Office of Naval Research

Compatible Reconstruction of Vectors

Blair Perot

Dept. of Mechanical & Industrial Engineering

Compatible Spatial Discretizations for Partial Differential Equations

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UNIVERSITY OF Massachusetts AMHERST

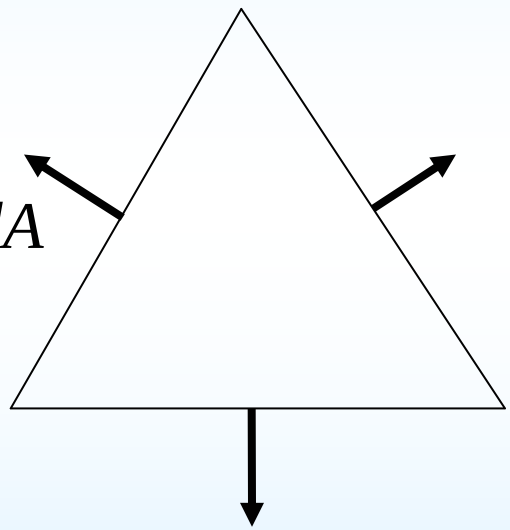
TU Delft
Delft University of Technology

Compatible Discretizations

- **Vector <u>Components</u> are Primary**

Normal components
(Face Elements)

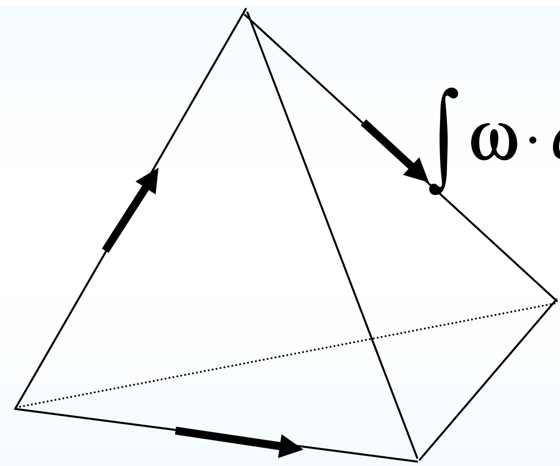
$$\int \mathbf{v} \cdot \mathbf{n} \, dA$$



Heat Flux
Magnetic Flux
Velocity Flux

Tangential components
(Edge Elements)

$$\int \boldsymbol{\omega} \cdot d\mathbf{l}$$



Temperature Gradient
Electric Field
Vorticity

Why Vector Components

- **Physics**
- **Mathematics**
- **Numerics**

Physics:

Measurements
Continuity Requirements
Boundary Conditions

Mathematics:

Differential Forms
Gauss/Stokes Theorems

Numerics:

