Multivariable Calculus – A Very Quick Review

Partial Derivatives:

Partial derivatives are relatively easy to understand. They are simply the derivative with respect to a particular independent variable, with all the other independent variables held constant. If $h(x,y)$ is a function that describes the height of the terrain as a function of the coordinates *x* and *y*, then $\frac{\partial h}{\partial x}$ is the slope along the x-direction and $\frac{\partial h}{\partial y}$ is the slope along the y-direction.

Total Derivative:

The total derivative is more complex. It is only relevant when the independent variables are not truly independent – but can be related to a single independent variable. For example, let $h(x,y)$ represent the height of a roller coaster. Now *x* and *y* are not actually independent. The roller coaster does not cover the entire *x-y* plane – just a curve within that plane. This curve can be represented by $x = x(s)$ and $y = y(s)$, where *s* is the distance along the roller coaster from some arbitrary starting point. Then $\frac{dh}{ds}$ (note it is no longer a partial derivative) is the slope of the roller coaster along the *direction* of the roller coaster (which is what you probably care about).

The total derivative is related to the partial derivatives via the chain rule, $\frac{dh}{ds} = \frac{\partial h}{\partial x}\frac{dx}{ds} + \frac{\partial h}{\partial y}\frac{dy}{ds}$. The key to remember is that the total derivative applies only to functions of a single variable. However, often the dependence is implicit - as in the above example.

Relation to Fluids:

The only total derivative that appears in fluids is the time derivative, $\frac{d}{dt}$. This is sometimes called the material or substantial derivative (you will see why soon), and often written as $\frac{D}{Dt}$ to make it obvious that it is not a partial derivative.

Imagine a particular (small) chunk of fluid that you are interested in. That chunk can move about as the fluid swirls, sloshes or whatever, but you are not concerned much with its motion. You just want to know how that chunk of fluid changes with time. How does its density, temperature, etc, vary with time. For example – you are stuck on a raft in the ocean, (to a good approximation – a raft is stuck to a particular chunk of fluid). And you want to know whether the water temperature is going to get warmer or colder with time.

You need the total (or material, or substantial) derivative with respect to time. It is called the material (or substantial) derivative, because it is the time derivative for a particular chunk of material (or substance). It is also the time derivative that occurs in a *Lagrangian* (stuck to the fluid) reference frame.

Say the temperature in the Caribbean ocean has been measured for you (via radar) and is given to you as a function of position and time, $T = T(x,y,t)$. Then you need to remember that your raft position (*x* and *y*) is also a function of time. So $\frac{dT}{dt} = \frac{\partial T}{\partial x}u_x + \frac{\partial T}{\partial y}u_y + \frac{\partial T}{\partial t}$, where we have used the fact that $\frac{dx}{dt} = u_x$, $\frac{dy}{dt} = u_y$ and $\frac{dt}{dt} = 1$. Note that you will also need the velocity of the Caribbean ocean as a function of position and time in order to calculate the change in time of the temperature $\frac{dT}{dt}$ on your raft. Note that $\frac{\partial T}{\partial t}$ is the change in the temperature with time at a fixed location (partial with x and y held constant).

What if you just built an oil rig in the middle of the Caribbean. Your position would be fixed and the change in the temperature with time would be much easier to calculate, just $\frac{\partial T}{\partial t}$ evaluated at the oil rig's position. Why an oil rig? Because this is the change in time that occurs in an Eulerian (pronounced Oilarian) frame of reference (which is a simple fixed to the earth frame).

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Why this is Important:
This is important because it is math and you should understand it. But furthermore, people generally think about fluids in an Eulerian frame of reference (with the coordinate system fixed to the earth). We care about the density or velocity at a particular time and place (Eulerian), not what is the density and velocity of this bit of fluid versus that bit of fluid (Lagrangian). However, Newton's laws of Mechanics apply to chunks of material – Lagrangian. Therefore, the whole trick to fluid mechanics is understanding how to transform Newton's (Lagrangian) equations into an Eulerian frame that makes more sense intuitively. The trick is we just use the total derivative (for Eularian) instead of just the partial time derivative (Lagrangian).

Note that physics is more intuitively Lagrangian – you follow the billiard ball about and calculate the forces on it. You rarely sit on the table, and wonder about what the forces are on the balls when they happen to pass underneath you (which would be Eulerian).

The forces on an object don't change as you go from Lagrangian to Eulerian, but the expression for the acceleration, $\frac{d\mathbf{v}}{dt}$, does change. In the Eularian frame you get some extra terms which we call convection terms. The convection terms account for the fact that fluids can move about underneath the point you are interested in. If you are stuck to a particular chunk of fluid (Lagrangian) the convective terms go away. The convection terms are non-linear which is why fluids move so very coolly. The nonlinearity is not obvious in the Lagrangian frame (in is implicit)

Partial Differential Operators:

All these partial derivatives will soon become a big nuisance. Fortunately, they always come as a group. So the notation can be simplified greatly by using the grad operator, ∇. This operator pops up in various forms (shown below).

