

CHAPTER 1
BRIEF REVIEW OF MATHEMATICAL PRELIMINARIES

References: Basic

1. H. Jeffreys, *Cartesian Tensors*, Cambridge, 1931.
2. L.A. Segel, *Mathematics Applied to Continuum Mechanics*, Dover, New York, 1987.

References: More Advanced

3. R.M. Bowen and C.C. Wang, *Introduction to Vectors and Tensors*, Volume 1, New York, Plenum, 1976.
4. I.M. Gelfand, *Lectures on Linear Algebra*, Wiley, New York, 1963.
5. P.R. Halmos, *Finite Dimensional Vector Spaces*, Van Nostrand, New Jersey, 1958.
6. J.K. Knowles, *Linear Vector Spaces and Cartesian Tensors*, to appear, 1996.

Notation:

a	scalar
$\{a\}$	3×1 column matrix
\mathbf{a}	vector
$[A]$	3×3 square matrix
\mathbf{A}	linear transformation

INDICIAL NOTATION

Consider, for example, the system of linear algebraic equations

$$\left. \begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1, \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= b_2, \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= b_3. \end{aligned} \right\} \quad (1)$$

We can simplify our writing by expressing this equivalently as

$$A_{i1}x_1 + A_{i2}x_2 + A_{i3}x_3 = b_i \quad (2)$$

with the understanding that (2) holds for each value of the subscript i in its range $i = 1, 2, 3$. This understanding is referred to as the *range convention*. The subscript i is called a *free*

subscript because it is free to take on each value in its range. The same free subscript (the single index i in the above example) must appear in each symbol grouping. Note that

$$A_{j1}x_1 + A_{j2}x_2 + A_{j3}x_3 = b_j$$

is identical to (2) because of the range convention. This illustrates the fact that the choice of index for the free subscript is not important provided that the same free subscript appears in every symbol grouping.

To simplify notation further, we write (2) as

$$\sum_{j=1}^3 A_{ij}x_j = b_i \quad (3)$$

and now agree to drop the summation sign while at the same time imposing the rule that summation is implied over any subscript that appears twice in a symbol grouping. Such a subscript is called a *repeated* or *dummy* subscript. Thus, using the *summation convention* we can write (3) as

$$A_{ij}x_j = b_i \quad (4)$$

with summation on the subscript j being implied. Note that

$$A_{ik}x_k = b_i$$

is identical to (4) in view of the implied summation on the dummy subscript k . Thus the choice of index for the dummy subscript is not important. In order to avoid ambiguity, no subscript can appear more than twice in any symbol grouping. Thus we shall never write, for example, $A_{ii}x_i = b_i$.

Summary of Rules:

1. Lower-case latin subscripts take on values in the range (1,2,3).
2. Summation is implied over a subscript that appears twice in a single symbol grouping.
3. The same subscript may not appear more than twice in the same symbol grouping.
4. All symbol groupings in an equation must have the same free subscripts. Free subscripts take on each value in its range.
5. Free and dummy indices may be changed without altering the meaning of an expression provided one does not violate the preceding rules.

Example(1): If $[A]$ and $[B]$ are 3×3 matrices and $\{x\}$, $\{y\}$, $\{z\}$ are 3×1 column matrices, express the matrix equation $\{y\} = [A]\{x\} + [B]\{z\}$ in component form.

Solution: By the rule for matrix multiplication, one has

$$y_i = A_{ij}x_j + B_{ij}z_j.$$

Note that the following are equivalent to the above (and are consistent with the rules for indicial notation):

$$y_k = A_{kj}x_j + B_{kj}z_j = A_{kp}x_p + B_{kp}z_p = A_{kp}x_p + B_{kj}z_j.$$

Example(2): If $[A]$, $[B]$, $[C]$, $[D]$ and $[E]$ are 3×3 matrices such that $[C] = [A][B]$, $[D] = [B][A]$ and $[E] = [A][B]^T$ express the elements of $[C]$, $[D]$ and $[E]$ in terms of the elements of $[A]$ and $[B]$.

