reference: Kreysig, Chapters 8 & 9
Note: We will introduce a notation not used in Kreysig.

IV. CALCULUS OF VECTORS, DYADICS & TENSORS

A. Introduction & Review

1. In this section we will review some elementary ideas of vector calculus which you have seen in earlier math, physics and mechanics (statics or dynamics) courses. We will then try to develop a more 'user-friendly' approach to the study of vector calculus and introduce the concept of tensors

In turn we will discuss

- scalar (or dot or inner) product
- vector (or cross) product
- index notation and the summation convention for making vector calculations fast & easy
- vector calculus → taking derivatives of scalar and vector functions
  → gradient, divergence, curl
- Lots of vector operations and identities
- Integral Theorems - Divergence and Stokes Theorem. Also, different forms of Green's Theorem, which will arise again in our study of pdes.
- Application of some of these ideas to derive a diffusion eqn, study geometry of curves (and surfaces), etc.
- vector calculus of orthogonal curvilinear coordinates such as cartesian, cylindrical and spherical coordinates, as well as a discussion of the general case which leads to such systems as oblate and prolate spheroidal coordinates.
- an introduction to tensors and dyadics

... and the answer to the question "What is a tensor?"

NOTE: There are a number of books which focus on teaching about vectors, tensors and vector calculus. Several of my favorite references for reading about ideas and examples are
F. Chorlton, Vector & Tensor Methods
H.M. Schey, Div, Grad, Curl and all That
J.G. Simmonds, A Brief on Tensor Analysis

2. scalars & vectors

a. A <u>scalar</u> is a quantity or function with <u>magnitude</u> only

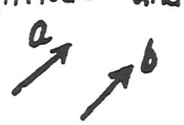
e.g. mass and temperature are common scalars.

b. A <u>vector</u> is characterized by <u>magnitude</u> and <u>direction</u>.

(i) The most common examples of vectors are a <u>force</u> applied to move a given object (clearly you must know both the magnitude of the force as well as its direction in order to describe what happens to the object) and the <u>velocity</u> with which a given object moves (e.g., if you compare driving in a straightline at 20 mph or driving at 20 mph on a circular track, then you realize in order to completely characterize the motion it is necessary to specify both the <u>magnitude</u> of the velocity - i.e., speed - as well as <u>direction</u>).

(ii) Vectors are typically represented geometrically with arrows ↗. (length denotes magnitude; orientation indicates direction)

(iii) Two vectors are equal if they have the same magnitude and same direction.



conclude $a = b$

sometimes this is called "parallel transport of vectors"

→ Nevertheless, it is very important to keep in mind that the <u>effect</u> of a given vector may, indeed often will, depend on its <u>location</u>.

(iv) NOTATION: I will typically indicate a vector quantity by an <u>underline</u>: $\underset{\sim}{a}$ or $\underset{\sim}{b}$ (books use bold-faced type)

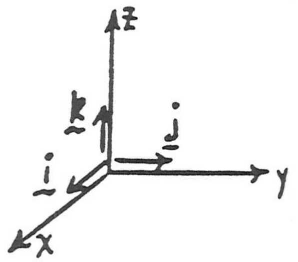
(Another common notation is to use arrows: $\vec{a}$ or $\vec{b}$)

one final comment on notation: a UNIT VECTOR by definition has length = 1

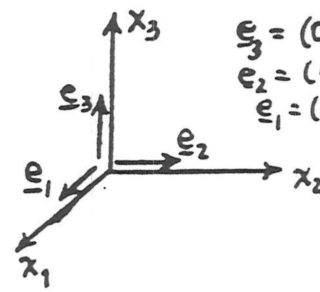
3 Cartesian Coordinate System

a. We will describe a vector by giving its components relative to a cartesian, or x, y, z, coordinate system.

b. We will indicate the base vectors as (by definition the base vectors have length = 1)



or



$\underset{\sim}{e}_3 = (0,0,1)$
$\underset{\sim}{e}_2 = (0,1,0)$
$\underset{\sim}{e}_1 = (1,0,0)$

"What you have probably seen" — "What we will use"

NOTE: We will indicate the 3 coordinate directions by x_1, x_2, x_3 where $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$.

C. In order to describe a vector you must give both the components and the base vectors:

For example:
$$\underset{\sim}{a} = a_x \underset{\sim}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k} \qquad (114)$$

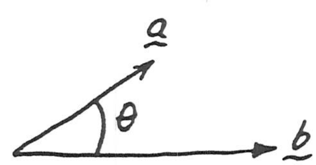
or we will write:
$$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3$$

→ length: $|\underset{\sim}{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

4. SCALAR product (also called the DOT or INNER product)

a. The scalar product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as

$$(115) \qquad \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$$



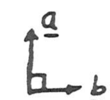
where $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$ are the lengths, or magnitudes, of the two vectors.

(scalar product continued)

b. The scalar product is commutative and distributive.

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{a} \quad \text{and} \quad (\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c} = \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{c}$$

c. Two vectors are <u>orthogonal</u> if $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$



If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are both nonzero vectors $(\underset{\sim}{a},\underset{\sim}{b} \neq \underset{\sim}{0})$
then $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \rightarrow \theta = \pi/2$ and the two vectors are perpendicular.

d. Since the cartesian unit vectors $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ satisfy
$\underset{\sim}{i}\cdot\underset{\sim}{i} = 1$, $\underset{\sim}{i}\cdot\underset{\sim}{j} = 0$, $\underset{\sim}{i}\cdot\underset{\sim}{k} = 0$, etc. then

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = a_x b_x + a_y b_y + a_z b_z \qquad (116)$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{in our new notation.}$$

Also, $\underset{\sim}{a}\cdot\underset{\sim}{a} = a_1^2 + a_2^2 + a_3^2 = |\underset{\sim}{a}|^2$ (117) (square of the length)

5. VECTOR (or CROSS) product of two vectors

a. The vector product of 2 vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as

NOTE
My notation for this operation is ^; many others write X

$$\underset{\sim}{a}\wedge\underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin\theta\ \underset{\sim}{e} \qquad (118)$$

where $\underset{\sim}{e}$ is a unit vector in the direction perpendicular to the plane formed by $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as given by the RIGHT-HAND RULE.

b. From the definition: $\underset{\sim}{a}\wedge\underset{\sim}{a} = \underset{\sim}{0}$ (cross product of a vector with itself $= \underset{\sim}{0}$)

Important to understand this

Also, $\boxed{\underset{\sim}{a}\wedge\underset{\sim}{b} = -\underset{\sim}{b}\wedge\underset{\sim}{a}}$ (119a) and $\underset{\sim}{a}\wedge(\underset{\sim}{b}\wedge\underset{\sim}{c}) = \underset{\sim}{a}\wedge\underset{\sim}{b} + \underset{\sim}{a}\wedge\underset{\sim}{c}$ (119b)

It follows that $\underset{\sim}{i}\wedge\underset{\sim}{j} = \underset{\sim}{k}$; $\underset{\sim}{j}\wedge\underset{\sim}{k} = \underset{\sim}{i}$; $\underset{\sim}{k}\wedge\underset{\sim}{i} = \underset{\sim}{j}$

c. You may also remember writing something like

$$\underset{\sim}{a}\wedge\underset{\sim}{b} = \det\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \underset{\sim}{i}(a_y b_z - a_z b_y) + \underset{\sim}{j}(a_z b_x - a_x b_z) + \underset{\sim}{k}(a_x b_y - a_y b_x) \qquad (119c)$$

← rather laborious to write!

Much of the above is cumbersome and frightfully lengthy to write. We now introduce a convenient shorthand notation which will simplify many manipulations.

B. EINSTEIN INDEX NOTATION & THE SUMMATION CONVENTION

1. Let us reconsider some of the above manipulations. From now on keep in mind that we are typically having to represent vectors in a three-dimensional world and so we will always write vectors with 3 components.

2. Label the (x,y,z) coordinates by $(1,2,3)$.

   a. Denote the vector $\underset{\sim}{a}$ with components a_i and base vectors $\underset{\sim}{e}_i$ where the subscript i may take on the 3 values $i=1,2$ or 3.
   We write

$$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3 = \sum_{i=1}^{3} a_i \underset{\sim}{e}_i \equiv a_i \underset{\sim}{e}_i \qquad (= a_j \underset{\sim}{e}_j) \tag{120}$$

↑ dummy summation symbol

IMPORTANT

   b. From now on we will <u>not</u> usually write the summation symbol. Instead we will invoke the SUMMATION CONVENTION —

SUMMATION CONVENTION

   if an index (i) appears TWICE in an expression, then we will know that we should perform a summation with $i=1,2$ and 3.

   Thus, the vector $\underset{\sim}{a}$ is expressed as $\underset{\sim}{a} = a_i \underset{\sim}{e}_i = a_m \underset{\sim}{e}_m$

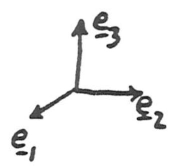
↑ dummy summation index; two sides of equal sign mean the same thing.

   → This idea of implied summation when an index appears TWICE must be clear in your mind before you move on.

   c. This notation simply allows us to proceed without carrying around the summation symbol.

3. inner product of base vectors and the Kronecker delta symbol

a. We previously saw that $\underset{\sim}{i} \cdot \underset{\sim}{i} = 1$, $\underset{\sim}{j} \cdot \underset{\sim}{j} = 1$, $\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$, $\underset{\sim}{i} \cdot \underset{\sim}{k} = 0$, etc or in terms of the $(1,2,3)$ notation we may write



$$\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1 = 1 \;;\quad \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_2 = 1 \;;\quad \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_1 = 0 \,,\; \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 = 0 \,,\; \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = 0, \text{ etc}$$

b. Thus, we see that

$$(121) \qquad \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = 1 \quad \text{if } i=j$$
$$= 0 \quad \text{if } i \neq j$$

orthogonality of the base vectors

where i and j may take on any of the values 1, 2 or 3.