Absence of Spurious Modes
Mimetic Properties

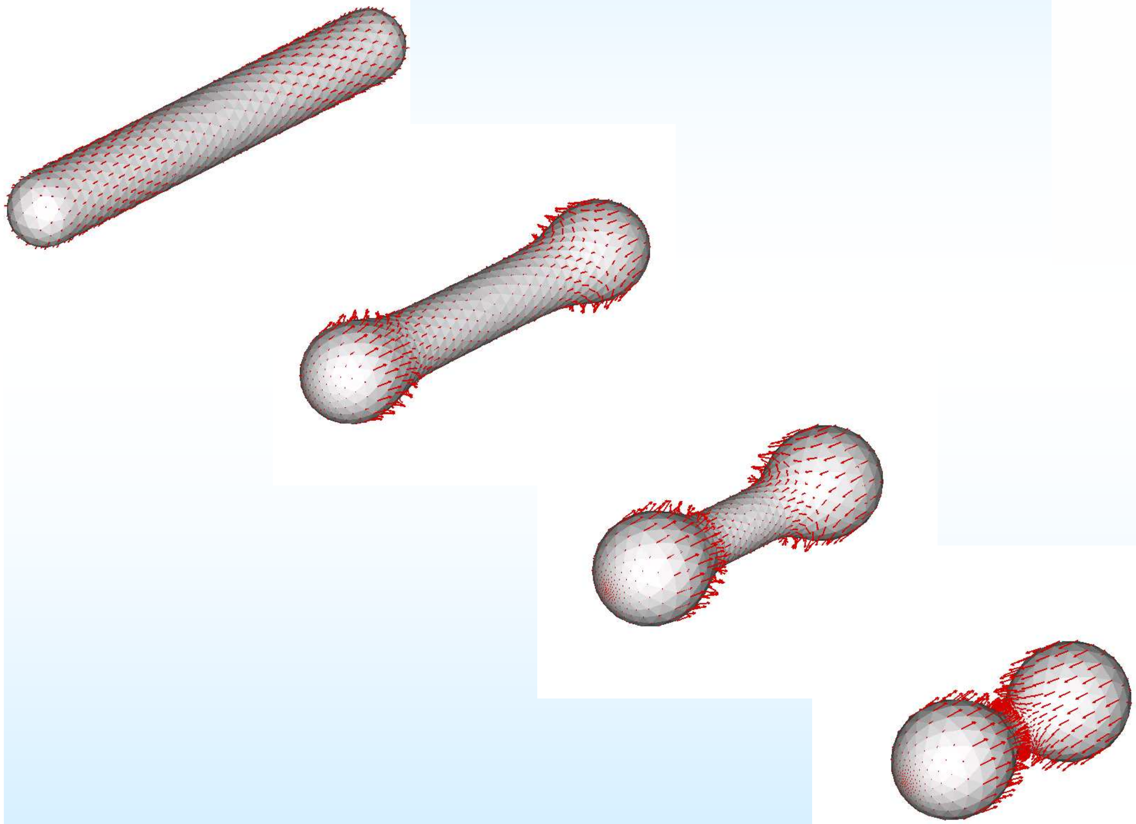
Unknowns should contain Geometry/Orientation Information

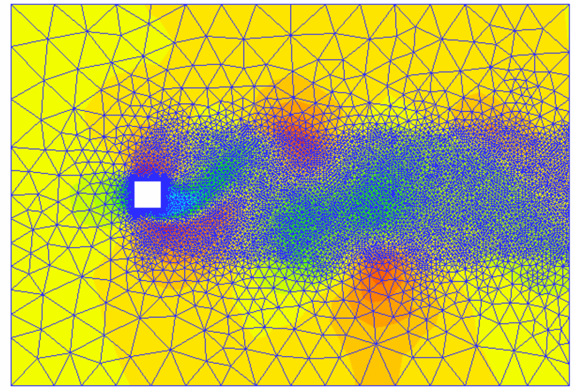
So Why Vector Reconstruction ?

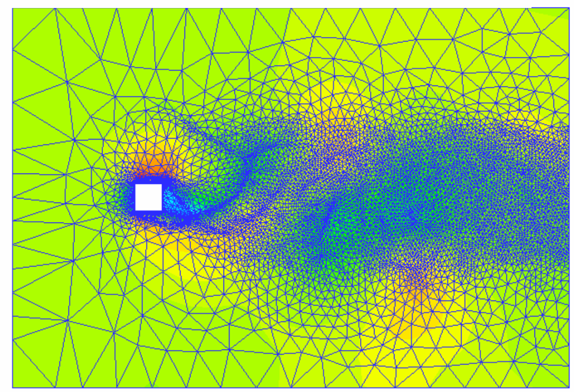
- **Convection** $\nabla \cdot (\rho \mathbf{v}\mathbf{v})$
- **Adaptation**
- **Formulation of Local Conservation Laws**
  - **(momentum, kinetic energy, vorticity/circulation)**
- **Construction of Hodge star operators**
- **Nonlinear constitutive relations**