Solution: By the rule for matrix multiplication, one has

$$C_{ij} = A_{ik}B_{kj}, \quad D_{ij} = B_{ik}A_{kj}, \quad E_{ij} = A_{ik}B_{jk}. \quad (5)$$

In obtaining the third of these, we have used the fact that the i, j -element of a matrix $[B]^T$ equals the j, i -element of the matrix $[B]$, i.e. B_{ji} . It is worth emphasizing that the preceding are scalar equations and that therefore, the order in which the terms appear in a symbol group is not important. Thus for example,

$$C_{ij} = A_{ik}B_{kj} = B_{kj}A_{ik}.$$

Kronecker Delta: The Kronecker Delta δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (6)$$

Note that it represents the elements of the Identity Matrix.

The following property of the Kronecker Delta, known as the *substitution rule*, is particularly useful. Since δ_{ij} is zero unless $i = j$, it follows that for any column matrix $\{u\}$ or square matrix $[A]$,

$$u_i \delta_{ij} = u_j, \quad A_{ip} \delta_{ij} = A_{jp}, \quad (7)$$

or more generally, for any quantity $T_{ipq\dots z}$,

$$T_{ipq\dots z} \delta_{ij} = T_{jpq\dots z}. \quad (8)$$

Thus, according to the substitution rule, if a quantity (e.g. T in preceding example) multiplying the Kronecker Delta has a common subscript (e.g. i above) with the Kronecker Delta, then one can substitute this repeated subscript in this quantity with the other subscript (e.g. j above) of the Kronecker Delta and then delete the Kronecker Delta.

Example(3): If $[Q]$ is an orthogonal matrix, use indicial notation to solve the matrix equation $\{a\} = [Q]\{b\}$ for $\{b\}$.

Solution: Since $[Q]$ is orthogonal, $[Q]^T[Q] = [Q][Q]^T = [I]$ or in indicial notation,

$$Q_{ki}Q_{kj} = Q_{ik}Q_{jk} = \delta_{ij}. \quad (9)$$

The given equation $\{a\} = [Q]\{b\}$ in indicial form reads

$$a_i = Q_{ij}b_j.$$

Multiplying both sides of this by Q_{ik} gives

$$Q_{ik}a_i = Q_{ik}Q_{ij}b_j = \delta_{ij}b_j = b_i,$$

where we have used (9) and the substitution rule. In matrix notation this result reads $\{b\} = [Q]^T\{a\}$ which could have been written down immediately.

Example(4): If $f(x_1, x_2, x_3) = A_{ij}x_i x_j$ where A_{ij} is constant, calculate $\partial f / \partial x_i$.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{\partial(A_{pq}x_p x_q)}{\partial x_i} = A_{pq} \frac{\partial(x_p x_q)}{\partial x_i} = A_{pq} x_q \frac{\partial x_p}{\partial x_i} + A_{pq} x_p \frac{\partial x_q}{\partial x_i} \\ &= A_{pq} x_q \delta_{pi} + A_{pq} x_p \delta_{qi} = A_{iq} x_q + A_{pi} x_p = (A_{ij} + A_{ji})x_j \end{aligned}$$

where we have used the substitution rule and the fact that $\partial x_i / \partial x_j = \delta_{ij}$.

The Alternator or Permutation Symbol: The alternator or permutation symbol is defined by

$$\begin{aligned} e_{ijk} &= \begin{cases} 0 & \text{if two or more subscripts } i, j, k, \text{ are equal,} \\ +1 & \text{if the subscripts } i, j, k, \text{ are in cyclic order,} \\ -1 & \text{if the subscripts } i, j, k, \text{ are in anticyclic order,} \end{cases} \\ &= \begin{cases} 0 & \text{if two or more subscripts } i, j, k, \text{ are equal,} \\ +1 & \text{for } (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 2, 1), \\ -1 & \text{for } (i, j, k) = (1, 3, 2), (2, 1, 3), (3, 1, 2). \end{cases} \end{aligned} \quad (10)$$

Observe that the sign of e_{ijk} is flipped if any two adjacent subscripts are switched:

$$e_{ijk} = -e_{jik} = e_{jki} \quad (11)$$

Two of the most important properties of the alternator are that it permits the determinant of a matrix $[A]$ to be written as

$$\det[A] = e_{ijk} A_{1i} A_{2j} A_{3k} = e_{ijk} A_{i1} A_{j2} A_{k3}, \quad (12)$$

and the following relation involving the alternator and the determinant, which for convenience we shall refer to (following Segel) as the "ed-rule"

$$e_{ijk} e_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}. \quad (13)$$

Proofs of these identities may be found, for example, in Jeffreys. They can, of course, be verified directly by simply writing out all of the terms in (12) and (13).