↓ a convenient shorthand notation

c. Kronecker delta δ_{ij}

let us define the symbol δ_{ij} as

$$(122) \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad \text{where } \begin{array}{l} i = 1,2,3 \\ j = 1,2,3 \end{array}$$

Therefore we see that

$$(123) \qquad \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij} \qquad \left(\begin{array}{lll} \text{which} & = 1 & \text{if } i=j \\ \text{and} & = 0 & \text{if } i \neq j \end{array}\right)$$

4. The scalar product revisited

a. $$\underset{\sim}{a} \cdot \underset{\sim}{b} = \sum_{i=1}^{3} a_i \underset{\sim}{e}_i \cdot \sum_{j=1}^{3} b_j \underset{\sim}{e}_j = \sum_{i=1}^{3} a_i b_i = a_i b_i \quad (= a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = 1 \text{ if } i=j$$
$$= 0 \text{ if } i \neq j$$

where we have invoked the summation convention

NOTE: in the second sum, $\sum_{j=1}^{3} b_j \underset{\sim}{e}_j$ we have used the dummy summation index j so as not to confuse operations with the first sum utilizing i as a summation index

b. Use the δ_{ij} shorthand and the summation convention

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i \underset{\sim}{e}_i \cdot b_j \underset{\sim}{e}_j = a_i b_j (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j) = \underbrace{a_i b_j \delta_{ij}}_{\substack{\text{actually represents the double sum} \\ \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \delta_{ij}}} = a_i b_i$$

only nonzero if $i=j$

vector operation; only acts on vectors and not the components.

$$\therefore \quad \boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_i \; (= a_j b_j = a_m b_m)} \qquad (124)$$

δ_{ij} is sometimes called the "replacement operator".

c. VERY IMPORTANT: We are using the summation convention to recognize that whenever we see a repeated index (e.g., $a_i \underset{\sim}{e}_i$, or $a_j b_j$) we mean an implied sum with the index taking the values 1, 2 and 3. Thus, it is important to use different summation labels when beginning operations: For example,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i \underset{\sim}{e}_i \cdot b_j \underset{\sim}{e}_j$$

a different summation index!

NEVER WRITE $a_i \underset{\sim}{e}_i \cdot b_i \underset{\sim}{e}_i$ which is very confusing.

d. A few remarks

(i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ where we have used the summation convention.

(ii) δ_{ij} : the replacement operator. $\Rightarrow \delta_{ij} c_j = c_i$

(iii) Sometimes people are sloppy and do not write the unit vectors i.e., instead of $\underset{\sim}{a} = a_i \underset{\sim}{e}_i$, occasionally you see written $\underset{\sim}{a} = a_i$ which you can recognize since an index will appear only once ('free index')

and could take on a value 1, 2 or 3

Whenever performing vector operations I will ALWAYS keep track of the unit vectors.

(iv) Example:

$$(\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{c} = a_i b_i \underset{\sim}{c} = a_i b_i c_j \underset{\sim}{e}_j$$

and you might call $(a_i b_i) c_j$ the j^{th} component of the vector.

** this symbol will be useful whenever vector products arise. **

5. Permutation Symbol ε_{ijk} $\quad i=1,2,3 \quad j=1,2,3 \quad k=1,2,3$

a. Definition:

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ or } -1 & \text{if } i,j,k \text{ are all different} \\ 0 & \text{if any two indices are the same} \end{cases} \qquad (125a)$$

In particular,

(125b) $\varepsilon_{ijk} = +1$ if if i,j,k are an EVEN permutation of $1,2,3$ (or $1,2,3$ are in cyclic order)

$\rightarrow \quad \varepsilon_{123} = 1 \quad \varepsilon_{312} = 1 \quad \varepsilon_{231} = 1$

(125c) $\varepsilon_{ijk} = -1$ if i,j,k are an ODD permutation of $1,2,3$

$\varepsilon_{213} = -1 \quad \varepsilon_{132} = -1 \quad \varepsilon_{321} = -1$

NOTE: By even permutation we mean that an even # of interchanges of the indices must occur to get back to the order 123; analogous for meaning of odd permutation.

b. This definition has the following cyclic and interchange property:

compare with above example: $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231}$ ←

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} \qquad (126a)$$

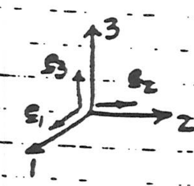
and if two indices are simply interchanged, the sign changes, **IMPORTANT**

$$(126b) \quad \varepsilon_{ijk} = -\varepsilon_{ikj} \quad \text{or} \quad \varepsilon_{ijk} = -\varepsilon_{jik}$$

Also, since i,j,k can each independently take on the values $1,2,3$, then ε_{ijk} represents 27 quantities.

c. We also have $\boxed{\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k}$ (127)

∴ cross-product of base vectors (or any 2 vectors) will always invoke the permutation symbol ε_{ijk}



and by referring to the figure at right, everything is ok.: $\underline{e}_1 \wedge \underline{e}_2 = +\underline{e}_3 = \underbrace{\varepsilon_{123}}_{+1} \underline{e}_3$ etc.

d. We now have an effective shorthand notation for representing the vector product.

let $\underline{c} = \underline{a} \wedge \underline{b}$; write $\underline{a} = a_i \underline{e}_i$, $\underline{b} = b_j \underline{e}_j$

$\rightarrow \quad = a_i \underline{e}_i \wedge b_j \underline{e}_j = a_i b_j (\underline{e}_i \wedge \underline{e}_j)$

(128) $\boxed{\underline{a} \wedge \underline{b} = a_i b_j\, \varepsilon_{ijk}\, \underline{e}_k}$ ← NOTE CAREFULLY THE ORDER OF THE INDICES

or with $\underline{c} = c_k \underline{e}_k$, we have $c_k = a_i b_j \varepsilon_{ijk}$ which represents 3 eqns for $k=1,2$ or 3

EXERCISE: Verify that this is in agreement with the 'matrix' definition ...

e. triple scalar product: $\underline{a}\cdot(\underline{b}\wedge\underline{c})$

Again we are careful to use different dummy indices for each vector so

$$\underline{a}\cdot(\underline{b}\wedge\underline{c}) = a_i\,\underline{e}_i\cdot(b_j\,\underline{e}_j\wedge c_k\,\underline{e}_k) = a_i\,\underline{e}_i\cdot(b_j c_k\,\varepsilon_{jkl}\,\underline{e}_l)$$

$$= a_i b_j c_k\,\varepsilon_{jkl}\,\underbrace{\underline{e}_i\cdot\underline{e}_l}_{\delta_{il}}$$

$$= \varepsilon_{jki}\,a_i b_j c_k = \varepsilon_{ijk}\,a_i b_j c_k = (\underline{a}\wedge\underline{b})\cdot c = (\underline{c}\wedge\underline{a})\cdot b$$

by using cyclic property of ε_{ijk}. Exercise - convince yourself that these last 2 identities follow from index expression.

Recall also that

$$\underline{a}\cdot(\underline{b}\wedge\underline{c}) = \det\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underbrace{\varepsilon_{ijk}\,a_i b_j c_k}$$

index representation of the 3x3 determinant.

5. Useful identities involving ε and δ

you are not expected to memorize this eqn. It will be frequently used however

$$(129)\qquad \boxed{\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}}$$

i,j,l,m can each independently take on values 1, 2, 3. Hence, this eqn corresponds to 81 quantities.

<u>Proof</u>: verify by brute force for each of the 81 eqns! However, it is best to make your life easier by noticing that both sides change sign if either $i \leftrightarrow j$ or $l \leftrightarrow m$ are interchanged. Also, both sides vanish if $i=j$ or $l=m$. Then, consider remaining terms like:

$$\varepsilon_{12k}\,\varepsilon_{k12} = \varepsilon_{121}\cancel{\varepsilon_{112}}^{\,0} + \varepsilon_{122}\cancel{\varepsilon_{212}}^{\,0} + \varepsilon_{123}\,\varepsilon_{312} = 1$$

and

$$\delta_{11}\,\delta_{22} - \cancel{\delta_{12}}^{\,0}\,\delta_{12} = 1 \quad \text{so o.k.}$$

Likewise

$$\varepsilon_{12k}\,\varepsilon_{k13} = \varepsilon_{121}\,\varepsilon_{113} + \varepsilon_{122}\,\varepsilon_{213} + \varepsilon_{123}\,\varepsilon_{313} = 0; \text{ also } \delta_{11}\,\delta_{23} - \delta_{13}\,\delta_{23} = 0 \text{ so o.k.}$$

etc. //

$\boxed{\text{Example 1}}$: show that $\underline{e}_i = \frac{1}{2}\,\varepsilon_{mni}\,\underline{e}_m\wedge\underline{e}_n$

Well,

$$\varepsilon_{mni}\,\underline{e}_m\wedge\underline{e}_n = \varepsilon_{mni}\,\varepsilon_{mnj}\,\underline{e}_j = \overbrace{\varepsilon_{nim}\,\varepsilon_{mnj}}^{\delta_{nn}\delta_{ij}-\delta_{nj}\delta_{ni}}\,\underline{e}_j$$

$$= (3\,\delta_{ij} - \delta_{ij})\,\underline{e}_j = 2\,\underline{e}_i$$

$\boxed{\text{Example 2}}$: show that $\underline{a}\wedge(\underline{b}\wedge\underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{c}(\underline{a}\cdot\underline{b})$ "BAC-CAB" identity

$$\underline{a}\wedge(\underline{b}\wedge\underline{c}) = a_i\,\underline{e}_i\wedge(b_j\,\underline{e}_j\wedge c_k\,\underline{e}_k) = a_i\,\underline{e}_i\wedge(b_j c_k\,\varepsilon_{jkl}\,\underline{e}_l) = a_i b_j c_k\,\varepsilon_{jkl}(\underline{e}_i\wedge\underline{e}_l)$$

$$= a_i b_j c_k\,\varepsilon_{jkl}\,\varepsilon_{ilm}\,\underline{e}_m = a_i b_j c_k\,\varepsilon_{jkl}\,\varepsilon_{lmi}\,\underline{e}_m = a_i b_j c_k(\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km})\underline{e}_m$$

$$= a_i b_m c_i\,\underline{e}_m - a_i b_i c_m\,\underline{e}_m = (a_i c_i)\,b_m\,\underline{e}_m - (a_i b_i)\,c_m\,\underline{e}_m = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c}$$

Some additional Examples of the use of index notation

First, a brief summary of the important ideas

(i) $\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

(ii) $\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k$ $\qquad \varepsilon_{ijk} = \begin{cases} +1 & i,j,k \text{ an even permutation of } 1,2,3 \\ -1 & i,j,k \text{ an odd permutation of } 1,2,3 \\ 0 & \text{any two indices the same} \end{cases}$

(iii) summation convention : whenever a subscript appears twice, a summation from $1 \to 3$ is implied.
(a subscript should __never__ appear more than twice)

Examples :

(i) $\delta_{ik}\delta_{jk} = \delta_{ij}$ since δ_{jk} is only nonzero when $j=k$ so the k in δ_{ik} may be replaced by j.