Convection/ Adaptation

M-adaptation







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Conservation

- **Wish to have discrete analogs of vector laws.**
  - **Conservation of Linear Momentum**
  - **Conservation of Kinetic Energy**

$$U_f = \int \mathbf{u}\cdot\mathbf{n}\,dA$$

$$R^T V^{-1} R \frac{\partial U_f}{\partial t} + R^T V^{-1} \mathbf{D} N^T \mathbf{a}(U_f) = -\mathbf{G} p_c$$

Component Equations

$$N R^T V^{-1} R \frac{\partial U_f}{\partial t} + N R^T V^{-1} \mathbf{D} N^T \mathbf{a}(U_f) = -N\mathbf{G} p_c$$

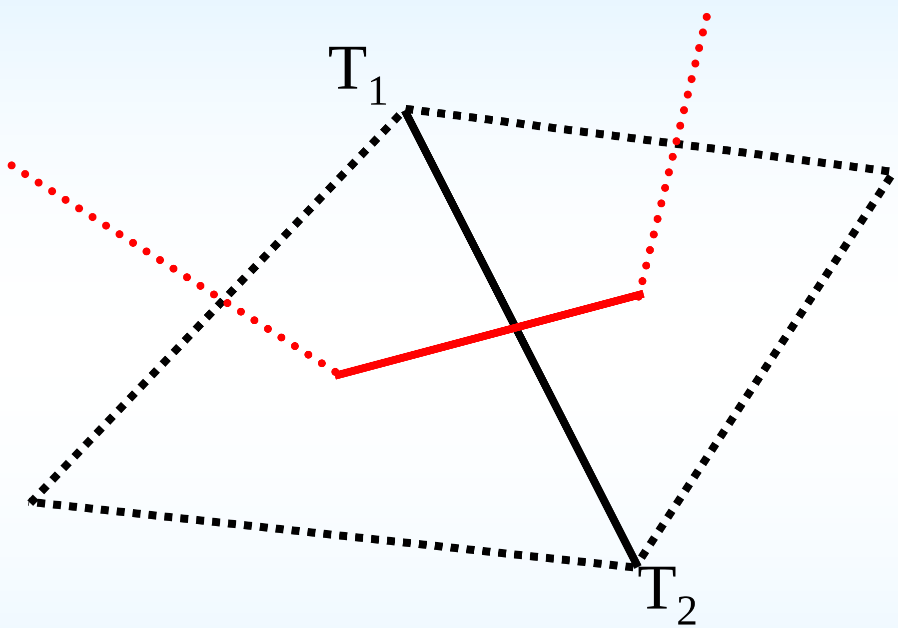
$$\sum_{cell(s)} V \frac{\partial \hat{\mathbf{u}}_c}{\partial t} = \sum_{boundary} (N p_b - \mathbf{a}_b)$$

Linear Momentum

$$\hat{\mathbf{u}}_c = V^{-1} R U_f$$

3-Form ?

Hodge Star Operators



$$\nabla \cdot k\nabla T = 0$$

$$\Downarrow$$

$$\mathbf{q} = -k\nabla T$$

$$\sum_{\substack{cell\\faces}} \int_{A_i} \mathbf{q}\cdot\mathbf{n}\,dA = 0$$

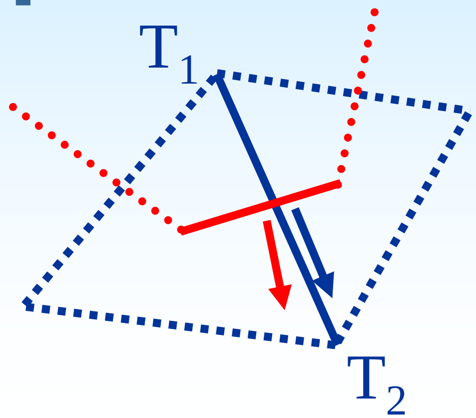
Have (tangential)	$\int \nabla T \cdot d\mathbf{l}$
Need (normal)	$\int \mathbf{q}\cdot\mathbf{n}\,dA$