Example(5): Show that

$$e_{ijk}S_{jk} = 0 \quad (14)$$

for every symmetric matrix [S].

Solution: We can write

$$e_{ijk}S_{jk} = \frac{1}{2}e_{ijk}S_{jk} + \frac{1}{2}e_{ijk}S_{jk} = \frac{1}{2}e_{ijk}S_{jk} + \frac{1}{2}e_{ikj}S_{kj}.$$

where, in the last step we have used the fact that j and k are dummy subscripts. However, since we are given that $S_{ij} = S_{ji}$, we can continue this calculation as

$$e_{ijk}S_{jk} = \frac{1}{2}e_{ijk}S_{jk} + \frac{1}{2}e_{ikj}S_{jk} = \frac{1}{2}(e_{ijk} + e_{ikj})S_{jk} = 0$$

where in the very last step we have used the property illustrated in (11). Note as a special case of (14) that

$$e_{ijk}v_jv_k = 0 \quad \text{for all } v_i. \quad (15)$$

VECTORS AND LINEAR TRANSFORMATIONS

Let V be a three-dimensional Euclidean vector space. Any set of three linearly independent vectors $\{e_1, e_2, e_3\}$ in V form a basis for V ; any set of three mutually orthogonal unit vectors forms an *orthonormal basis* for V . Thus, for an orthonormal basis,

$$e_i \cdot e_j = \delta_{ij} \quad (16)$$

where δ_{ij} denotes the Kronecker delta. In these notes we shall restrict attention to orthonormal bases. If the basis is right-handed, one has in addition

$$e_i \cdot (e_j \times e_k) = e_{ijk}. \quad (17)$$

The *components* v_i of a vector v in a basis $\{e_1, e_2, e_3\}$ are defined by

$$v_i = v \cdot e_i. \quad (18)$$

They may be assembled into a column matrix

$$\{v\} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \quad (19)$$

The vector \mathbf{v} can be expressed in terms of its components v_i and the basis vectors \mathbf{e}_i as

$$\mathbf{v} = v_i \mathbf{e}_i. \quad (20)$$

It is important to emphasize the fact that the components v_i of a vector depend on both the vector \mathbf{v} and the choice of basis. The components v'_i of the vector \mathbf{v} in a second basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are given by

$$v'_i = \mathbf{v} \cdot \mathbf{e}'_i.$$

In general, $v_i \neq v'_i$. The (same) vector \mathbf{v} can be expressed in this second basis as

$$\mathbf{v} = v'_i \mathbf{e}'_i.$$

Given any column matrix $\{u\}$ and a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, there is a unique vector \mathbf{u} associated with them such that the components of \mathbf{u} in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are $\{u\}$.

If u_i and v_i are the components of two vectors \mathbf{u} and \mathbf{v} in a given basis, then the scalar product $\mathbf{u} \cdot \mathbf{v}$ can be expressed as

$$\mathbf{u} \cdot \mathbf{v} = (u_i \mathbf{e}_i) \cdot (v_j \mathbf{e}_j) = u_i v_j \mathbf{e}_i \cdot \mathbf{e}_j = u_i v_j \delta_{ij} = u_i v_i; \quad (21)$$

the vector product $\mathbf{u} \times \mathbf{v}$ can be expressed as

$$\mathbf{u} \times \mathbf{v} = (e_{ijk} u_j v_k) \mathbf{e}_i \quad \text{or equivalently as} \quad (\mathbf{u} \times \mathbf{v})_i = e_{ijk} u_j v_k \quad (22)$$

which follows from the the property (12) of the alternator and the usual representation of the vector product in the form of a determinant.