(ii) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$ note: since i was a dummy index, $\delta_{ii} = \delta_{kk} = \delta_{mm}$ etc.

(iii) $\delta_{ij}\,\varepsilon_{ijk} = \varepsilon_{iik} = 0$ since two of the indices are the same.

(iv) $\varepsilon_{ijk}\varepsilon_{njk} = \varepsilon_{ijk}\varepsilon_{knj}$ by first rotating the indices on the second ε.

Next, use the identity : $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$\Rightarrow \varepsilon_{ijk}\varepsilon_{knj} = \delta_{in}\underbrace{\delta_{jj}}_{3} - \delta_{ij}\delta_{nj} = 3\delta_{in} - \delta_{in} = \underline{2\delta_{in}}$

(v) $a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = ?$

$\rightarrow$ m & n appear twice in each term so summation is implied.

But, m & n are simply dummy variables, i.e., we could just as well use another letter.

So, examine the second term $a_n b_m \varepsilon_{mnq}$.

$a_n b_m \varepsilon_{mnq} = -a_n b_m \varepsilon_{nmq}$; now let $j=n$, $m=k$

$= -a_j b_k \varepsilon_{jkq}$ which is the same as the above since summation on j,k is implied.

$= -a_m b_n \varepsilon_{mnq}$ by letting $j=m$ $k=n$

So, we see that

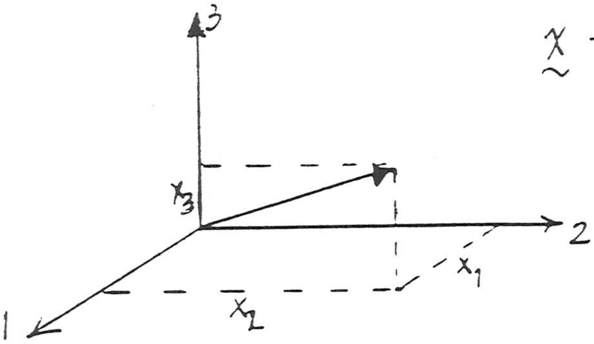
$a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = 2\underbrace{a_m b_n \varepsilon_{mnq}}_{(\underline{a}\wedge\underline{b})_q}$ $\leftarrow$ q^{th} component of $\underline{a}\wedge\underline{b}$

→ + DIV, GRAD, CURL.

C. SOME VECTOR CALCULUS (taking derivatives of vector functions)

1. We now wish to consider the derivatives of (i) vector functions which depend on a scalar variable, e.g., calculate rate-of-change of the velocity $\underset{\sim}{v}(t)$ with respect to time, (ii) scalar functions which depend on position, e.g., the temperature $T(x,y,z)$ at spatial location (x,y,z), and (iii) vector functions which depend on position, e.g., the electric field $\underset{\sim}{E}(x,y,z)$ or velocity field of a fluid $\underset{\sim}{v}(x,y,z)$ at position (x,y,z).

2. Notation: we will typically use the vector $\underset{\sim}{x}$ to denote the position vector locating a point in space.



$$\underset{\sim}{x} = (x, y, z) = (x_1, x_2, x_3)$$

the value of ϕ depends on location $\underset{\sim}{x}$ in space

One can discuss SCALAR fields $\phi(\underset{\sim}{x}) = \phi(x_1, x_2, x_3)$ or just ϕ

one can discuss VECTOR fields of position $\underset{\sim}{a}(\underset{\sim}{x}) = a_1(x_1, x_2, x_3)\,\underset{\sim}{e}_1 + a_2(x_1, x_2, x_3)\,\underset{\sim}{e}_2$
$+ a_3(x_1, x_2, x_3)\,\underset{\sim}{e}_3$

each of the components of the vector $\underset{\sim}{a}$ depends on location in space. ⇒ sometimes one sees this written $a_j(x)$

and one can discuss vector fields of a single variable, e.g., $\underset{\sim}{b}(t)$.

3. Differentiation of vectors

a. Suppose $\underset{\sim}{a} = \underset{\sim}{a}(t) = a_i(t)\, \underline{e}_i$ — cartesian base vectors; constant length (= 1) and constant direction.

Then,[‡] $\frac{d}{dt}\underset{\sim}{a}(t) = \frac{da_i(t)}{dt}\underline{e}_i$ since the cartesian base vectors are constant vectors, $\frac{d}{dt}\underline{e}_i = 0$.

b. By the product rule

$$\frac{d}{dt}\left[\underset{\sim}{a}(t)\cdot\underset{\sim}{b}(t)\right] = \frac{d\underset{\sim}{a}}{dt}\cdot\underline{b} + \underset{\sim}{a}\cdot\frac{d\underset{\sim}{b}}{dt}$$

c. Exercise: Evaluate $\frac{d}{dt}\left[\phi(t)\,\underline{a}(t)\right]$.

We will now consider spatial derivatives of scalar functions, e.g., $\frac{\partial}{\partial x}\phi(x,y)$, $\frac{\partial\phi}{\partial y}(x,y)$ or $\frac{\partial}{\partial x}\underset{\sim}{b}(x,y,z)$.

4. Gradient operator Kreysig §8.e-9; Greenberg §9.3; Hildebrand §6.

a. Let $\phi(\underset{\sim}{x})$ be a scalar function which varies with position x, y, z in space.

The rate of variation of ϕ in the x-direction is $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x_1}$, in the y-direction is $\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x_2}$, and in the z-direction is $\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial x_3}$.

b. We introduce the vector

$$(130)\quad \text{grad}\,\phi \equiv \underline{\nabla}\phi = \underline{e}_1\frac{\partial\phi}{\partial x_1} + \underline{e}_2\frac{\partial\phi}{\partial x_2} + \underline{e}_3\frac{\partial\phi}{\partial x_3} = \underline{e}_i\frac{\partial\phi}{\partial x_i} = \underline{e}_i\,\underbrace{\phi_{,i}}$$

(called the gradient of ϕ)

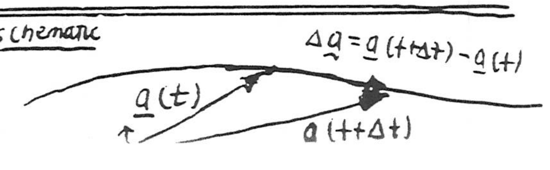
Component in $\underline{e}_i$ is the derivative with respect to x_i

"comma" notation to indicate differentiation with respect to x_i

$$\text{gradient operator} \equiv \underset{\sim}{\nabla} = \underline{e}_1\frac{\partial}{\partial x_1} + \underline{e}_2\frac{\partial}{\partial x_2} = \underline{e}_3\frac{\partial}{\partial x_3} \quad\text{or}\quad \boxed{\underset{\sim}{\nabla} = \underline{e}_i\frac{\partial}{\partial x_i}} \quad (131)$$

---

‡ $\frac{d}{dt}\underset{\sim}{a}(t) = \lim_{t\to 0}\frac{\underline{a}(t+\Delta t) - \underline{a}(t)}{\Delta t}$

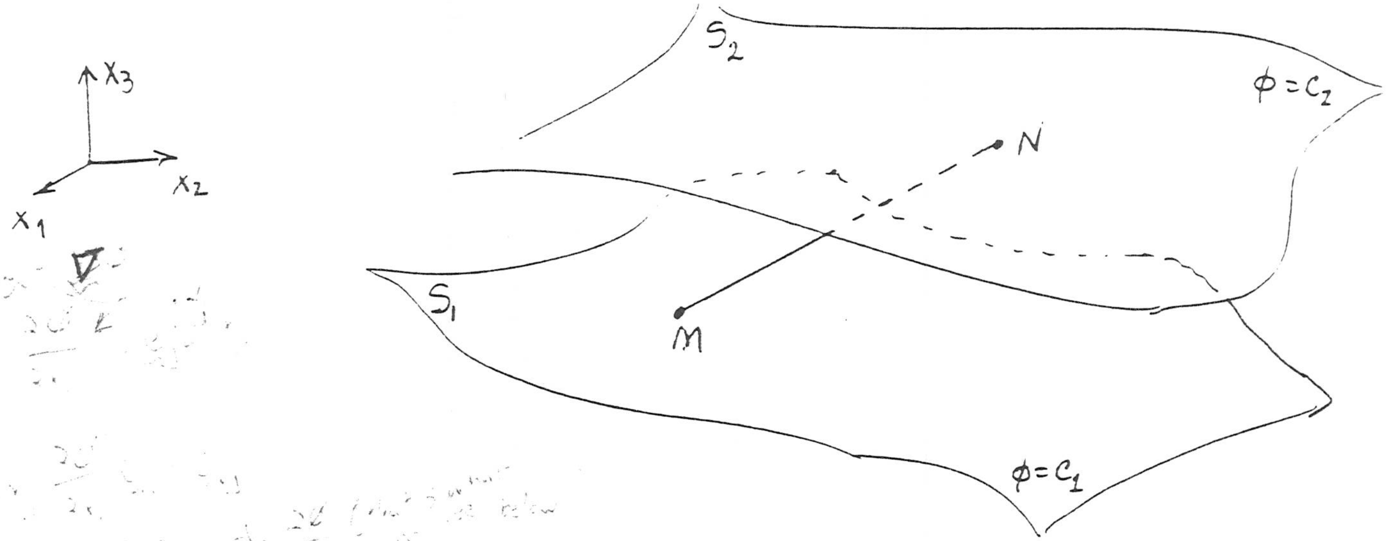


} schematic interpreta[tion] of the derivative

For those who are interested in a more geometric interpretation of the gradient operator:

Consider a scalar function of position $\phi(\underset{\sim}{x}) = \phi(x,y,z)$.