- **Discrete Hodge Star**
  - **Interpolate / Integrate**
  - **Least Squares**

$$\int \mathbf{q}\cdot\mathbf{n}\,dA = H * \int \nabla T \cdot d\mathbf{l}$$

Notation

Compatible/Mimetic Discretization



$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad \frac{\partial \rho C T}{\partial t} = \nabla \cdot k \nabla T$$

de Rham-like complex

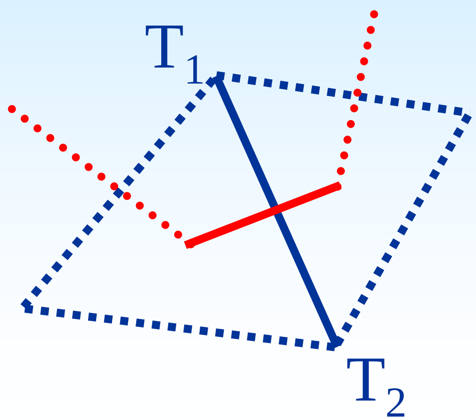
$$\Rightarrow \; T_n \; \xrightarrow{\mathbf{G}_{n2e}} \; \int \nabla T \cdot d\mathbf{l} \; \xrightarrow{\mathbf{C}_{e2f}} \; 0 \; \xrightarrow{\mathbf{D}_{f2c}} \; P_{cell} \; \Rightarrow$$

$$S = \rho C_v T \qquad \Updownarrow \; \mathbf{q} = -k \nabla T$$

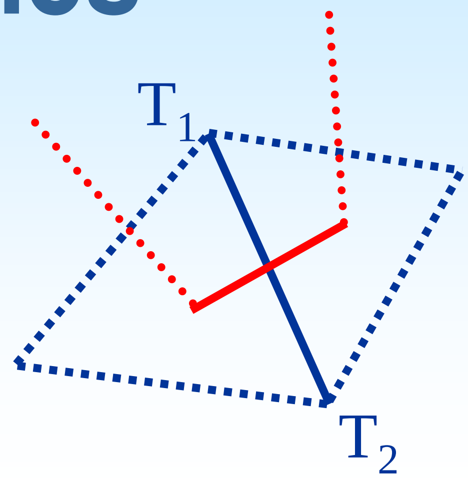
$$\Leftarrow \; \int S dV \; \xleftarrow{\mathbf{D}_{e2n}} \; \int \mathbf{q} \cdot \mathbf{n} dA \; \xleftarrow{\mathbf{C}_{f2e}} \; D_{edge} \; \xleftarrow{\mathbf{G}_{c2f}} \; D_{node} \; \Leftarrow$$

Exact
Connectivity Matrices
Transposes

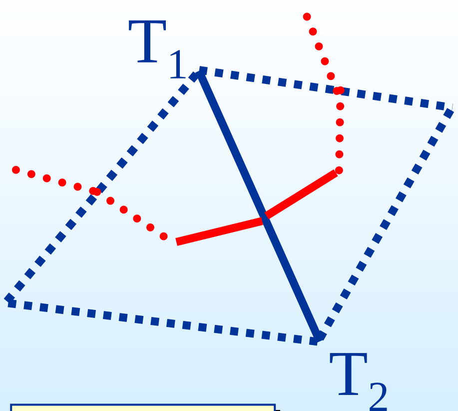
Dual Meshes



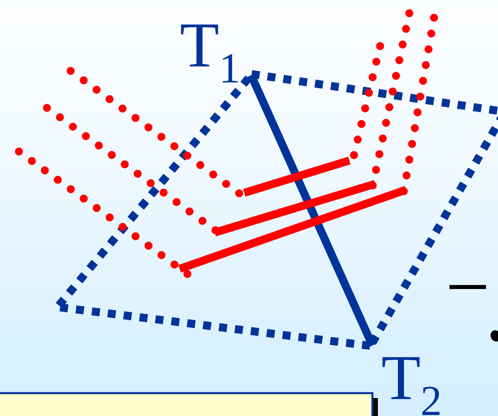
Circumcenter (Voronio)



