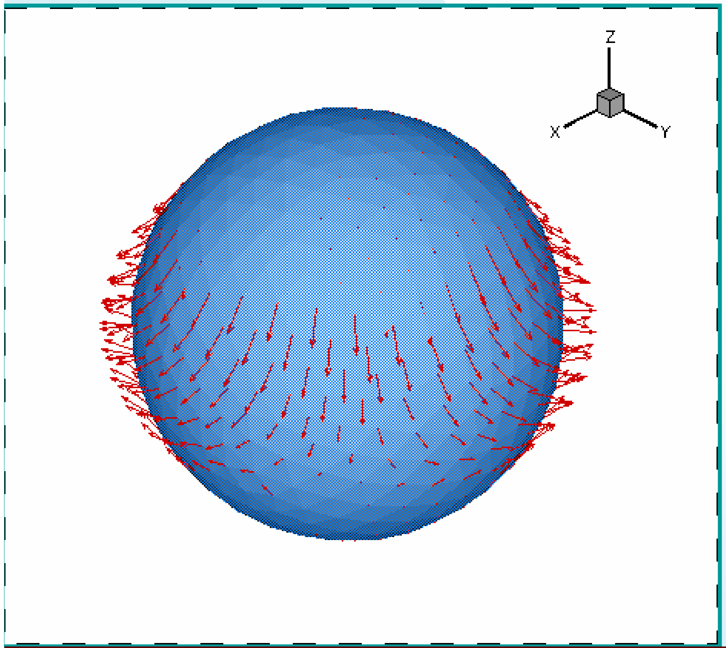
Solution of Incompressible Flow Problems on Moving Staggered Meshes



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Dept. of Mechanical Engineering

MIMETIC DISCRETIZATIONS OF COTINUUM MECHANICS

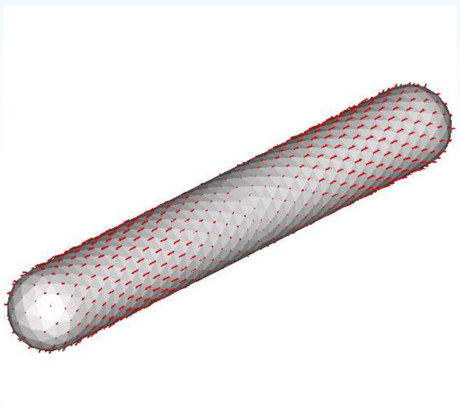
July 9-11, 2003

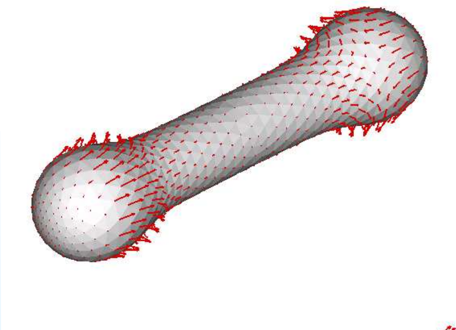
University of Massachusetts Amherst

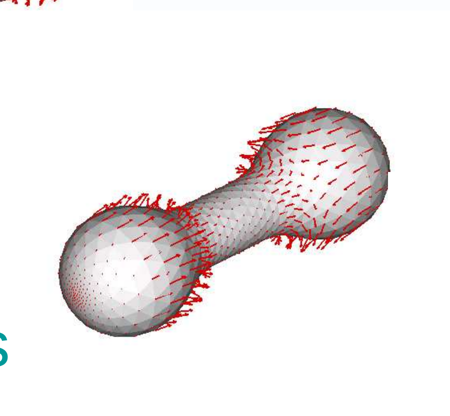
Moving Mesh Applications

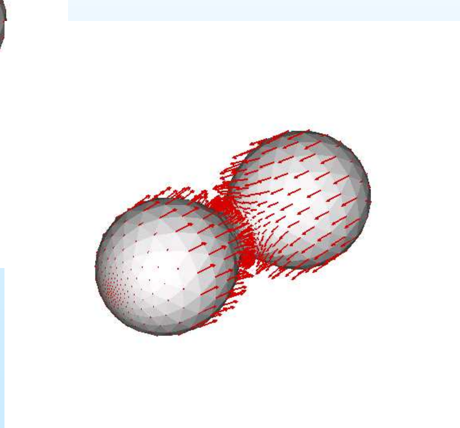
Collapse of a Water Filament

(Re=3, We=3)





