C.2 Differential Operations in Curvilinear Coordinates

TABLE C.3
Differential Operations in the Cylindrical Coordinate System r, θ, z

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_z \end{pmatrix}_{r\theta z} \tag{C.3-1}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \tag{C.3-2}$$

$$\nabla a = \begin{pmatrix} \dfrac{\partial a}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial a}{\partial \theta} \\ \dfrac{\partial a}{\partial z} \end{pmatrix}_{r\theta z} \tag{C.3-3}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2} \tag{C.3-4}$$

$$\nabla \cdot \underline{w} = \frac{1}{r}\frac{\partial}{\partial r}(r w_r) + \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z} \tag{C.3-5}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial w_z}{\partial \theta} - \dfrac{\partial w_\theta}{\partial z} \\ \dfrac{\partial w_r}{\partial z} - \dfrac{\partial w_z}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial (r w_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta z} \tag{C.3-6}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z} \tag{C.3-7}$$

$$\nabla \underline{w} = \begin{pmatrix} \dfrac{\partial w_r}{\partial r} & \dfrac{\partial w_\theta}{\partial r} & \dfrac{\partial w_z}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial w_r}{\partial \theta} - \dfrac{w_\theta}{r} & \dfrac{1}{r}\dfrac{\partial w_\theta}{\partial \theta} + \dfrac{w_r}{r} & \dfrac{1}{r}\dfrac{\partial w_z}{\partial \theta} \\ \dfrac{\partial w_r}{\partial z} & \dfrac{\partial w_\theta}{\partial z} & \dfrac{\partial w_z}{\partial z} \end{pmatrix}_{r\theta z} \tag{C.3-8}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial (r w_r)}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2 w_r}{\partial \theta^2} + \dfrac{\partial^2 w_r}{\partial z^2} - \dfrac{2}{r^2}\dfrac{\partial w_\theta}{\partial \theta} \\ \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial (r w_\theta)}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2 w_\theta}{\partial \theta^2} + \dfrac{\partial^2 w_\theta}{\partial z^2} + \dfrac{2}{r^2}\dfrac{\partial w_r}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w_z}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 w_z}{\partial \theta^2} + \dfrac{\partial^2 w_z}{\partial z^2} \end{pmatrix}_{r\theta z} \tag{C.3-9}$$

continued

$$\nabla\cdot\underline{\underline{A}} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(rA_{rr}) + \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{\partial A_{zr}}{\partial z} - \frac{A_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_{r\theta}) + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\partial A_{z\theta}}{\partial z} + \frac{A_{\theta r} - A_{r\theta}}{r} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_{rz}) + \frac{1}{r}\frac{\partial A_{\theta z}}{\partial \theta} + \frac{\partial A_{zz}}{\partial z} \end{pmatrix}_{r\theta z} \quad \text{(C.3-10)}$$

$$\underline{u}\cdot\nabla\underline{w} = \begin{pmatrix} u_r\left(\frac{\partial w_r}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + u_z\left(\frac{\partial w_r}{\partial z}\right) \\ u_r\left(\frac{\partial w_\theta}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + u_z\left(\frac{\partial w_\theta}{\partial z}\right) \\ u_r\left(\frac{\partial w_z}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_z}{\partial \theta}\right) + u_z\left(\frac{\partial w_z}{\partial z}\right) \end{pmatrix}_{r\theta z} \quad \text{(C.3-11)}$$

TABLE C.4
Differential Operations in the Spherical Coordinate System r, θ, ϕ

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_\phi \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-1)}$$

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \quad \text{(C.4-2)}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial a}{\partial \phi} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-3)}$$

$$\nabla\cdot\nabla a = \nabla^2 a = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial a}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial a}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 a}{\partial \phi^2} \quad \text{(C.4-4)}$$

$$\nabla\cdot\underline{w} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(w_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial \phi} \quad \text{(C.4-5)}$$

$$\nabla\times\underline{w} = \begin{pmatrix} \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(w_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial w_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rw_\phi) \\ \frac{1}{r}\frac{\partial}{\partial r}(rw_\theta) - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-6)}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{r\phi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\phi} \\ A_{\phi r} & A_{\phi\theta} & A_{\phi\phi} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-7)}$$