Centroid (center of gravity)



Median

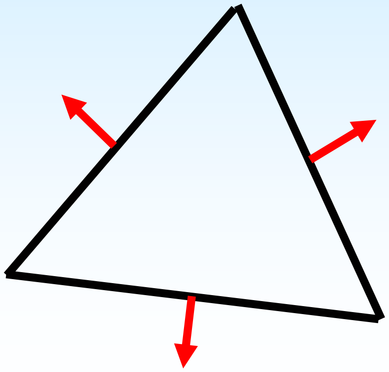


FE (smeared)

$$-\int k\nabla T \cdot \nabla w dV = 0$$

$$\int \mathbf{q} \cdot \mathbf{n} dA dx = 0$$

FE Reconstruction



- 2D: Raviart-Thomas
- 3D: Nedelec

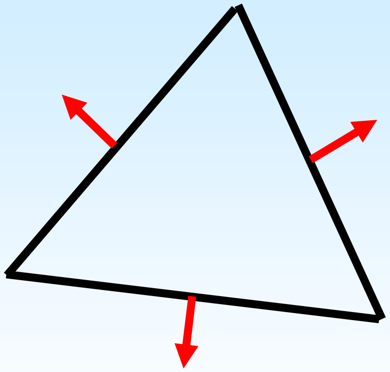
$$\mathbf{v}(\mathbf{x}) = \mathbf{a}_n(\mathbf{x}) + b_n(\mathbf{x})\mathbf{x} \quad \textbf{Face}$$

$$\mathbf{v}(\mathbf{x}) = \mathbf{c}_n(\mathbf{x}) + \mathbf{d}_n(\mathbf{x}) \times \mathbf{x} \quad \textbf{Edge}$$

- Compute Coefficients in the Interpolation
- Compute Integrals

No dual – because FE is an average over all duals.
Quadrature rule is a way of weighting the duals.
(which is how you can get other methods)

SOM Reconstruction



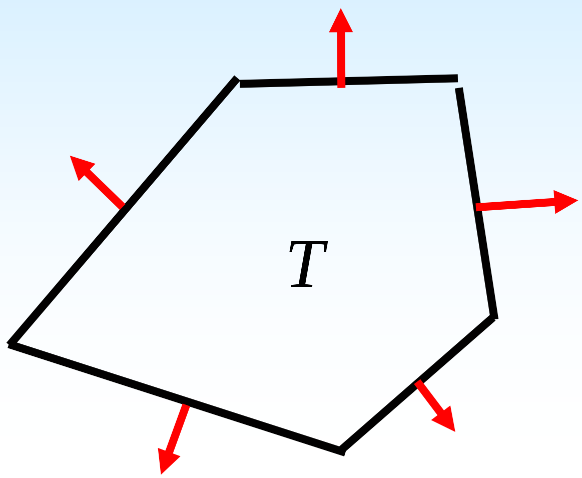
- Shashkov, et al.
  - Reconstruct local node values
  - then interpolate
- Arbitrary Polygons
- When Incompressible and Simplex = FE interpolation

$$\Rightarrow T_n \xrightarrow{\mathbf{G}_{n2e}} \int \nabla T \cdot d\mathbf{l} \xrightarrow{\mathbf{C}_{e2f}} 0 \xrightarrow{\mathbf{D}_{f2c}} P_{cell} \Rightarrow$$

Discrete Hodge Stars $\quad \downarrow N \qquad \downarrow L \qquad \downarrow S \qquad \downarrow R$

$$\Leftarrow \int S dV \xleftarrow{\mathbf{D}_{e2n}} \int \mathbf{q} \cdot \mathbf{n} dA \xleftarrow{\mathbf{C}_{f2e}} D_{edge} \xleftarrow{\mathbf{G}_{c2f}} D_{node} \Leftarrow$$