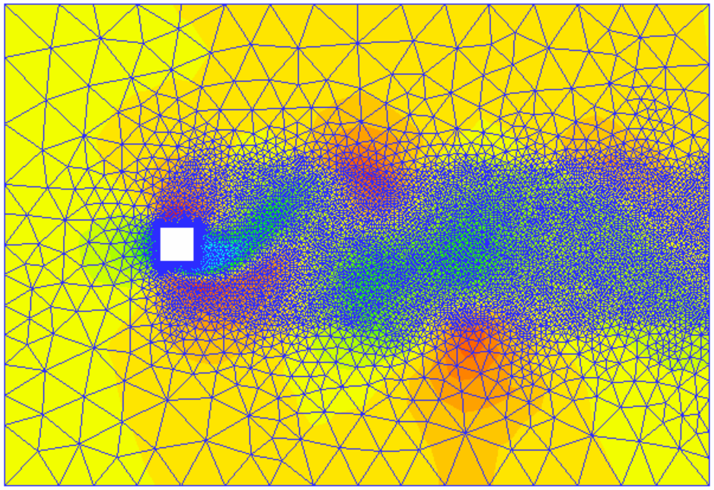




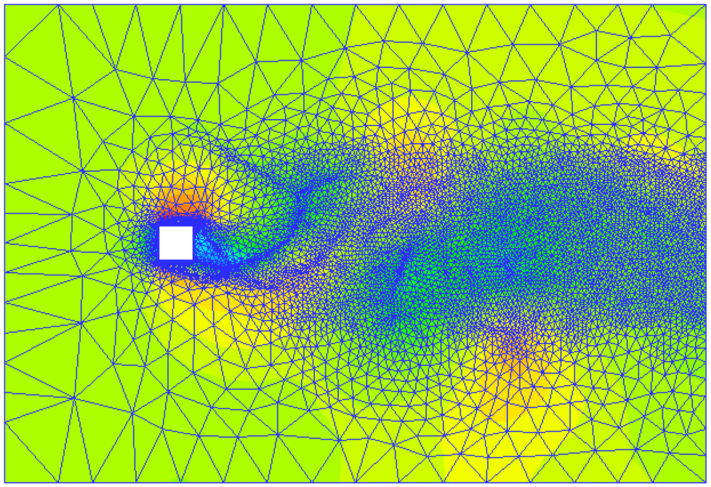
- Large Deformations
- Critical (Surface) Physics

Video Clip

Moving Mesh Applications



Turbulent Flow Around a Square Cylinder



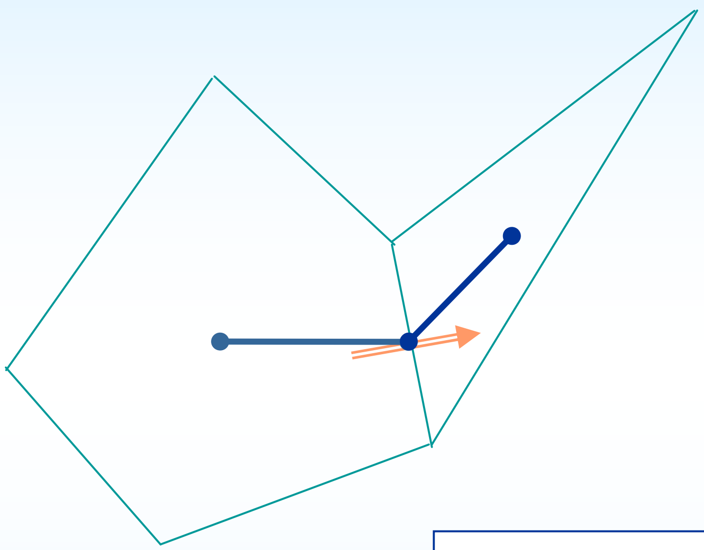
- Fixed Cost Mesh Adaptation
- Capture Critical (Transient) Physics

Video Clip

Moving Mesh Issues

- How to prescribe the mesh motion and local reconnection.
  - Lagrangian
  - Adaptive
  - Quality Preserving
- Geometry is time dependent.
- Conservation statements can apply to entire vectors - not components.