Imagine drawing two surfaces in space, labelled S_1 and S_2, where $\phi = c_1$ = constant along S_1 and $\phi = c_2$ = some other constant along S_2



" An exaggerated view of two surfaces along which $\phi(\underset{\sim}{x})$ = constant "

Consider the two nearby points labelled M and N. Clearly, you should recognize and appreciate that since the 'spacing' or distance between the two surfaces changes at different locations along either of the surfaces, then the rate-of-change of ϕ with respect to changes in spatial location will depend on the direction in which the change is meas

Now, for small (infinitesimal) changes, construct a $\boxed{\text{Taylor series}}$ about the point M:

$$\phi(N) = \phi(M) + dx_1 \left.\frac{\partial\phi}{\partial x_1}\right|_M + dx_2 \left.\frac{\partial\phi}{\partial x_2}\right|_M + dx_3 \left.\frac{\partial\phi}{\partial x_3}\right|_M + \cdots$$

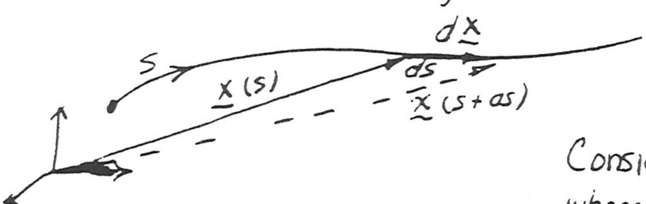
or $\boxed{d\phi = \phi(N) - \phi(M) = d\underset{\sim}{x} \cdot \underset{\sim}{\nabla}\phi}$ where $\underset{\sim}{\nabla}\phi = \underset{\sim}{e}_i \frac{\partial\phi}{\partial x_i}$ and $d\underset{\sim}{x} = \underset{\sim}{e}_j\, dx_j$

change in ϕ given displacement $d\underset{\sim}{x}$.

5. The directional derivative and the gradient operator

a. Suppose you are interested in some scalar function of position $\phi(\underset{\sim}{x})$ which may describe the temperature distribution in a component in your stereo or computer. The change in temperature if you are displaced by a small amount $d\underset{\sim}{x}$ (has magnitude and direction) is given by the 'directional derivative'.

b. Preliminary: consider a path, or curve, in space. Measure distance along the curve by s, which is the arclength.



where $d\underset{\sim}{x} = \underset{\sim}{e}_i\, dx_i = \underset{\sim}{e}_1\, dx_1 + \underset{\sim}{e}_2\, dx_2 + \underset{\sim}{e}_3\, dx_3$

$(d\underset{\sim}{x}\cdot d\underset{\sim}{x} = |d\underset{\sim}{x}|^2 = dx_1^2 + dx_2^2 + dx_3^2 = ds$

Consider a small displacement $d\underset{\sim}{x}$ where $|d\underset{\sim}{x}| = ds$ (distance along path)

$\Rightarrow$ The unit tangent vector $\underset{\sim}{t}$ in the direction of $d\underset{\sim}{x}$ is

$$\underset{\sim}{t} = \frac{d\underset{\sim}{x}(s)}{ds} = \underset{\sim}{e}_i \frac{dx_i(s)}{ds}$$

c. Ask how ϕ varies along the curve? $\Rightarrow$ $\dfrac{d\phi}{ds} = ?$

Use the chain-rule:

$$\frac{d}{ds}\phi(\underset{\sim}{x}(s)) = \frac{dx_1}{ds}\frac{\partial\phi}{\partial x_1} + \frac{dx_2}{ds}\frac{\partial\phi}{\partial x_2} + \frac{dx_3}{ds}\frac{\partial\phi}{\partial x_3} = \frac{dx_i}{ds}\frac{\partial\phi}{\partial x_i}$$

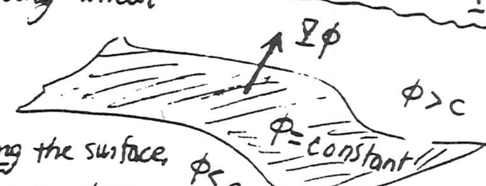
position along path

$$\boxed{\frac{d\phi}{ds} = \underset{\sim}{t}\cdot\underline{\nabla}\phi} \quad (132)$$

$\leftarrow$ directional derivative of ϕ in the $\underset{\sim}{t}$-direction.

Exercise: Show using index notati[on] that $\underline{t}\cdot\underline{\nabla}\phi = t_i \dfrac{\partial\phi}{\partial x_i}$

d. Now consider a surface along which $\phi(\underset{\sim}{x}) = \text{constant}$.



Now, as ϕ = constant on this surface, then $d\phi = 0$ for any displacement along the surface.

interpretation of $\underline{\nabla}\phi$

Then, from (132) $\underline{t}\cdot\underline{\nabla}\phi = 0$ for movements along or tangent to the surface. It necessarily follows that $\underline{t} \perp \underline{\nabla}\phi$ ... $\underline{\nabla}\phi$ is ...

6. Divergence of a vector field : $\underset{\sim}{\nabla}\cdot\underset{\sim}{f}$ or $\operatorname{div}\underset{\sim}{f}$

a. Simply compute using standard ideas.

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \left(\underset{\sim}{e}_i \frac{\partial}{\partial x_i}\right)\cdot\left(f_j\, \underset{\sim}{e}_j\right) = \underset{\sim}{e}_i \cdot \frac{\partial f_j}{\partial x_i}\, \underset{\sim}{e}_j + f_j\, \underset{\sim}{e}_i \cdot \frac{\partial \underset{\sim}{e}_j}{\partial x_i}$$ using the product rule

$\frac{\partial \underset{\sim}{e}_j}{\partial x_i} = 0$ since the $\underset{\sim}{e}_j$ are unit vectors which do not vary with position in space

$$= \delta_{ij}\frac{\partial f_j}{\partial x_i}$$

comma notation sometimes used

(133) $$\boxed{\underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \frac{\partial f_j}{\partial x_j}} \left(= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) = f_{j,j}$$

b. NOTE: Now that you have gone through this, make your life easier. The $\underset{\sim}{e}_j$ are constant vectors with respect to differentiation so we know it is ok to simply write

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \underset{\sim}{e}_i \frac{\partial}{\partial x_i}\cdot\left(f_j\, \underset{\sim}{e}_j\right) = \underset{\sim}{e}_i\cdot\underset{\sim}{e}_j \frac{\partial f_j}{\partial x_i} = \frac{\partial f_j}{\partial x_j}$$

same ind

⇒ Also, whenever you see a term like $\frac{\partial f_k}{\partial x_k}$, you now know $\frac{\partial f_k}{\partial x_k} = \underset{\sim}{\nabla}\cdot\underset{\sim}{f}$.

c. An identity using index notation : let $\phi(\underset{\sim}{x})$ be a scalar field

$$\underset{\sim}{\nabla}\cdot\left(\phi\, \underset{\sim}{f}\right) = \underset{\sim}{e}_i \frac{\partial}{\partial x_i}\cdot\left(\phi f_j\, \underset{\sim}{e}_j\right)$$ ← (i) use product rule (ii) $\underset{\sim}{e}_j$ are constant vectors (iii) the inner product (•) only operates on vectors, not the scalar components

$$= \left(\underset{\sim}{e}_i\cdot\underset{\sim}{e}_j\right)\frac{\partial}{\partial x_i}\left(\phi f_j\right)$$

$$= \delta_{ij}\left[\frac{\partial \phi}{\partial x_i} f_j + \phi \frac{\partial f_j}{\partial x_i}\right]$$

$$= \frac{\partial \phi}{\partial x_j} f_j + \phi\frac{\partial f_j}{\partial x_j} = \left(\underset{\sim}{\nabla}\phi\right)\cdot\underset{\sim}{f} + \phi\, \underset{\sim}{\nabla}\cdot\underset{\sim}{f}$$

$$\therefore\ \underset{\sim}{\nabla}\cdot\left(\phi\, \underset{\sim}{f}\right) = \left(\underset{\sim}{\nabla}\phi\right)\cdot\underset{\sim}{f} + \phi\, \underset{\sim}{\nabla}\cdot\underset{\sim}{f} \qquad (134)$$ ← Notice how similar this is to the normal product rule of differentiation

Exercise: Evaluate (i) $\underset{\sim}{\nabla}\cdot\left(\underset{\sim}{a}\wedge\underset{\sim}{b}\right)$

(ii) $\underset{\sim}{\nabla}\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$

where $a(x)$ and $b(x)$ are vector functions of position

7. Curl of a vector field: $\underline{\nabla} \wedge \underset{\sim}{f}$ or sometimes curl $\underset{\sim}{f}$

a. Let us first calculate using standard ideas

$$\underline{\nabla} \wedge \underline{f} = \underline{e}_i \frac{\partial}{\partial x_i} \wedge (f_j \, \underline{e}_j)$$

← As before, the $\underline{e}_j$ are constant vectors. Also, the curl ($\wedge$) operation only affects vecto

$$= (\underline{e}_i \wedge \underline{e}_j) \frac{\partial f_j}{\partial x_i}$$

$$\therefore \boxed{\underline{\nabla} \wedge \underline{f} = \epsilon_{ijk} \frac{\partial f_j}{\partial x_i} \underline{e}_k} \qquad (135)$$

(remember: the summation convention is in effect. Write out what eqn 135 represents.)

NOTE: Sometimes you will ~~find that people write~~

$$(\underline{\nabla} \wedge \underline{f})_k = \epsilon_{ijk} \frac{\partial f}{\partial x_i}$$

↑ indicates the k^{th} component of the vector $\underline{\nabla} \wedge \underline{f}$.

b. Also, since $\varepsilon_{ijk} = \varepsilon_{kij}$ we may write

$$\underline{\nabla} \wedge \underline{f} = \varepsilon_{kij} \frac{\partial f_j}{\partial x_i} \underline{e}_k$$

c. Alternatively, let's simply show that eqn (135) agrees with what you have seen in your earlier vector calculus courses.

