underline{A}} \cdot \underline{\underline{A}}^T$ is symmetric.

Using Einstein notation, show that $\underline{\underline{A}}^T + \underline{\underline{B}}^T = (\underline{\underline{A}} + \underline{\underline{B}})^T$.

sing Einstein notation, show that $(\underline{\underline{A}} \cdot \underline{\underline{B}} \cdot \underline{\underline{C}})^T = \ldots^T \cdot \underline{\underline{B}}^T \cdot \underline{\underline{A}}^T$.

ing Einstein notation, show that the tensor $\underline{\underline{A}} + \underline{\underline{A}}^T$ ymmetric. Show that $\underline{\underline{A}} - \underline{\underline{A}}^T$ is antisymmetric.

$\underline{a} \cdot \underline{\underline{B}} = \underline{\underline{B}} \cdot \underline{a}$ in general? Show why or why sing Einstein notation.

that $\underline{\underline{A}} : \underline{\underline{A}}^T > 0$. Use Einstein notation.

agnitude of a tensor $\underline{\underline{A}}$ is defined by

$$\left|\underline{\underline{A}}\right| = \text{abs}\left(\sqrt{\frac{\underline{\underline{A}} : \underline{\underline{A}}}{2}}\right)$$

the magnitude of the tensor $\underline{\underline{A}}$ given below?

$$\ldots_1 + 3\hat{e}_1\hat{e}_2 - 3\hat{e}_1\hat{e}_3 - \hat{e}_2\hat{e}_1 - \hat{e}_2\hat{e}_2 + 2\hat{e}_2\hat{e}_3$$
$$3\hat{e}_3\hat{e}_1$$

e Laplacian of a tensor $\underline{\underline{B}}$ in Einstein hat is the order of this quantity?

e rules of algebra for taking the divergence of a vector (e.g., $\nabla \cdot \underline{w}$) and for taking the divergence of a tensor (e.g., $\nabla \cdot \underline{\underline{B}}$)? Show that your rules hold by working out the expressions in Einstein notation.

**2.26** Prove that the following equality holds:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} v_i = \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f$$

**2.27** Using Einstein notation, show that for a symmetric tensor $\underline{\underline{A}}$:

$$\underline{\underline{A}} : \nabla\underline{v} = \nabla \cdot (\underline{\underline{A}} \cdot \underline{v}) - \underline{v} \cdot (\nabla \cdot \underline{\underline{A}})$$

**2.28** For the vector $\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$, what is $\nabla(\underline{x} \cdot \underline{x})$? Write your final answer in Gibbs notation.

**2.29** Prove that the following equality holds:

$$\nabla \cdot \underline{a}\,\underline{b} = \underline{a} \cdot \nabla\underline{b} + (\nabla \cdot \underline{a})\underline{b}$$

**2.30** For $\underline{v} = \begin{pmatrix} ax_2 \\ bx_1 + x_2^2 \\ cx_3 \end{pmatrix}_{123}$, what is $\nabla \cdot \underline{v}$?

**2.31** Expand $\nabla \cdot (\alpha\underline{v})$, where $\alpha$ is a scalar but is not constant. Write your answers in Einstein notation and vector form.

**2.32** Using Einstein notation, show that $\nabla \cdot \underline{v}\,\underline{w} = \underline{v} \cdot \nabla\underline{w} + \underline{w}(\nabla \cdot \underline{v})$.

**2.33** What is $\nabla \cdot a\underline{\underline{I}}$? Express your answer in Einstein notation and in vector form.

**2.34** Simplify the expression $\underline{\underline{I}} : \nabla\underline{v}$.

**2.35** What is the $x_2$-component of $\nabla \cdot \nabla\underline{v}$?

**2.36** The trace of a tensor is the sum of the elements on the diagonal, as shown below:

$$\underline{\underline{A}} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{xyz}$$

$$\text{trace}\left(\underline{\underline{A}}\right) = A_{11} + A_{22} + A_{33} = a + e + i$$

Show that the trace of $\nabla\underline{v}$ is equal to $\nabla \cdot \underline{v}$.

