Multivariable Calculus – A Very Quick Review

Partial Derivatives:

Partial derivatives are relatively easy to understand. They are simply the derivative with respect to a particular independent variable, with all the other independent variables held constant. If $h(x,y)$ is a function that describes the height of the terrain as a function of the coordinates x and y, then $\frac{\partial h}{\partial x}$ is the slope along the x-direction and $\frac{\partial h}{\partial y}$ is the slope along the y-direction.

Partial Differential Operators:

All these partial derivatives can soon become a big nuisance. Fortunately, they always come in only three combinations – the gradient ∇, the divergence $\nabla\cdot$, and the curl $\nabla\times$. So the notation can be simplified greatly by using these operators. Though these operators look similar (and they all look like a form of multiplication) **they are not the same**. Do **not** interchange them.

Cartesian Coordinate Systems:

The ***gradient*** of a scalar, *s(x,y,z),* is a vector

$$\nabla s = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)s = \left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial s}{\partial z}\right)$$

The ***divergence*** of a vector **v** is a scalar quantity. (it is the opposite of the gradient – but not the inverse). It represents net outflow.

$$\nabla \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(v_x, v_y, v_z\right) = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

The ***curl*** of a vector **v** and is another vector. It represents the net spin (and the rotation axis).

$$\nabla \times \mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times\left(v_x, v_y, v_z\right) = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

All other multi-dimensional calculus operators are just combinations of these first three. For example:

> The dot product of a vector with the gradient operator is the scalar convection *operator* found in the Navier-Stokes equations.
>
> $$\mathbf{v}\cdot\nabla = \left(v_x, v_y, v_z\right)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)$$
>
> The combination of the divergence operator with the gradient operator is a scalar *operator* called the Laplacian, ∇^2.
>
> $$\nabla\cdot\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

Cylindrical Coordinate Systems:

In cylindrical coordinates, the gradient, divergence, curl, and other operators are more complex to compute because the coordinate system orientation depends on θ. So in cylindrical coordinates:

$$\nabla s = \left(\frac{\partial s}{\partial r}, \frac{\partial s}{r\partial \theta}, \frac{\partial s}{\partial z}\right)$$
$$\nabla \cdot \mathbf{v} = \left(\frac{\partial (r v_r)}{r\partial r} + \frac{\partial v_\theta}{r\partial \theta} + \frac{\partial v_z}{\partial z}\right)$$
$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{r\partial \theta} - \frac{\partial v_\theta}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \frac{\partial (r v_\theta)}{r\partial r} - \frac{\partial v_r}{r\partial \theta}\right)$$

Similarly, in a spherical coordinate system the definitions for the primary operators are again slightly different. The book and google know what they are…

But what does it really mean?

Let us think in two dimensions and Cartesian coordinates for the moment.

The **gradient** is the slope in both directions (x and y). It is a vector. If the function is height – its gradient *vector* points directly uphill and its magnitude is proportional to the slope of the hill (hence gradient) at that point.

The **divergence** is a property of vector fields. Given a vector field (lots of vectors at lots of positions), do those vectors tend to point outwards or inwards at any particular location. For example, draw vectors indicating the size and direction of the fluid flow at lots of locations in the flow field. Do the vectors tend to point inwards or diverge outwards? (For incompressible flow they will do neither – zero divergence).

The **curl** is also a property of vector fields. Given a vector field (lots of vectors at lots of positions), do those vectors tend to point clockwise (negative curl) or counterclockwise (positive curl – right hand rule). We will see that many velocity fields (but certainly not all) have zero curl. The curl of the velocity vector is so important it gets a special name in fluid mechanics – the **vorticity**. The vorticity is a vector quantity which points along the axis of rotation (use right hand rule to get direction). In 2D all the axis of rotation must point out of the board and the vorticity vector looks like $(0,0,\omega_z)$.

What about Cylindrical Coordinates?

The key to cylindrical coordinates is to *always* remember that the r and θ directions actually depend on where you are. The r direction always points outwards, and the θ direction always points 90 degrees to the left of outwards (counterclockwise – right hand rule). As you move around, the outwards direction changes.

So if $h(r,\theta)$ is a function of height. Then ∇h is *still* the vector which points along the direction of the steepest slope (and uphill). However, $\frac{\partial h}{\partial r}$ is the slope in the direction pointing away from the origin, and $\frac{1}{r}\frac{\partial h}{\partial \theta}$ is the slope in the direction pointing counterclockwise to the origin. And those particular direction are different depending on where you are standing relative to the origin.