continued

$$\nabla \underline{w} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{\partial w_\theta}{\partial r} & \frac{\partial w_\phi}{\partial r} \\ \frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} & \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} & \frac{1}{r}\frac{\partial w_\phi}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial w_r}{\partial \phi} - \frac{w_\phi}{r} & \frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial \phi} - \frac{w_\phi}{r}\cot\theta & \frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial \phi} + \frac{w_\phi}{r} + \frac{w_\theta}{r}\cot\theta \end{pmatrix}_{r\theta\phi} \tag{C.4-8}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \left\{ \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w_r)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w_r}{\partial \phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}(w_\theta \sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial w_\phi}{\partial \phi} \right\} \\ \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(w_\theta \sin\theta)\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial w_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial w_\phi}{\partial \phi} \right\} \\ \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(w_\phi \sin\theta)\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial w_r}{\partial \phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial w_\theta}{\partial \phi} \right) \end{pmatrix}_{r\theta\phi} \tag{C.4-9}$$

$$\nabla \cdot \underline{\underline{A}} = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_{\theta r}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_{\phi r}}{\partial \phi} - \frac{A_{\theta\theta} + A_{\phi\phi}}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}(r^3 A_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\theta}}{\partial \phi} + \frac{(A_{\theta r} - A_{r\theta}) - A_{\phi\phi}\cot\theta}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}(r^3 A_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_{\theta\phi}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{(A_{\phi r} - A_{r\phi}) + A_{\phi\theta}\cot\theta}{r} \end{pmatrix}_{r\theta\phi} \tag{C.4-10}$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r\left(\frac{\partial w_r}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_r}{\partial \phi} - \frac{w_\phi}{r}\right) \\ u_r\left(\frac{\partial w_\theta}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial \phi} - \frac{w_\phi}{r}\cot\theta\right) \\ u_r\left(\frac{\partial w_\phi}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_\phi}{\partial \theta}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial \phi} + \frac{w_r}{r} + \frac{w_\theta}{r}\cot\theta\right) \end{pmatrix}_{r\theta\phi} \tag{C.4-11}$$

TABLE C.5
Continuity Equation in Three Coordinate Systems

Continuity equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0 \tag{C.5-1}$$

continued

Continuity equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{C.5-2}$$

Continuity equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi} = 0 \tag{C.5-3}$$

TABLE C.6
Equation of Motion for Incompressible Fluids in Three Coordinate Systems

Equation of motion for incompressible fluid, Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \tag{C.6-1}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y \tag{C.6-2}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \tag{C.6-3}$$

Equation of motion for incompressible fluid, cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right] + \rho g_r \tag{C.6-4}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right] + \rho g_\theta \tag{C.6-5}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_z \tag{C.6-6}$$

Equation of motion for incompressible fluid, spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{r\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] + \rho g_r \tag{C.6-7}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{(\cot\theta)\tau_{\phi\phi}}{r}\right] + \rho g_\theta \tag{C.6-8}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} - \frac{v_\phi v_\theta \cot\theta}{r}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\phi})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} - \frac{(2\cot\theta)\tau_{\theta\phi}}{r}\right] + \rho g_\phi \tag{C.6-9}$$

TABLE C.7
Equation of Motion for Incompressible Newtonian Fluids: The Navier–Stokes Equations in Three Coordinate Systems

$\underline{\underline{\tau}} \equiv \mu \dot{\underline{\underline{\gamma}}} = \mu(\nabla \underline{u} + (\nabla \underline{u})^{+})$

Equation of motion for incompressible Newtonian fluid (Navier–Stokes equation), Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x \qquad \text{(C.7-1)}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y \qquad \text{(C.7-2)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z \qquad \text{(C.7-3)}$$

Equation of motion for incompressible Newtonian fluid (Navier–Stokes equation), cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad \text{(C.7-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad \text{(C.7-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad \text{(C.7-6)}$$

Equation of motion for incompressible Newtonian fluid (Navier–Stokes equation), spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{v_\theta}{\partial \theta} - \frac{2v_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_r \qquad \text{(C.7-7)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_\theta \qquad \text{(C.7-8)}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right) + \rho g_\phi \qquad \text{(C.7-9)}$$

where

$$\nabla \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial \phi^2}\right)$$