The components A_{ij} of a linear transformation \mathbf{A} in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are defined by

$$A_{ij} = \mathbf{e}_i \cdot (\mathbf{A} \mathbf{e}_j), \quad (23)$$

so that

$$\mathbf{A} \mathbf{e}_j = A_{ij} \mathbf{e}_i. \quad (24)$$

The components A_{ij} can be assembled into a square matrix:

$$[A] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

We draw attention to the fact that the components A_{ij} of a linear transformation depend on both the linear transformation \mathbf{A} and the choice of basis. The components A'_{ij} of \mathbf{A} in a second basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are given by

$$A'_{ij} = \mathbf{e}'_i \cdot (\mathbf{A} \mathbf{e}'_j). \quad (25)$$

Given any square matrix $[M]$ and a basis $\{e_1, e_2, e_3\}$, there is a unique linear transformation M associated with them such that the components of M in $\{e_1, e_2, e_3\}$ are $[M]$.

If A is a linear transformation and x and y are vectors such that $y = Ax$, then the component matrices $\{x\}$, $\{y\}$, and $[A]$ are related by

$$\{y\} = [A]\{x\} \quad \text{or} \quad y_i = A_{ij}x_j.$$

Similarly, if A, B , and C are linear transformations such that $C = AB$, then their component matrices $[A]$, $[B]$, and $[C]$ are related by

$$[C] = [A][B] \quad \text{or} \quad C_{ij} = A_{ik}B_{kj}.$$

The component matrix of the identity linear transformation in any orthonormal basis is the unit matrix; its components are therefore given by the Kronecker Delta δ_{ij} .

Let S be a symmetric linear transformation. A vector $e \neq 0$ and a scalar λ are said to be an eigenvector and corresponding eigenvalue of S if $Se = \lambda e$. A symmetric linear transformation has three real (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding orthonormal eigenvectors e_1, e_2, e_3 . The particular basis consisting of the eigenvectors is called a *principal basis* for S . The component matrix of S in this basis is

$$[S] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

CARTESIAN TENSORS

Consider a 3-dimensional Euclidean vector space. A triplet of orthonormal unit vectors $\{e_1, e_2, e_3\}$ forms an orthonormal basis (or a rectangular cartesian coordinate frame). If $\{e'_1, e'_2, e'_3\}$ is a second orthonormal basis, the numbers Q_{ij} defined by

$$Q_{ij} = e_i \cdot e'_j \tag{26}$$

are the elements of an orthogonal matrix $[Q]$. If both bases are right-handed or both are left-handed, $[Q]$ is proper orthogonal, i.e. it represents a rotation and $\det[Q] = +1$. Given an orthonormal basis $\{e_1, e_2, e_3\}$, a second triplet of vectors $\{e'_1, e'_2, e'_3\}$ form an orthonormal basis if and only if

$$e'_i = Q_{ij}e_j \tag{27}$$

for some orthogonal matrix $[Q]$. As mentioned previously we shall only consider orthonormal bases in these notes and thus shall not continue to use the adjective "orthonormal".

Let T be a physical entity (such as, for example, a scalar, vector or linear transformation) which, in a given basis $\{e_1, e_2, e_3\}$, is defined completely by a set of 3^n ordered numbers $T_{i_1 i_2 \dots i_n}$. The numbers $T_{i_1 i_2 \dots i_n}$ are called the *components* of T in the basis $\{e_1, e_2, e_3\}$. If, for example, T is a scalar, vector or linear transformation, it is represented by $3^0, 3^1$ and 3^2 components respectively in the given basis. Let $\{e'_1, e'_2, e'_3\}$ be a second basis related to the first one by the orthogonal matrix $[Q]$, and let $T'_{i_1 i_2 \dots i_n}$ be the components of the entity T in the second basis. Then, if for every choice of such bases, these two sets of components are related by

$$T'_{i_1 i_2 \dots i_n} = Q_{i_1 j_1} Q_{i_2 j_2} \dots Q_{i_n j_n} T_{j_1 j_2 \dots j_n}, \quad (28)$$

the entity T is called a *Cartesian tensor of rank n* or more simply an *n -tensor*. Thus, the components of a tensor in every basis may be determined if its components in any single (convenient) basis are known.