Since

$$\underline{\nabla} \wedge \underset{\sim}{f} = (\underline{e}_i \wedge \underline{e}_j) \frac{\partial f_j}{\partial x_i}$$

and since the summation convention has been assumed and the variable i, j appear twice, we must sum $i = 1 \to 3$, $j = 1 \to 3$.

then

$$\underline{\nabla} \wedge \underline{f} = \cancelto{0}{(\underline{e}_1 \wedge \underline{e}_1)} \frac{\partial f_1}{\partial x_1} + \underbrace{(\underline{e}_1 \wedge \underline{e}_2)}_{= \underline{e}_3} \frac{\partial f_2}{\partial x_1} + \underbrace{(\underline{e}_1 \wedge \underline{e}_3)}_{= -\underline{e}_2} \frac{\partial f_3}{\partial x_1} + \underbrace{(\underline{e}_2 \wedge \underline{e}_1)}_{= -\underline{e}_3} \frac{\partial f_1}{\partial x_2} + \cancelto{0}{(\underline{e}_2 \wedge \underline{e}_2)} \frac{\partial f_2}{\partial x_2}$$

$$= \underline{e}_1 \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) + \underline{e}_2 \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) + \underline{e}_3 \left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \right)$$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

which is probably how you previously saw it represented.

8. Two examples

a. An important identity : Evaluate $\nabla \wedge \nabla\phi(\underline{x})$

Using index notation we have

$$\underline{\nabla} \wedge \underline{\nabla}\phi = \underline{e}_i \frac{\partial}{\partial x_i} \wedge \left(\underline{e}_j \frac{\partial\phi}{\partial x_j}\right) = \underline{e}_i \wedge \underline{e}_j \frac{\partial^2\phi}{\partial x_i \partial x_j}$$

$$= \epsilon_{ijk} \frac{\partial^2\phi}{\partial x_i \partial x_j} \underline{e}_k$$

Next, notice that by using the 'indices-interchange' property of ϵ_{ijk} we may write

$$\nabla \wedge \nabla\phi = \epsilon_{ijk} \frac{\partial^2\phi}{\partial x_i \partial x_j} \underline{e}_k = -\epsilon_{jik} \frac{\partial^2\phi}{\partial x_i \partial x_j} \underline{e}_k = -\epsilon_{ijk} \frac{\partial^2\phi}{\partial x_j \partial x_i} \underline{e}_k$$

relabel dummy variables: $i \to j$, $j \to i$

By comparing the first and last equalities:

$$\epsilon_{ijk} \frac{\partial^2\phi}{\partial x_i \partial x_k} = -\epsilon_{ijk} \frac{\partial^2\phi}{\partial x_k \partial x_i} = 0 \qquad \left(\text{provided } \frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{\partial^2\phi}{\partial x_j \partial x_i}\right)$$

Exercise: As $\underline{a} \wedge \underline{a} = \underline{0}$, then $\underline{a} \wedge \underline{a} = a_i a_j \epsilon_{ijk} \underline{e}_k = \underset{\sim}{0}$. You should recognize this combination as $= 0$.

$$\therefore \quad \boxed{\underline{\nabla} \wedge \underline{\nabla}\phi = \underset{\sim}{0} \quad \text{for } \underline{\text{any}} \text{ twice continuously differentiable scalar function } \phi(\underline{x})}$$

b. Consider two vectors $\underset{\sim}{a}(\underline{x})$ and $\underline{b}(\underline{x})$.

$\Rightarrow$ Evaluate $\underset{\sim}{\nabla} \cdot (\underset{\sim}{a} \wedge \underline{b})$

<u>SOLN</u>

$$\underline{e}_i \frac{\partial}{\partial x_i} \cdot \left(a_j \underline{e}_j \wedge b_k \underline{e}_k\right) = \underline{e}_i \frac{\partial}{\partial x_i} \cdot (a_j b_k)\, \epsilon_{jkl}\, \underline{e}_l$$

constant, so may be taken outside of parenthesis

$$= (\underline{e}_i \cdot \underline{e}_l) \frac{\partial}{\partial x_i}(a_j b_k)\, \epsilon_{jkl} = \frac{\partial}{\partial x_i}(a_j b_k)\, \epsilon_{jki}$$

$\underline{e}_i \cdot \underline{e}_l = \delta_{il}$, $\qquad \epsilon_{jki} = \epsilon_{ijk} =$

$$= \left(\frac{\partial a_j}{\partial x_i}\right) \epsilon_{ijk} b_k + \frac{\partial b_k}{\partial x_i} \epsilon_{kij} a_j$$

$$\underline{\nabla} \cdot (\underset{\sim}{a} \wedge \underset{\sim}{b}) = \frac{\partial a_j}{\partial x_i} \epsilon_{ijk} b_k - \frac{\partial b_k}{\partial x_i} \epsilon_{ikj} a_j = (\underset{\sim}{\nabla} \wedge \underset{\sim}{a})_k b_k - (\underset{\sim}{\nabla} \wedge \underset{\sim}{b})_j a_j$$

or in vector notation,

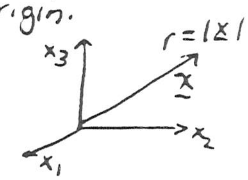
$$\underset{\sim}{\nabla} \cdot (\underset{\sim}{a} \wedge \underset{\sim}{b}) = (\underset{\sim}{\nabla} \wedge \underset{\sim}{a}) \cdot \underset{\sim}{b} - (\underset{\sim}{\nabla} \wedge \underset{\sim}{b}) \cdot \underset{\sim}{a}$$

9. A menagerie of typical calculations

a. In many problems a scalar or vector function, $\phi(\underset{\sim}{x})$ or $\underline{f}(\underline{x})$ say, depends on position $\underset{\sim}{x}$.

let $\underline{x}$ = position vector

let $r^2 = \underline{x}\cdot\underline{x} = x_i^2$ denote distance from the origin.



b. What is $\underset{\sim}{\nabla} r^2$?

$$\underline{\nabla} r^2 = \underline{e}_j \frac{\partial}{\partial x_j} r^2 = \underline{e}_j\, 2r \underbrace{\frac{\partial r}{\partial x_j}}_{\text{what is this?}} \qquad (*)$$

... continue in a moment, but first lets address the question, what is $\underset{\sim}{\nabla} r$ or in component form $\frac{\partial r}{\partial x_j} = ?$

c. Since $r = \sqrt{x^2+y^2+z^2} = \sqrt{x_1^2+x_2^2+x_3^2}$ we see that

$$\frac{\partial r}{\partial x_1} = \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} = \frac{x_1}{r}\ ; \text{ similarly } \frac{\partial r}{\partial x_2} = \frac{x_2}{r}\ ;\ \frac{\partial r}{\partial x_3} = \frac{x_3}{r}$$

and so $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$ or $\underline{\nabla} r = \underline{e}_j \frac{\partial r}{\partial x_j} = \underline{e}_j \frac{x_j}{r} = \frac{\underline{x}}{r}$

$$\therefore \boxed{\frac{\partial r}{\partial x_j} = \frac{x_j}{r}} \ (136a)\ ; \quad \boxed{\underset{\sim}{\nabla} r = \frac{\underline{x}}{r}} \ (136b)$$

Alternatively, we may note that $r^2 = x_i x_i$
so taking $\partial/\partial x_j$ gives

$$2r\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j}(x_i x_i) = 2x_i \overset{\delta_{ij}}{\frac{\partial x_i}{\partial x_j}} = 2x_j$$

$$\therefore \boxed{\frac{\partial r}{\partial x_j} = \frac{x_j}{r}}$$

(note the simplicity of the index notation calculation.

d. .. and so now we return to $(*)$ to see that $\underset{\sim}{\nabla} r^2 = \underline{e}_j\, 2r\cdot\frac{x_j}{r} = 2\underset{\sim}{x}$

e. Evaluate $\underline{\nabla}\cdot(\underset{\sim}{x}e^r) = (\overset{3}{\cancel{\underline{\nabla}\cdot\underset{\sim}{x}}})e^r + \underline{x}\cdot\underline{\nabla}e^r = 3e^r + x_i\frac{\partial}{\partial x_i}e^r$ — use chain r... + results ab...

$$= 3e^r + x_i\frac{\partial r}{\partial x_i}\frac{\partial}{\partial r}e^r = 3e^r + x_i\cdot\frac{x_i}{r}e^r = (3+r)e^r.$$

f. Exercise: Evaluate $\underline{\nabla}\wedge\underline{x}$.

D. Integral Theorems

1. DIVERGENCE THEOREM (also, sometimes referred to as Gauss' (GREEN'S THEOREM) Green's or Ostrogradsky's Theorem.

⇒ This theorem relates integrals over volumes to integrals over the bounding surface(s).

a. The Divergence Theorem states that given a continu[ous] vector function $\underset{\sim}{f}(\underset{\sim}{x})$ with continuous first partial derivatives, then

DIVERGENCE THEOREM [1,2]

$$\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{f} \; dV = \int_S \underline{n} \cdot \underset{\sim}{f} \; dS \qquad (137)$$

volume → V ; S ← denotes ALL bounding surfaces

where $\underset{\sim}{n}$ is the unit outward normal from V

b. You may wish to read & recall the proof of the theorem given in Math 21. We will provide a proof for the two dimensional situation.

c. Eqn (137), written out in 3D, is

$$\int_V \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dV = \int_S \left(n_1 f_1 + n_2 f_2 + n_3 f_3 \right) dS$$

or using index notation

$$\int_V \frac{\partial f_i}{\partial x_i} \, dV = \int_S n_i f_i \, dS$$

notation: Sometimes y[ou] see $d\underset{\sim}{S} = \underline{n}\, d$[S]

---

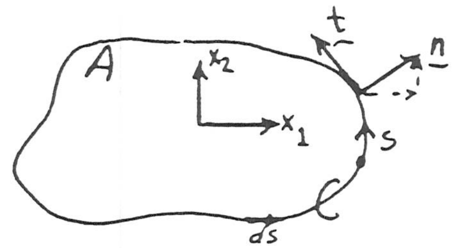
1. When writing this theorem in vector form, it is useful to get into the habit of writing $\underline{n}$ on the left. The order will be important when we apply the Divergence Theorem to tensor functions.