Voronio Reconstruction



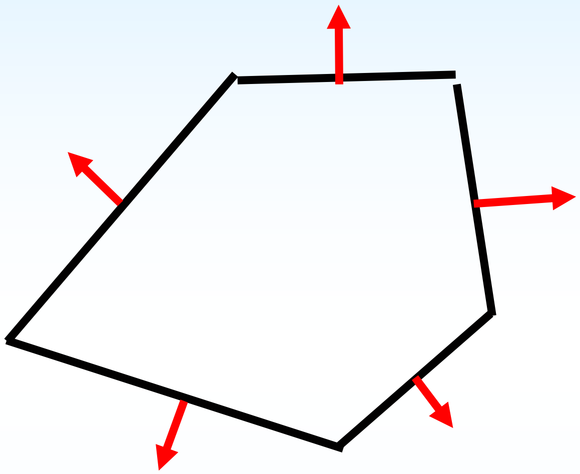
$$\int \mathbf{q} \cdot \mathbf{n} dA = \left(k_f \frac{A_f}{L_f} \right) \int \nabla T \cdot d\mathbf{l}$$

Diagonal Hodge star operator (due to local orthogonality)

- **CoVolume method (when simplices).**
- **Used in ‘meshless’ methods (material science)**
- **Locally Conservative (N.S. momemtum and KE).**

Discrete Maximum Principal in 3D for Delaunay mesh (not true for FE).

Staggered Mesh Reconstruction



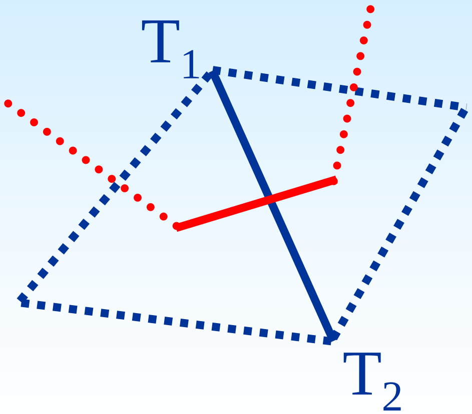
$$\bar{\mathbf{v}}_c = \frac{1}{V} \sum_{\substack{cell \\ faces}} \{\mathbf{x}_f^{CG} - \mathbf{x}_c^{CG}\} U_f$$

Dilitation = constant

Face normal velocity is constant

- **Conserves Momentum and Kinetic Energy.**
- **Arbitrary mesh connectivity.**
- **No locally orthogonality between mesh and dual.**
- **Hodge is now sparse sym pos def matrix.** $\int \mathbf{q} \cdot \mathbf{n} dA = M \int \nabla T \cdot d\mathbf{l}$

Vector Reconstruction



Expand the Hodge star operation

Nonlinear Constitutive Relations are no problem

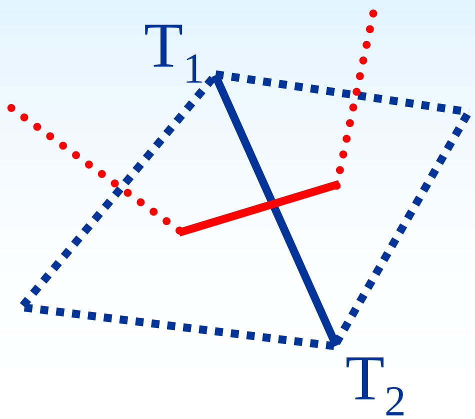
$$const \Rightarrow T_n \xrightarrow{\mathbf{G}_{n2e}} \int \nabla T \cdot d\mathbf{l} \xrightarrow{\mathbf{C}_{e2f}} 0 \xrightarrow{\mathbf{D}_{f2c}} P_{cell} \Rightarrow 0$$

$$\nabla T_n \Leftrightarrow \mathbf{q}_n \qquad \nabla T_c \Leftrightarrow \mathbf{q}_c$$

$$0 \Leftarrow \int S dV \xleftarrow{\mathbf{D}_{e2n}} \int \mathbf{q} \cdot \mathbf{n} dA \xleftarrow{\mathbf{C}_{f2e}} D_{edge} \xleftarrow{\mathbf{G}_{c2f}} D_{node} \Leftarrow const$$