Unstructured Mimetic Methods



- HAVE $\int_A \mathbf{u} \cdot \mathbf{n} dA$
- WANT $\int_L \mathbf{u} \cdot d\mathbf{l}$

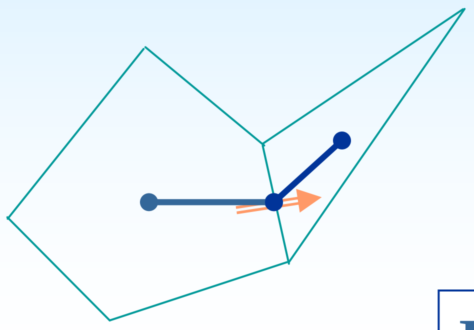
Faces to Dual Edges (*fe')

Lowest order

$$\int_L \mathbf{u} \cdot d\mathbf{l} = \frac{L}{A} \int_A \mathbf{u} \cdot \mathbf{n} dA$$

Only reasonable for orthogonal dual meshes

Unstructured Mimetic Methods



First order for general dual meshes

Faces to Cells³ (*fc³) : reconstruction

$$\mathbf{u}_{cell\_CG}V \approx \int_V (u_i + x_i u_{j,j})dV = \int_V (x_i u_j)_{,j} dV = \sum_{faces} \int_A x_i u_j n_j dA \approx \sum_{faces} \hat{\mathbf{x}}_{face\_CG} U$$

Cells³ to dual edges (*c³e') : recovery

$$\int_{L_1} \mathbf{u} \cdot d\mathbf{l} = \mathbf{u}_{cell\_cg} \cdot \mathbf{t}'_1 = \mathbf{u}_{cell\_cg} \cdot \hat{\mathbf{x}}_{face\_CG}$$

Symmetric pos. def.

1st order in general.

$$u_e = (X^T V^{-1} X)\, U_f$$

Moving Staggered Mesh Equations

- Normals, edges, etc are now time dependent.

Reynolds Transport Theorem

$$\frac{\partial}{\partial t}\int_{CV(t)} dV = \int_{CS(t)} \mathbf{u}_{mesh}\cdot\mathbf{n}dA$$

Constraints

$$\frac{\partial}{\partial t}\int_{CV(t)} \rho dV + \int_{CS(t)} \rho(\mathbf{u}-\mathbf{u}_{mesh})\cdot\mathbf{n}dA = 0$$

$$\frac{\partial}{\partial t}\int_{CV(t)} \rho\mathbf{u} dV + \int_{CS(t)} \rho\mathbf{u}(\mathbf{u}-\mathbf{u}_{mesh})\cdot\mathbf{n}dA = -\int_{CS(t)} p\mathbf{n}dA + \int_{CS(t)} \mu\frac{\partial\mathbf{u}}{\partial n}dA$$

Constraints

$$\frac{V^{n+1} - V^{n}}{\Delta t} = \sum_{faces} U_{mesh}$$

$$\sum_{faces} U = 0$$

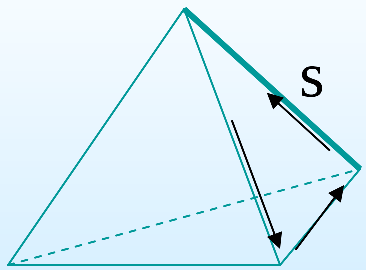
- Consistency requires that the mesh flux exactly satisfy the volume equation

$$U_{mesh} = \mathbf{u}_{face\_CG} \cdot \left\{ \tfrac{1}{2}(\mathbf{n}^{n+1} A^{n+1} + \mathbf{n}^{n} A^{n}) - \tfrac{\Delta t^2}{12} \sum_{edges} (\mathbf{u}_{mesh\_n1} \times \mathbf{u}_{mesh\_n2}) \right\}$$

- Divergence constraint satisfied by using a discrete curl

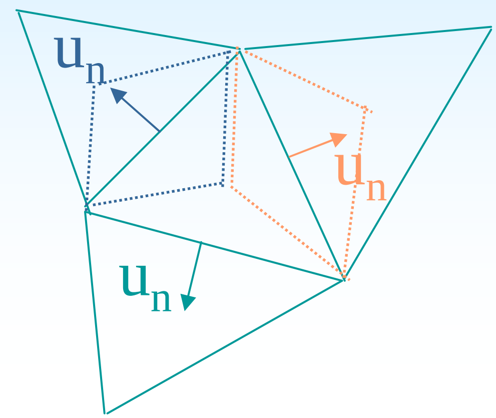
$$U = \sum_{edges} \pm s$$

$$s = \int_{edge} \vec{\psi} \cdot dl$$



Momentum Equation

- Integrate momentum equation over the **Cell** CV.