2. It is common to see the volume integral indicated as $\iiint_V$ and the surface integral as $\iint_S$. I will always use the simpler notation shown in eqn (136) and clearly indicate volume or surface integration with V, dV or S, dS respectively.

2. Planar (2D) versions of the Divergence Theorem (common to discuss these in introductory vector calculus courses.)

a. Consider some area A in the plane bounded by a curve C.

The Divergence Theorem (eqn 13) is now



(138a) $$\int_A \underset{\sim}{\nabla} \cdot \underset{\sim}{f}\, dA = \oint_C \underset{\sim}{n} \cdot \underset{\sim}{f}\, ds$$

C ← integral around boundary

b. You have probably seen the Divergence Theorem in a somewhat different form.

Let $\underset{\sim}{n}$ and $\underset{\sim}{t}$ denote the unit outward normal and tangent vectors along the boundary C (see sketch).

Let $d\underset{\sim}{x}$ indicate a small displacement along the boundary ($ds = |d\underset{\sim}{x}|$)

Since $\underset{\sim}{t} = \dfrac{d\underset{\sim}{x}}{ds} \Rightarrow \underset{\sim}{t}\, ds = dx_1\, \underset{\sim}{e}_1 + dx_2\, \underset{\sim}{e}_2$

Also, as $\underset{\sim}{n} \cdot \underset{\sim}{t} = 0$, then

$$\underset{\sim}{n} = \frac{dx_2}{ds}\underset{\sim}{e}_1 - \frac{dx_1}{ds}\underset{\sim}{e}_2 \Rightarrow \underset{\sim}{n}\, ds = dx_2\, \underset{\sim}{e}_1 - dx_1\, \underset{\sim}{e}_2$$

c. If we simply expand eqn (138a) then

$$\int_A \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dA = \oint_S \underset{\sim}{n} \cdot \underset{\sim}{f}\, ds = \oint_C \left(f_1\, dx_2 - f_2\, dx_1\right) \quad (138b)$$

which is a common eqn to see.

Alternatively, if we let $f_1(x_1, x_2) = N(x_1, x_2)$
$f_2(x_1, x_2) = -M(x_1, x_2)$ then

then we may write

$$\int_A \left(\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}\right) \overbrace{dA}^{dx_1\, dx_2 = dx\, dy} = \oint_C \left(M\, dx_1 + N\, dx_2\right) \quad (138c)$$

which also commonly appears in introductory vector calculus courses.

d. Proof the Divergence Theorem in 2D

a. A result we will need: $\displaystyle\int_{x=\alpha}^{\beta} \frac{\partial g}{\partial x}(x,y)\, dx = g(\beta,y) - g(\alpha,y).$

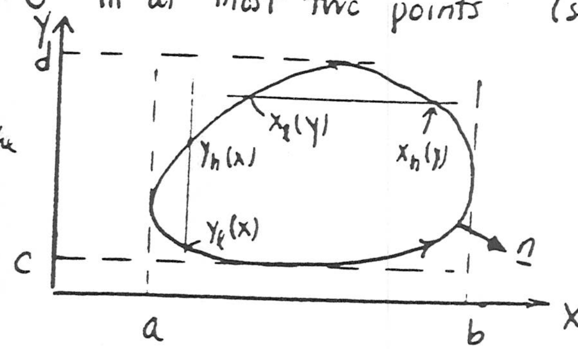
integration with respect to x of a function of x, y

b. Consider $\displaystyle\int_A \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) \underbrace{dx\, dy}_{dA}$

Let the region A be such that any vertical line or horizontal line intersects the boundary C in at most two points (see sketch)

→ identify functions $x_\ell(y)$, $x_h(y)$ which describe the boundary as the y-direction is traversed from c to d.

→ Similarly, $y_\ell(x)$, $y_h(x)$ describe the boundary as the x-direction is traversed from a to b.



c. Then,

$$\int_A \nabla\cdot f\, dA = \int_{y=c}^{d}\int_{x=x_\ell(y)}^{x_h(y)} \frac{\partial f_x}{\partial x}\, dx\, dy + \int_{x=a}^{b}\int_{y=y_\ell(x)}^{y_h(x)} \frac{\partial f_y}{\partial y}\, dy\, dx$$

$$= \int_{y=c}^{d}\left[f_x(x_h,y) - f_x(x_\ell,y)\right] dy + \int_{x=a}^{b}\left[f_y(x,y_h) - f_y(x,y_\ell)\right] dy$$

note the form of the rearrangement

$$= \int_{y=c}^{d} f_x(x_h(y),y)\, dy + \int_{d}^{c} f_x(x_\ell(y),y)\, dy - \int_{x=a}^{b} f_y(x,y_\ell(x))\, dy - \int_{x=b}^{a} f_y(x,y_h(x))\, dx$$

$$= \oint_C f_x(x,y)\, dy - \oint_C f_y(x,y)\, dx$$

$$\therefore \int_A \nabla\cdot f\, dy = \oint_C \left(f_x\, dy - f_y\, dx\right) = \oint_C n\cdot f\, ds$$

proof complete

where $\oint$ denotes integration in the counter-clockwise direction and $n = (dy/ds, -dx/ds)$.

Remark: It is straightforward to extend the theorem to more general regions by subdividing the region into smaller regions similar to the figure above.

Note: It is not terribly useful to attempt to memorize all the different versions in which the Divergence Theorem may be written. Rather simply remember

$$\int_V \nabla \cdot \underset{\sim}{f}\, dV = \int_S \underset{\sim}{n} \cdot \underset{\sim}{f}\, dS \quad \text{in 3D} \qquad \text{or} \qquad \int_A \nabla \cdot \underset{\sim}{f}\, dA = \int_C \underset{\sim}{n} \cdot f\, ds \quad \text{in 2D}$$

and recognize that with a little algebra & geometry you can derive most intermediate formulas.

e. A planar version of Stokes' Theorem

Recall eqn (138c) on the bottom of pg. 141.

Identify the scalar function $N(x_1, x_2)$ and $M(x_1, x_2)$ with the components of a vector according to

$$\underset{\sim}{a}(\underset{\sim}{x}) = a_1(x_1, x_2)\, \underset{\sim}{e}_1 + a_2(x_1, x_2)\, \underset{\sim}{e}_2 = M(x_1, x_2)\, \underset{\sim}{e}_1 + N(x_1, x_2)\, \underset{\sim}{e}_2$$

Then, from (138c) we have

$$\int_A \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) dA = \oint_C a_1\, dx_1 + a_2\, dx_2$$

which we may express in a simple form using vector notation

$$(139) \qquad \int_A (\nabla \wedge \underset{\sim}{a}) \cdot \underset{\sim}{e}_3 \; dA = \oint_C \underset{\sim}{a} \cdot \underset{\sim}{t}\, ds$$

where $\underset{\sim}{t} = \frac{dx_1}{ds} \underset{\sim}{e}_1 + \frac{dx_2}{ds} \underset{\sim}{e}_2$
$= \frac{d\underset{\sim}{x}}{ds}$ is the unit tangent vector

Planar version of Stokes Theorem.

we'll come back to this in a few pages.

3. Some theorems, or identities, which follow directly from the Divergence Theorem

Begin with $\displaystyle\int_V \nabla \cdot \underset{\sim}{f}\, dV = \int_S \underset{\sim}{n} \cdot \underset{\sim}{f}\, dS$ (*)

or you may wish to write $\displaystyle\int_V \frac{\partial f_i}{\partial x_i}\, dV = \int_S n_i f_i\, dS$

→ In the calculation presented below I will work in vector notation. You may wish instead to work in index notation.

a. Let $\underset{\sim}{f} = \phi\, \underset{\sim}{b}$ where $\phi = \phi(\underset{\sim}{x})$ and $\underset{\sim}{b}$ is an arbitrary <u>CONSTANT</u> vec

Substitute into (*),
Since $\underset{\sim}{b}$ = constant vector, $\qquad \nabla \cdot (\phi \underset{\sim}{b}) = \underset{\sim}{b} \cdot \nabla \phi$
So, we arrive at

(**) $$\boxed{\underset{\sim}{b} \cdot \int_V \nabla \phi\, dV = \underset{\sim}{b} \cdot \int_S \underset{\sim}{n}\, \phi\, dS}$$

where $\underset{\sim}{b}$ can be taken outside of the integrals becau[se] it is a CONSTANT vector.

Because this eqn must be true for arbitrary $\underset{\sim}{b}$, we conclude

$$\boxed{\int_V \nabla \phi\, dV = \int_S \underset{\sim}{n}\, \phi\, dS} \qquad (140)$$

which you may li[ke] to refer to as the 'Divergence' Theorem fo[r] scalar functions.

We will make use of this argument several times in this course.

c. The formal argument to go from (**) to (140) goes like this:
Rearrange (*) as

(141) $$\underset{\sim}{b} \cdot \left[\int_V \nabla \phi\, dV - \int_S \underset{\sim}{n}\, \phi\, dS \right] = 0$$

where the quantity inside the brackets [ ] is clearly a vector.