Cartesian Coordinate System:

The ***gradient*** of a scalar, *s*, is a vector
$$\nabla s = \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right)s = \left(\tfrac{\partial s}{\partial x}, \tfrac{\partial s}{\partial y}, \tfrac{\partial s}{\partial z}\right)$$
The ***divergence*** of a vector **v** is a scalar.
$$\nabla \cdot \mathbf{v} = \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right)\cdot\left(v_x, v_y, v_z\right) = \left(\tfrac{\partial v_x}{\partial x} + \tfrac{\partial v_y}{\partial y} + \tfrac{\partial v_z}{\partial z}\right)$$
The ***curl*** of a vector **v** and is another vector.
$$\nabla \times \mathbf{v} = \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right)\times\left(v_x, v_y, v_z\right) = \left(\tfrac{\partial v_z}{\partial y} - \tfrac{\partial v_y}{\partial z}, \tfrac{\partial v_x}{\partial z} - \tfrac{\partial v_z}{\partial x}, \tfrac{\partial v_y}{\partial x} - \tfrac{\partial v_x}{\partial y}\right)$$

In addition, we can form more operators… (which work on both scalars and vectors).

> The dot product of a vector with the gradient operator is a scalar convection *operator* (see page 1).
> $$\mathbf{v}\cdot\nabla = \left(v_x, v_y, v_z\right)\cdot\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right) = \left(v_x \tfrac{\partial}{\partial x} + v_y \tfrac{\partial}{\partial y} + v_z \tfrac{\partial}{\partial z}\right)$$
> The dot product of the gradient operator with itself, is a scalar *operator* called the Laplacian, ∇^2.
> $$\nabla\cdot\nabla = \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right)\cdot\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right) = \left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2} + \tfrac{\partial^2}{\partial z^2}\right)$$

Cylindrical Coordinate System

In Cylindrical Coordinates, the gradient, divergence, curl, and other operators are more complex, because the coordinate system orientation depends on θ. So in cylindrical coordinates:

$$\nabla s = \left(\tfrac{\partial s}{\partial r}, \tfrac{\partial s}{r\partial\theta}, \tfrac{\partial s}{\partial z}\right)$$
$$\nabla\cdot\mathbf{v} = \left(\tfrac{\partial(r v_r)}{r\partial r} + \tfrac{\partial v_\theta}{r\partial\theta} + \tfrac{\partial v_z}{\partial z}\right)$$
$$\nabla\times\mathbf{v} = \left(\tfrac{\partial v_z}{r\partial\theta} - \tfrac{\partial v_\theta}{\partial z}, \tfrac{\partial v_r}{\partial z} - \tfrac{\partial v_z}{\partial r}, \tfrac{\partial(r v_\theta)}{r\partial r} - \tfrac{\partial v_r}{r\partial\theta}\right)$$

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But what do they mean?

Let us think in two dimensions for the moment.

The **gradient** is the slope in both directions (x and y). It is a vector. If the function is height – the gradient vector points directly uphill at each point on the hill and its magnitude is proportional to the slope of the hill (hence gradient) at that point.

The **divergence** is a property of vector fields. Given a vector field (lots of vectors at lots of positions), do those vectors tend to point outwards or inwards at any particular location. For example, draw vectors indicating the size and direction of the fluid flow at lots of locations in the flow field. Do the vectors tend to point inwards or diverge outwards? (For incompressible flow they will do neither – zero divergence).

The **curl** is also a property of vector fields. Given a vector field (lots of vectors at lots of positions), do those vectors tend to point clockwise (negative curl) or counterclockwise (positive curl – right hand rule). We will see that many velocity fields (but certainly not all) have zero curl. The curl of the velocity vector is so important it gets a special name in fluid mechanics – the **vorticity**. The vorticity is a vector quantity which points along the axis of rotation (use right hand rule to get direction). In 2D, the axis of rotation must point out of the board and the vorticity vector always looks like $(0,0,\omega_z)$.

What about Cylindrical Coordinates?

The key to cylindrical coordinates is to *always* remember that the r and θ directions actually depend on where you are. The r direction always points outwards, and the θ direction always points 90 degrees to the left of outwards (counterclockwise – right hand rule).

So if $h(r,\theta)$ represents the height of a hill, then ∇h is still the vector which points along the direction of steepest slope (the 'grad' notation is coordinate system independent). However, $\frac{\partial h}{\partial r}$ is the slope in the direction pointing away from the origin, and $\frac{1}{r}\frac{\partial h}{\partial \theta}$ is the slope in the direction pointing counterclockwise to the origin.