Constant Pressure

Linear⁺ Velocity

$$\frac{\mathbf{u}_c^{n+1}V^{n+1} - \mathbf{u}_c^{n}V^{n}}{\Delta t} + \sum_{faces} \mathbf{u}_f (U - U_{mesh}) = -V\nabla p + \sum_{faces} \nu \nabla \mathbf{u}_f \cdot \mathbf{N}_f$$

$$\mathbf{u}_c V = \sum_{faces} \hat{\mathbf{x}}_{face\_CG} U$$

Evolution equation for integral along the edge

$$\frac{\mathbf{X}^{T\,n+1} V^{-1} \mathbf{X}^{n+1} U^{n+1} - \mathbf{X}^{T\,n+1} V^{-1} \mathbf{X}^{n} U^{n}}{\Delta t} + \mathbf{X}^{T\,n+1} V^{-1} \mathbf{a}_c = -\mathbf{G} p^{n+1}$$

Unknowns at faces

Eqn. on dual edge

Momentum Equation

- Two step symmetric Hodge* operator

$$s(edges) \xrightarrow{curl} U(faces) \xrightarrow{div} V\nabla \bullet \mathbf{u}(cells) = 0$$

$$\searrow^{X}$$

$$V\mathbf{u}(cells)$$

$$\swarrow^{X^T}$$

$$w(dual\_face) \xleftarrow{curl} u_e(dual\_edges)$$

Equation System

- Saddle Point Problem

$$\begin{bmatrix} (\frac{1}{\Delta t}\mathbf{M}^{n+1} - \mathbf{X}^{n+1}\mathbf{A}^{n+1}) & \mathbf{G} \\ \mathbf{D} & 0 \end{bmatrix} \begin{pmatrix} U^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{X}^{n+1} r^{n} \\ 0 \end{pmatrix}$$

$U = \mathbf{C}s$ where $\mathbf{D}\mathbf{C} = 0$ and so $\mathbf{C}^T\mathbf{G} = 0$

- Symmetric Pos. Def.

$$\mathbf{C}^T(\tfrac{1}{\Delta t}\mathbf{M}^{n+1} - \mathbf{X}^{n+1}\mathbf{A}^{n+1})\mathbf{C}s = \mathbf{C}^T\mathbf{X}^{n+1}r_c^{\,n}$$

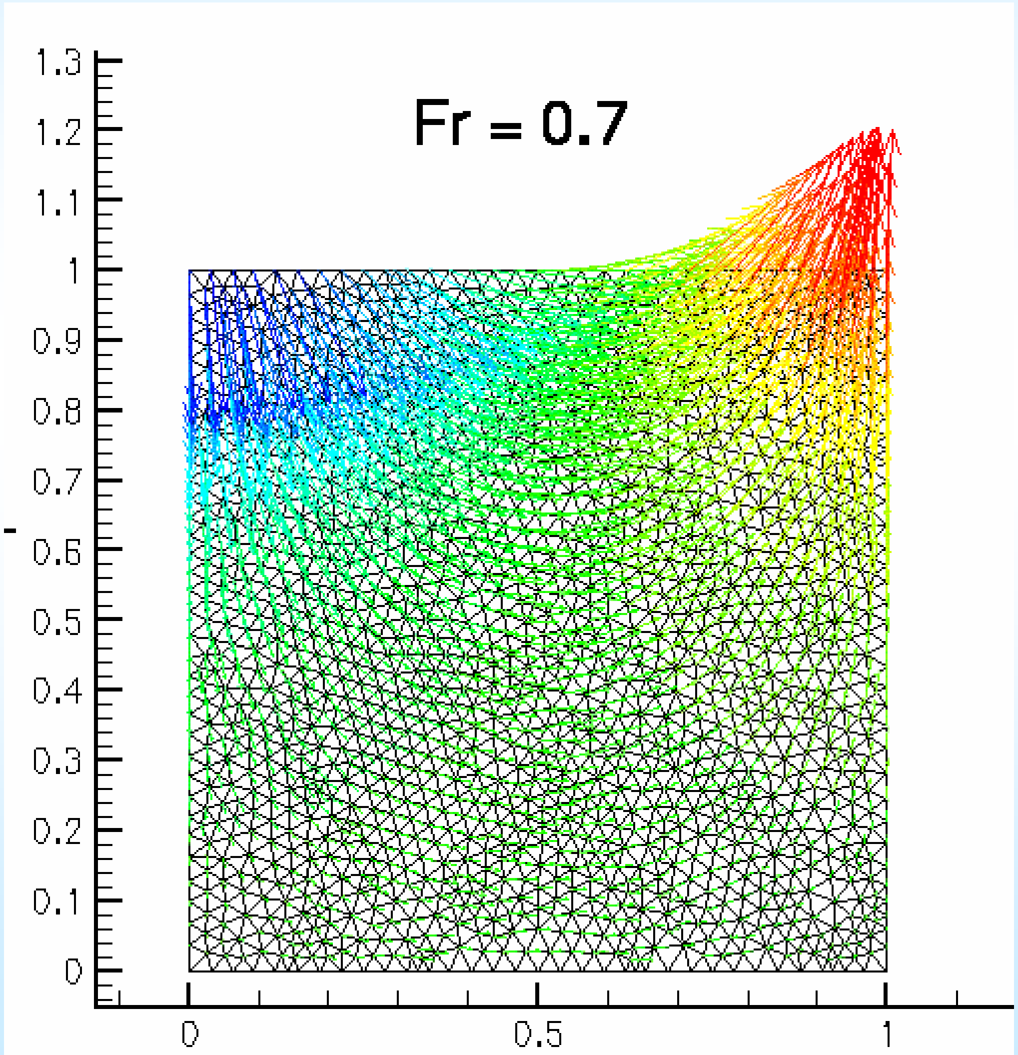
•Reduced number of unknowns.