Remember $\underset{\sim}{b}$ has been assumed to be an <u>arbitrary</u> <u>constant</u> vector

Thus from (141) we must either have

(i) $[\,] = \underset{\sim}{0}$
(ii) $\underset{\sim}{b} = \underset{\sim}{0}$
(iii) $\underset{\sim}{b} \perp \underset{\sim}{[\,]}$

} but $\underset{\sim}{b}$ is arbitrary so (ii) & (iii) cannot hold.
Therefore $[\,] = \underset{\sim}{0}$

vector equ[ation] holds for ea[ch] component

$\therefore \displaystyle\int_V \nabla \phi\, dV = \int_S \underset{\sim}{n}\, \phi\, dS$ or in index notation $\displaystyle\int_S \frac{\partial \phi}{\partial x_i}\, \underset{\sim}{e}_i = \int_S \underset{\sim}{e}_i\, n_i\, \phi\, d[S]$

d. Another theorem which follows from the Divergence Theorem is obtained by letting

$$\underset{\sim}{f} = \underset{\sim}{\nabla}\phi$$

Then since

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\phi) = \overbrace{\underset{\sim}{e}_i \frac{\partial}{\partial x_i}\cdot\left(\underset{\sim}{e}_j\right.}^{\delta_{ij}}\left.\frac{\partial\phi}{\partial x_j}\right) \qquad \frac{\partial^2\phi}{\partial x_i^2} \Downarrow$$

$$= \frac{\partial^2\phi}{\partial x_i \partial x_i} = \frac{\partial^2\phi}{\partial x_i^2} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} \equiv \underbrace{\nabla^2\phi}_{\text{the LAPLACIAN}}$$

we have

$$\boxed{\int_V \nabla^2\phi \, dV = \int_S \underbrace{\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\phi}_{\frac{\partial\phi}{\partial n}} \, dS} \qquad (141)$$

← gradient of ϕ in the direction of the unit normal $\underset{\sim}{n}$ (i.e., directional derivative along $\underset{\sim}{n}$)

4. GREEN'S THEOREMS (Hildebrand, p. 301-2; Kreysig §9.8)

a. Begin with the Divergence Theorem and let $\underset{\sim}{f} = \psi\,\underset{\sim}{\nabla}\phi$

Then,

$$\int_S \psi(\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\phi)\, dS = \int_V \underset{\sim}{\nabla}\cdot(\psi\,\underset{\sim}{\nabla}\phi)\, dV = \int_V \overbrace{\underset{\sim}{e}_i \frac{\partial}{\partial x_i}\cdot\left(\psi \frac{\partial\phi}{\partial x_j}\underset{\sim}{e}_j\right)}^{\delta_{ij}}$$

$$= \int_V \frac{\partial}{\partial x_i}\left(\psi\frac{\partial\phi}{\partial x_i}\right) dV = \int_V \left(\frac{\partial\psi}{\partial x_i}\frac{\partial\phi}{\partial x_i} + \psi\frac{\partial^2\phi}{\partial x_i^2}\right) dV$$

$$\therefore \boxed{\int_S \psi \underbrace{\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\phi}_{\partial\phi/\partial n}\, dS = \int_V \left[\underset{\sim}{\nabla}\psi\cdot\underset{\sim}{\nabla}\phi + \psi\nabla^2\phi\right] dV} \qquad (142)$$

Green's 1st form or Green's 1st Identity

b. Interchange ψ, ϕ in the above eqn, then subtract from the eqn just derived.

$$\Rightarrow \boxed{\int_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS = \int_V \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) dV} \qquad (143)$$

Green's 2nd form or Green's Second Identity

Exercise: derive this eqn

An interesting aside : We will return to this idea when we study pdes. Just skim for now.

c. Green's Theorems are often useful for proving some very general results.

For example, begin with Green's first form; let $\psi = \phi$

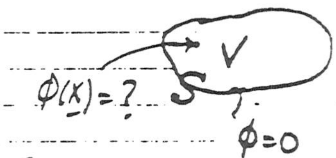
Then,

$$\int_S \phi \frac{\partial \phi}{\partial n} \, dS = \int_V \left[\nabla \phi \cdot \nabla \phi + \phi \nabla^2 \phi \right] dV$$

a p.d.e.

⟹ Now suppose you wish to solve $\nabla^2 \phi = 0$ (Laplace's eqn) in V subject to the b.c $\phi = 0$ on S

What is $\phi(\underset{\sim}{x})$ for $\underset{\sim}{x} \in V$?



Well, we are given $\phi = 0$ on the boundary S so,

$$\int_S \phi \frac{\partial \phi}{\partial n} \, dS = 0.$$

Furthermore, since $\nabla^2 \phi = 0$ in V, the eqn above reduces to

$$\int_V \left[\nabla \phi \cdot \nabla \phi \right] dV = 0$$

But the integrand $\nabla \phi \cdot \nabla \phi$ is the sum of squares, so is always positive. For the integral to vanish we must consequently require $\nabla \phi \cdot \nabla \phi = 0$ everywhere in V

or $\quad \nabla \phi = 0$, i.e., ϕ doesn't vary with spatial position in

$\Rightarrow \quad \phi =$ constant in V

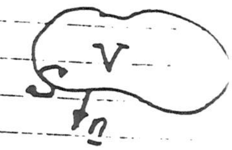
But $\phi = 0$ on S and so therefore $\phi = 0$ throughout V

#

5. A further generalization of the Divergence Theorem:

a. We began by stating the Divergence Theorem $\int_V \nabla \cdot \underset{\sim}{f}\, dV = \int_S \underset{\sim}{n} \cdot \underset{\sim}{f}\, dS$

Where S represents the closed surface enclosing the volume V and $\underset{\sim}{n}$ is the unit outward normal from the volume.



We then proved a form suitable for scalar functions,

$$\int_V \nabla \phi \, dV = \int_S \underset{\sim}{n}\, \phi \, dS$$ ← eqn (140), p. 144.

or using index notation

$$\int_V \frac{\partial \phi}{\partial x_i} \, dV \, \underset{\sim}{e}_i = \int_S n_i \, \phi \, dS \, \underset{\sim}{e}_i$$

If two vectors are equal, their corresponding components are equal, so we see that

$$(*) \qquad \int_V \frac{\partial \phi}{\partial x_i} \, dV = \int_S n_i \, \phi \, dS \qquad \text{for } i = 1, 2 \text{ or } 3.$$

b. We now construct a form useful for when the cross-product appears.

For example, consider integrals of the form $\int_V \nabla \wedge \underset{\sim}{f}\, dV$

It is simplest to work in index notation. Notice that $\nabla \wedge \underset{\sim}{f}$ represents a vector so the result of the integration is also a vector. Basically, we then proceed by considering each component of the vector separately.

So using index notation

$$\int_V \nabla \wedge \underset{\sim}{f}\, dV = \int_V \frac{\partial f_k}{\partial x_j} \varepsilon_{jki} \, \underset{\sim}{e}_i \, dV = \int_V \frac{\partial f_k}{\partial x_j} \, dV \, \underbrace{\varepsilon_{jki} \, \underset{\sim}{e}_i}_{\text{these are just constants}}$$

(remember – the summation convention is in effect so this actually represents the sum of lots of terms)

But for each k and i we know that

$$\int_V \frac{\partial f_k}{\partial x_j} \, dV = \int_S n_j \, f_k \, dS$$

In other words, for each k, i use eqn $(*)$ and let $f_k = \phi$ so that

$$\int_V \frac{\partial \phi}{\partial x_j} \, dV \, \varepsilon_{jki} \, \underset{\sim}{e}_i = \int_S n_j \, \phi \, dS \, \varepsilon_{jki} \, \underset{\sim}{e}_i$$

Hence,

$$\int_V \nabla \wedge \underset{\sim}{f}\, dV = \int_V \frac{\partial f_k}{\partial x_j} \, dV \, \varepsilon_{jki} \, \underset{\sim}{e}_i = \int_S n_j \, f_k \, dS \, \varepsilon_{jki} \, \underset{\sim}{e}_i = \int_S \underset{\sim}{n} \wedge \underset{\sim}{f}\, dS \qquad (144$$

#

c. Notice, that we can conveniently summarize all of the above results concerning the Divergence Theorem as follows:

$$\int_V \underset{\sim}{\nabla} * \Phi \, dV = \int_S \underset{\sim}{n} * \Phi \, dS \qquad (145)$$

where Φ is any quantity, scalar or vector, and $*$ is any operation (scalar product, vector product or a simple gradient operation) that <u>makes</u> <u>sense</u>.

d. EXAMPLES:

(i) let $\underset{\sim}{a} \equiv$ constant vector $\int_S \underset{\sim}{n} \cdot \underset{\sim}{a} \, dS = \int_V \underbrace{\underset{\sim}{\nabla} \cdot \underset{\sim}{a}}_{0 \text{ if } \underset{\sim}{a} = \text{const vect}} dV = 0$

(ii) Evaluate $\int_S \underset{\sim}{n} \cdot (\underset{\sim}{\nabla} \wedge \underset{\sim}{f}) \, dS$

By the Divergence Theorem,

$$\int_S \underset{\sim}{n} \cdot (\underset{\sim}{\nabla} \wedge \underset{\sim}{f}) \, dS = \int_V \underbrace{\underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \wedge \underset{\sim}{f})}_{0} dV = 0.$$

0 for any twice continuously differentiable vector function

(iii) Evaluate $\int_S \underset{\sim}{n} \cdot \underset{\sim}{\nabla} r^2 \, dS$

Using index notation, $\underset{\sim}{n} \cdot \underset{\sim}{\nabla} r^2 = n_i \frac{\partial}{\partial x_i}(r^2) \rightarrow \frac{\partial}{\partial x_i}(r^2) = 2r \frac{\partial r}{\partial x_i} = 2x_i$

So,

$$\int_S \underset{\sim}{n} \cdot \underset{\sim}{\nabla} r^2 \, dS = \int_S n_i \frac{\partial}{\partial x_i}(r^2) \, dS \qquad \text{since } \frac{\partial r}{\partial x_i} = x_i / r \text{ (pg. 139)}$$

$$= \int_S 2 n_i x_i \, dS = 2 \int_V \underbrace{\frac{\partial x_i}{\partial x_i}}_{\delta_{ii}} dV \text{ by the Divergence Theorem}$$

$$= 2 \cdot \delta_{ii} \int_V \underbrace{dV}_{= V} \quad V \equiv \text{volume of domain} ; \delta_{ii}$$

$\therefore$ $\int_S \underset{\sim}{n} \cdot \underset{\sim}{\nabla} r^2 dS = 6V$ where $V \equiv$ volume of domain bounded

c. Notice, that we can conveniently summarize all of the above results concerning the Divergence Theorem as follows:

$$\int_V \underline{\nabla} * \Phi \, dV = \int_S \underline{n} * \Phi \, dS \qquad (145)$$

where Φ is any quantity, scalar or vector, and $*$ is any operation (scalar product, vector product or a simple gradient operation) that makes sense.

d. EXAMPLES:

(i) let $\underline{a} \equiv$ constant vector $\int_S \underline{n} \cdot \underline{a} \, dS = \int_V \underbrace{\underline{\nabla} \cdot \underline{a}}_{0 \text{ if } \underline{a} = \text{constant vector}} dV = 0$

(ii) Evaluate $\int_S \underline{n} \cdot (\underline{\nabla} \wedge \underline{f}) \, dS$

By the Divergence Theorem,

$$\int_S \underline{n} \cdot (\underline{\nabla} \wedge \underline{f}) \, dS = \int_V \underbrace{\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{f})}_{0} dV = 0.$$

0 for any twice continuously differentiable vector function $\underline{f}$.