**2.37** What are the orders of the following quantities? Prove your answers using Einstein notation.

(a) $\underline{\underline{A}} : \underline{\underline{B}}$

(b) $(\underline{\underline{A}} \cdot \underline{b}) \cdot \underline{v}$

(c) $\underline{a}\,\underline{b} \cdot \underline{\underline{C}}$

(d) $\nabla^2\underline{\underline{A}}$

(e) $(\underline{a} \cdot \underline{\underline{B}}) \times \underline{c}$

owing equations valid? Explain

$\underline{b} = c$

$\underline{b} = \underline{\underline{C}}$

$\nabla \cdot \underline{b} = \alpha$

the invariants of the tensor $\underline{\underline{B}}$ =

$$\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}_{xyz}$$

t

$$\underline{\underline{B}} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}B_{ji} = \text{trace}(\underline{\underline{B}} \cdot \underline{\underline{B}})$$

at

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3} B_{ij}B_{jm}B_{mi} = \text{trace}(\underline{\underline{B}} \cdot \underline{\underline{B}} \cdot \underline{\underline{B}})$$

that the cross product written as

$$\underline{v} \times \underline{u} = v_p u_s \varepsilon_{psj} \hat{e}_j$$

ivalent to the cross product carried out with the minant method [Equation (2.80)].

that the $\theta$-derivative of the cylindrical unit or $e_r$ is given by $\partial\hat{e}_r/\partial\theta = \hat{e}_\theta$.

t is $\partial\hat{e}_\theta/\partial\theta$ equal to in the cylindrical coordinate em? Derive your answer.

**2.45** Calculate $\partial\hat{e}_r/\partial\phi$, $\partial\hat{e}_\theta/\partial\phi$, and $\partial\hat{e}_\phi/\partial\phi$ for the spherical coordinate system. Your final answer will be in terms of $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$.

**2.46** For the points listed below in Cartesian coordinates $x_1, x_2, x_3$, write the cylindrical system unit vectors at that point, $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$. Express the unit vectors with respect to the 123 Cartesian coordinate system. Sketch the results in the $x_1x_2$-plane.

(a) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}_{123}$

(b) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{123}$

**2.47** Consider the steady flow of a fluid of density $\rho$ in which the velocity at every point is given by the vector field $\underline{v}(x, y, z)$. What is the mass flow rate (mass/time) through a surface of area $A$ located at point $P$? The unit normal to the surface considered is given by the unit vector $\hat{n}$ (Figure 2.15).

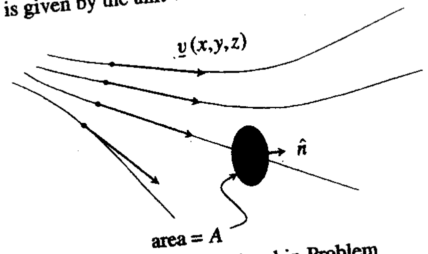

**Figure 2.15** Flow considered in Problem 2.47.

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CHAPTER

3

# Newtonian Fluid Mechanics

It is now helpful to recall the big picture of what we are trying to accomplish. The purpose of this text is to help the reader to understand rheology. Our goal is to be able to take knowledge of material properties and interest in a flow situation and to be able to predict stresses, strains, velocities, or any other variable of interest that will result from the ensuing flow or deformation. Once material properties and the flow situation are supplied, certain stresses, strains, and so on, will be produced because some physical laws are known to hold when matter flows. The physical laws form mathematical constraints on the variables in the problem and allow only particular solutions, that is, once we decide on the material and the flow situation, a particular, nonarbitrary value of the stress, for example, will be produced.

The two physical laws governing the isothermal deformation of matter are the law of conservation of mass and the law of conservation of linear momentum.[1] To obtain the conservation equations in a form compatible with the mathematics we have discussed so far, we will derive equations for both of these laws. These two equations are sometimes called the equations of change. After this we will introduce the Newtonian constitutive equation, which captures, mathematically, how simple fluids respond to stresses and deformation. We spend the latter half of this chapter solving flow problems for Newtonian fluids.

The goal of this chapter, then, is to introduce Newtonian fluid mechanics as a stepping-off point for the study of non-Newtonian fluid mechanics in the remainder of the text. If you are already familiar with Newtonian fluid mechanics, you may wish to skip this chapter.

We begin now by deriving the equations of change.

## ervation of Mass

The usual engineering problem-solving procedure for applying the principle of conservation of mass is to choose a system (a mixer, for example), identify the streams passing into and out of the system, and set up an equation where

$$\text{mass in} - \text{mass out} = 0$$

---

[1] A third conservation law is conservation of energy, and it is essential in solving nonisothermal problems; see [26]. A fourth conservation law, conservation of angular momentum, will be invoked in our discussion of the stress tensor.