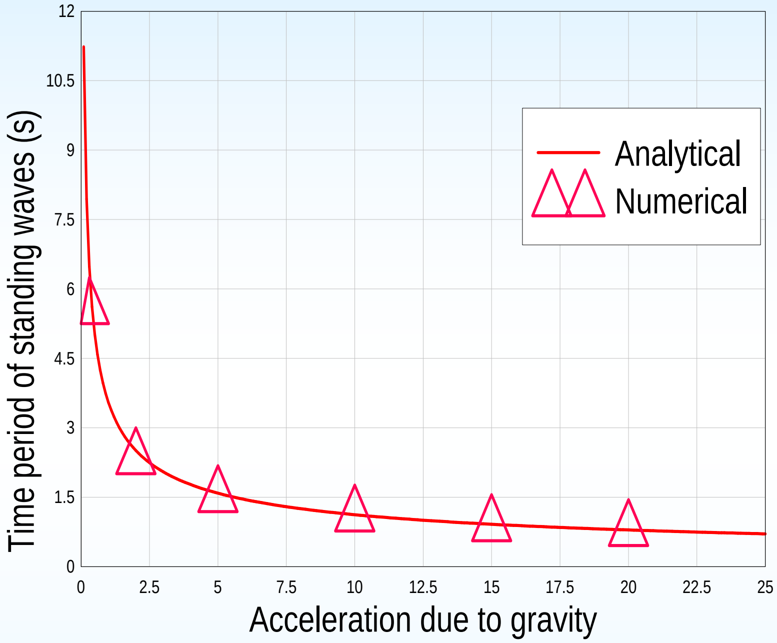
•Easier to solve.