(iii) Evaluate $\int_S \underline{n} \cdot \underline{\nabla} r^2 \, dS$

Using index notation, $\underline{n} \cdot \underline{\nabla} r^2 = n_i \frac{\partial}{\partial x_i}(r^2) \rightarrow \frac{\partial}{\partial x_i}(r^2) = 2r \frac{\partial r}{\partial x_i} = 2x_i$

So, $\int_S \underline{n} \cdot \underline{\nabla} r^2 \, dS = \int_S n_i \frac{\partial}{\partial x_i}(r^2) \, dS$ since $\frac{\partial r}{\partial x_i} = x_i/r$ (pg. 139)

$= \int_S 2 n_i x_i \, dS = 2 \int_V \underbrace{\frac{\partial x_i}{\partial x_i}}_{\delta_{ii}} dV$ by the Divergence Theorem

$= 2 \cdot \delta_{ii} \int_V \underbrace{dV}_{= V \equiv \text{volume of domain}}$; δ_{ii}

$\therefore$ $\int_S \underline{n} \cdot \underline{\nabla} r^2 \, dS = 6V$ where $V \equiv$ volume of domain bounded by S

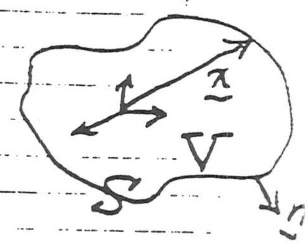
e. For those of you who would like more exercises, show

(i) $$\int_S \underset{\sim}{n} \wedge \underset{\sim}{\nabla} r^2 \, dS = 0$$

(ii) $$\int_S \underset{\sim}{n} \cdot \underset{\sim}{\nabla} (\underset{\sim}{x} \wedge \underset{\sim}{a}) \, dS = 0$$

where $\underset{\sim}{a}$ is a constant vector and $\underset{\sim}{x}$ is a position vector to a pt on the surface

$r = |\underset{\sim}{x}|$

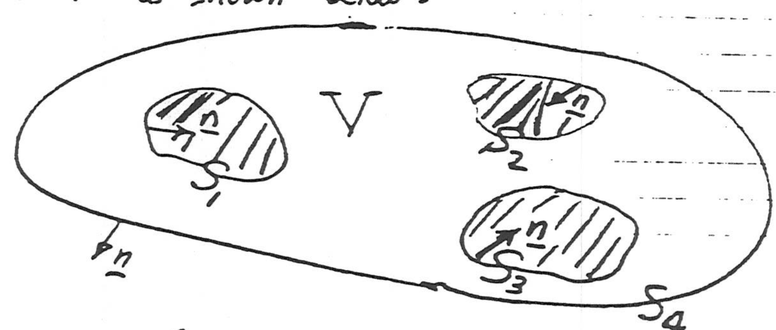


6. There is one final important point concerning the Divergence Theorem and that is if there are multiple bounding surfaces, you must include all of them.

In other words $$\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{f} \, dV = \int_S \underset{\sim}{n} \cdot \underset{\sim}{f} \, dS$$

S ← all bounding surfaces

So, if V is as shown below:

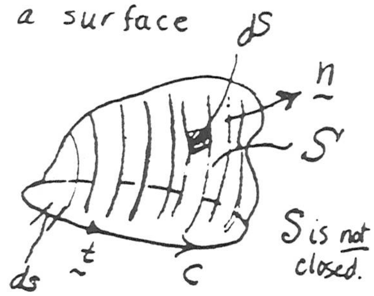


Then
$$\int_S \underset{\sim}{n} \cdot \underset{\sim}{f} \, dS = \int_{S_1 + S_2 + S_3 + S_4} \underset{\sim}{n} \cdot \underset{\sim}{f} \, dS$$

and notice how on each surface, $\underset{\sim}{n}$ points outward from V

7. Stokes' Theorem – This allows one to express an integral around a <u>closed curve</u> as an integral over that <u>area</u> having the curve as a boundary.

a. Let C be a closed curve and let S be a surface which has C as a bounding edge.
(imagine a 'hat-shaped' surface)



let $\underset{\sim}{n} \equiv$ unit normal to S with direction given by the right-hand rule – as you curl your hand in the direction indicated about C, your thumb points in the direction of $\underset{\sim}{n}$.

$\underset{\sim}{t} \equiv$ unit tangent vector to C.

sometimes people write $d\underset{\sim}{s} = \underset{\sim}{t}\, ds$

Stokes Theorem:

$$\oint_C \underset{\sim}{f} \cdot \underset{\sim}{t}\, ds = \int_S \underset{\sim}{n} \cdot (\underset{\sim}{\nabla} \wedge \underset{\sim}{f})\, dS \qquad (146)$$

ds ↑ differential element along C

Recall proof given in Math 21.

← distinguish 'little s' and 'big S'

Using index notation:

$$\oint_C f_i t_i\, ds = \int_S n_i \frac{\partial f_k}{\partial x_j} \varepsilon_{ijk}\, dS \qquad (147)$$

Writing this all out

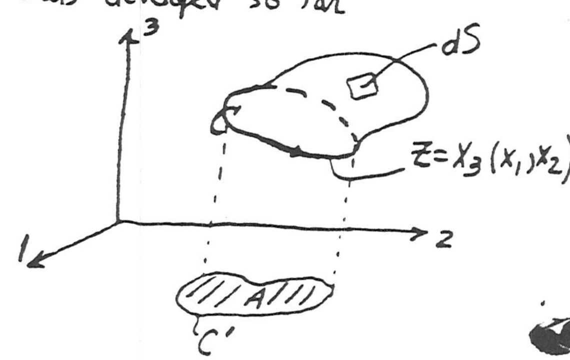
$$\oint_C (f_1 t_1 + f_2 t_2 + f_3 t_3)\, ds = \iint_S \left[n_1\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) + n_2\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) + n_3\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\right] dS$$

NOTE: This is true for an arbitrary surface S with C as a bounding edge.

b. We can actually construct a simple proof using results developed so far

Using s to denote arclength along the bounding curve C,

$$\frac{d\underset{\sim}{x}}{ds} = \underset{\sim}{t} \quad \text{then} \quad dx_i = t_i\, ds$$



So, considering the f_1 term:

$$\oint_C f_1 t_1\, ds = \oint_C f_1\, dx_1 = \oint_{C'} \mathcal{F}(x_1(x_3), x_2(x_3))\, dx_1$$

f_1 ↑ function of x, y, z

$z = z(x, y)$ represents curve C given C' in xy-plane

<u>and</u> we have written $f_1(x_1, x_2, x_3(x_1, x_2)) = \mathcal{F}(x_1, x_2)$

where we have essentially <u>projected</u> information down to the ...

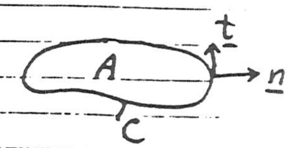
b. "simple proof" (continued)

With information "in the plane" we can now use the identity given on pg. 147 :

one form of the 'Divergence Theorem' was $\int_S n_i \phi \, dS = \int_V \frac{\partial \phi}{\partial x_i} \, dV$

which has the 'planar version'

$$\int_A \frac{\partial \phi}{\partial x_i} \, dA = \int_C \phi \, n_i \, ds$$



or

$$\int_A \frac{\partial \phi}{\partial x_1} \, dA = \int_C \phi \, dx_2 \quad \text{and} \quad \int_A \frac{\partial \phi}{\partial x_2} \, dA = -\int_C \phi \, dx_1$$

$$\begin{cases} \underline{t} \, ds = dx_1 \, \underline{e}_1 + dx_2 \, \underline{e}_2 \\ \underline{n} \, ds = dx_2 \, \underline{e}_1 - dx_1 \, \underline{e}_1 \end{cases}$$

Hence, beginning with the eqn on the bottom of the last page

$$\oint_{C'} \mathcal{F} \, dx_1 = -\int_A \frac{\partial \mathcal{F}}{\partial x_2} \, dA \qquad \text{but} \quad \frac{\partial}{\partial x_2} \mathcal{F}(x_1, x_2) = \frac{\partial}{\partial x_2} f_1(x_1, x_2, x_3(x_1 x_2))$$

$$= -\int_A \left[\frac{\partial f_1}{\partial x_2} + \frac{\partial x_3}{\partial x_2} \frac{\partial f_1}{\partial x_3} \right] dA \qquad \text{by the chain-rule}$$

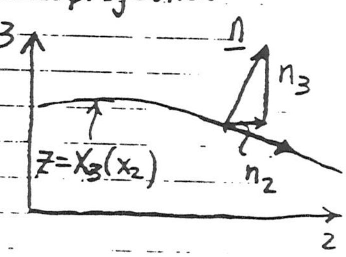
Now, a differential element dS in space is related to its projection in the xy-plane by

$$dA = n_3 \, dS$$

and furthermore one can show

i.e.,

$$\frac{\partial x_3}{\partial x_2} = -\frac{n_2}{n_3}$$



so that we have

$$\oint_{C'} \mathcal{F} \, dx_1 = -\int_S \left[\frac{\partial f_1}{\partial x_2} n_3 - \frac{\partial f_1}{\partial x_3} n_2 \right] dS$$

or

$$\oint_C f_1 \, dx_1 = \oint_C f_1 \, t_1 \, ds = \int_S \left(\frac{\partial f_1}{\partial x_3} n_2 - \frac{\partial f_1}{\partial x_2} n_3 \right) dS \quad \Leftarrow$$

which accounts for two of the terms in Stokes theorem. In a similar manner one can account for the other 2 terms. As in standard versions of these proofs, it is necessary to imagine that S is subdivided into sections which can be projected onto the xy-plane, even when the whole of S does not have a one-to-one projection.