Example (1): Consider a vector v . Its components v_i in the basis $\{e_1, e_2, e_3\}$ are defined by

$$v_i = v \cdot e_i,$$

and its components v'_i in the second basis $\{e'_1, e'_2, e'_3\}$ are defined by

$$v'_i = v \cdot e'_i.$$

It readily follows this and (27) that

$$v'_i = v \cdot e'_i = v \cdot (Q_{ij} e_j) = Q_{ij} v \cdot e_j = Q_{ij} v_j. \quad (29)$$

Thus, a vector is a 1-tensor.

Example (2): Consider a linear transformation A on a 3-dimensional Euclidean vector space. Its components A_{ij} in the basis $\{e_1, e_2, e_3\}$ are defined by

$$A_{ij} = e_i \cdot (Ae_j),$$

and its components A'_{ij} in the second basis $\{e'_1, e'_2, e'_3\}$ are defined by

$$A'_{ij} = e'_i \cdot (Ae'_j).$$

On making use of (27) we can write this as

$$A'_{ij} = e'_i \cdot (Ae'_j) = Q_{ip} e_p \cdot (AQ_{jq} e_q) = Q_{ip} Q_{jq} e_p \cdot (Ae_q) = Q_{ip} Q_{jq} A_{pq}. \quad (30)$$

Thus, a linear transformation is a 2-tensor.

Three common tensor operations:

1. Two tensors of the same rank are *added or subtracted* by adding or subtracting corresponding components.
2. Given an n -tensor A and an m -tensor B their *outer product* is the $(m + n)$ -tensor C whose components are given by

$$C_{i_1 i_2 \dots i_n j_1 j_2 \dots j_m} = A_{i_1 i_2 \dots i_n} B_{j_1 j_2 \dots j_m}.$$

3. Let A be a n -tensor with components $A_{i_1 i_2 \dots i_n}$ in some basis. Then “*contracting*” A over two of its subscripts, say the i_j th and i_k th subscripts, leads to the $(n - 2)$ -tensor with whose components in this basis are $A_{i_1 i_2 \dots i_{j-1} i_{j+1} \dots i_{k-1} i_{k+1} \dots i_n}$. Contracting over two subscripts involves setting those two subscripts equal, and therefore summing over them.

The components of the *identity 2-tensor* I are δ_{ij} in every basis.

The following property is known as the *quotient rule*: Let a and b be 1-tensors and suppose that their components in a basis are related by

$$a_i = T_{ij} b_j$$

for some numbers T_{ij} . Then T_{ij} are the components of a 2-tensor. This result generalizes naturally to tensors of more general order.

A tensor T is said to be an *isotropic tensor* if its components have the same values in all bases, i.e. if

$$A'_{i_1 i_2 \dots i_n} = A_{i_1 i_2 \dots i_n} \quad (31)$$

in all bases $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$. Equivalently, for an isotropic tensor

$$T_{i_1 i_2 \dots i_n} = Q_{i_1 j_1} Q_{i_2 j_2} \dots Q_{i_n j_n} T_{j_1 j_2 \dots j_n} \quad \text{for all orthogonal matrices } [Q]. \quad (32)$$

The only isotropic 1-tensor is the null vector o .

The most general isotropic 2-tensor is a scalar multiple of the identity tensor, αI .

The most general isotropic 3-tensor is the null 3-tensor o . We note however that if we define a 3-tensor e by picking a right-handed basis $\{e_1, e_2, e_3\}$ and letting the components of e in this basis be the alternator e_{ijk} , the components of this tensor in every other right-handed basis are also the alternator e_{ijk} . Thus the alternator satisfies the above definition for isotropic tensors for all *proper* orthogonal matrices $[Q]$ (but not for all orthogonal matrices $[Q]$).

The most general isotropic 4-tensor T has components (in any basis)

$$T_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \quad (33)$$

where α, β, γ are arbitrary scalars.

$\lambda_{ir} + i\lambda_{ic} \rightarrow$
 $f(x) = e^{\lambda_{ir}}$

~~Cosh~~
 $\frac{\cosh \lambda_{ir}}{\cosh \lambda_{ir}}$

$A = \langle \lambda_{ir} \rangle \cosh \lambda_{ir} + \langle \lambda_{ic} \rangle \sinh \lambda_{ic} \times$

Cosh Cos Cosh Sin