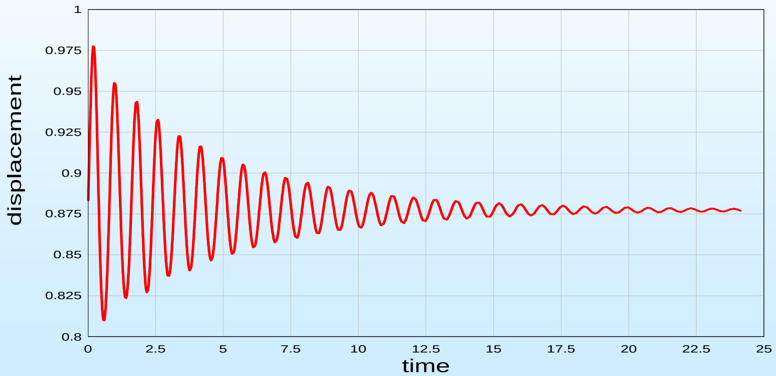
•BCs on U and p

Validation

Sloshing Tank

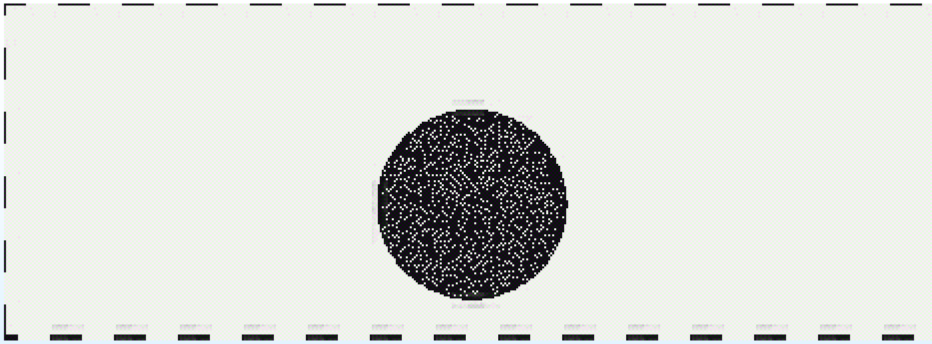


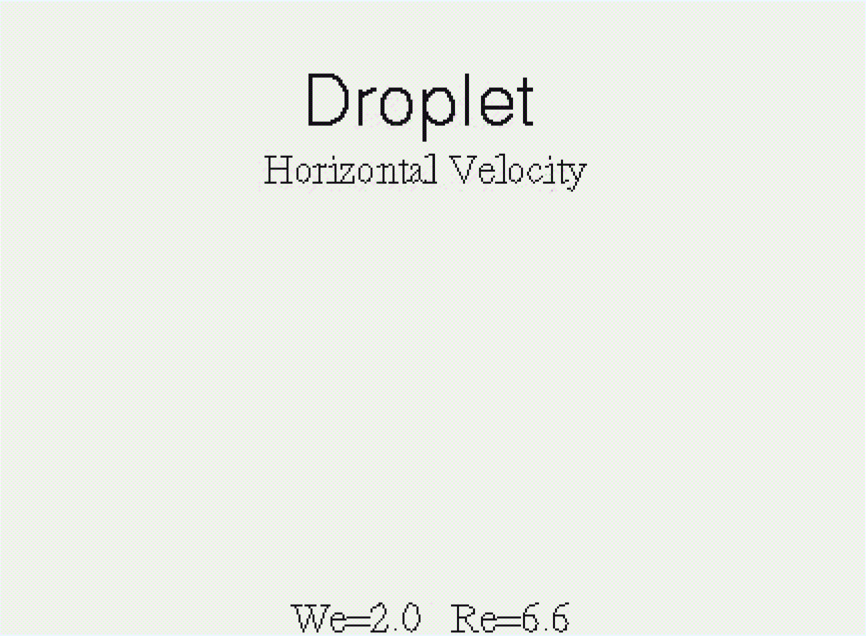




Validation

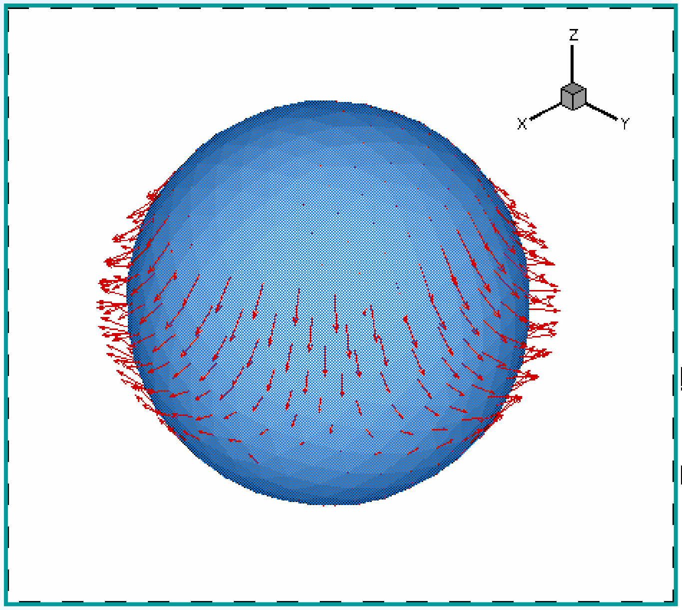
Droplets hitting a wall

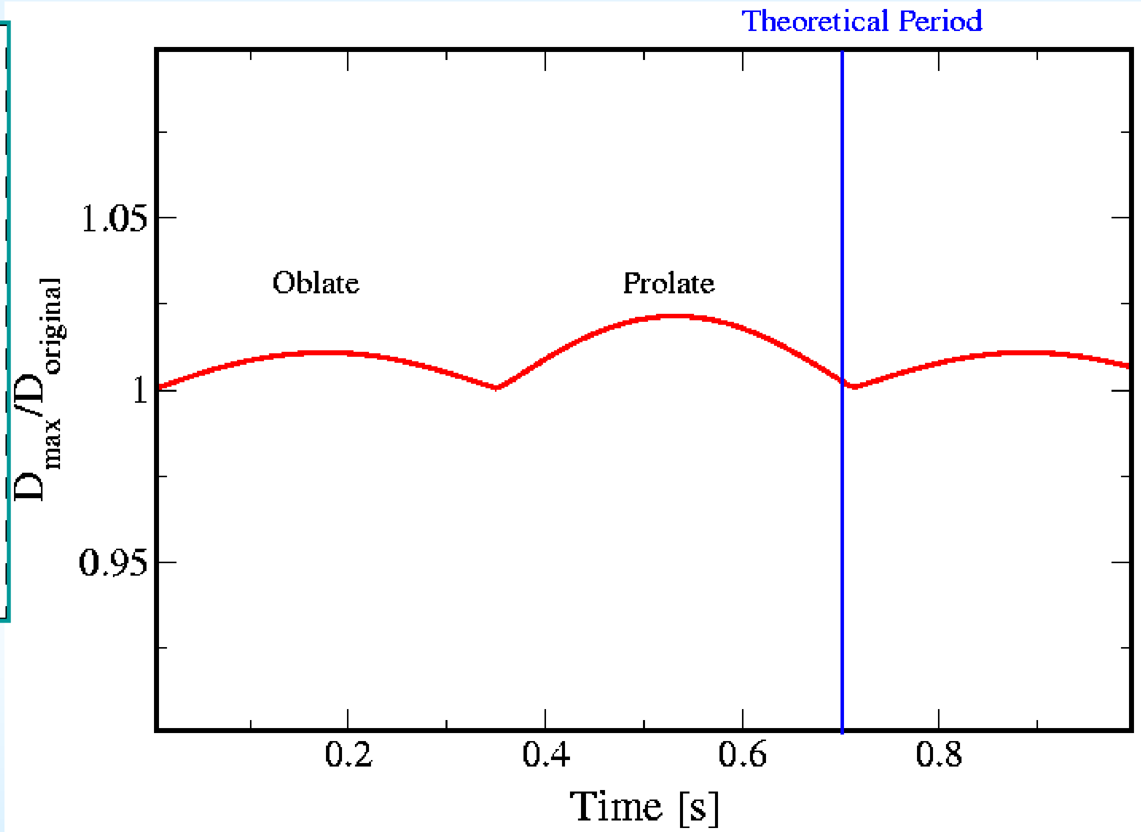




Validation

Oscillating 3D Droplet

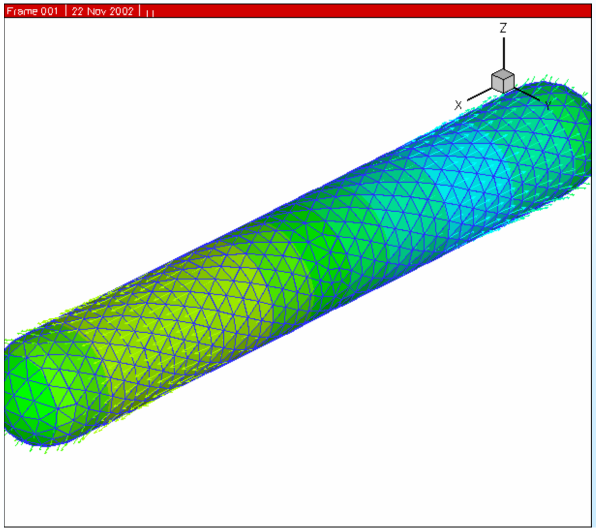




- With perturbation analyses good to 0.02%

Conclusion

- Conservative moving mesh Mimetic methods are possible.
- Mimetic methods make incompressibility easy to handle.
- It is useful to define vector quantities.



- Conjecture: a good Hodge* operator should be sym. pos. def. and factor into the operator that defines vectors.

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