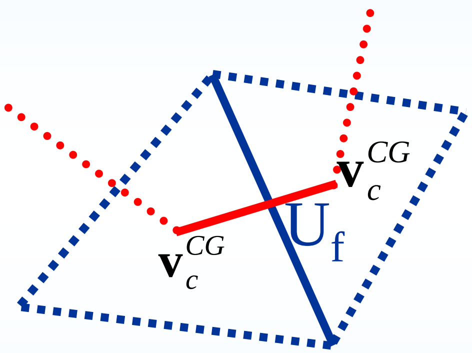
Other Methods

- **CVFEM**
  - **Linear in elements (sharp dual)**
  - **Local conservation**
- **Classic FEM**
  - **Linear in element (spread dual)**
- **Discontinuous Galerkin / Finite Volume**
  - **Reconstruct in the Voronio Cell**



Methods Differ in:
Interpolation Assumptions
Integration Assumptions

Staggered Mesh Reconstruction



Interpolate

$$\int (\mathbf{v} + \mathbf{x}\nabla \cdot \mathbf{v})dV = \int (x_i v_j)_{,j} dV = \sum_{\substack{cell \\ faces}} \int \mathbf{x}\mathbf{v} \cdot \mathbf{n} dA$$

$$\mathbf{v}_c^{CG} \approx \tfrac{1}{V} \sum_{\substack{cell \\ faces}} \pm(\mathbf{x}_f^{CG} - \mathbf{x}_c^{CG})U_f = V^{-1}\mathbf{X}U_f$$

Integrate

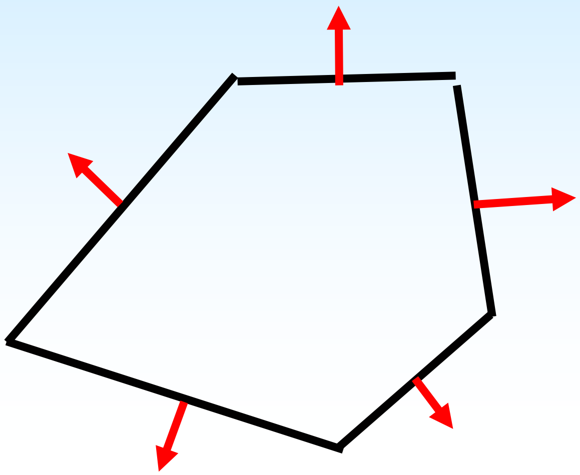
$$\int \rho\mathbf{v} \cdot d\mathbf{l} \approx \sum_{\substack{face \\ cells}} \pm(\mathbf{x}_f^{CG} - \mathbf{x}_c^{CG})\rho_c \mathbf{v}_c^{CG} = \mathbf{X}^T \rho_c \mathbf{v}_c^{CG}$$

$$\int \rho\mathbf{v} \cdot d\mathbf{l} \approx (\mathbf{X}^T \rho_c V^{-1} \mathbf{X}) \int \mathbf{v} \cdot \mathbf{n} dA$$

Symmetric Pos. Def. sparse discrete Hodge star operator

X has same sparsity pattern as **D**

Staggered Mesh Conservation



Exact Geometric Identities

$$V = \int x_{i,j} dV = \sum_{\substack{cell \\ faces}} \int \mathbf{xn} dA = \sum_{\substack{cell \\ faces}} \mathbf{x}_f^{CG} \mathbf{n} A_f$$

$$0 = \int 1_{,j} dV = \sum_{\substack{cell \\ faces}} \int \mathbf{n} dA = \sum_{\substack{cell \\ faces}} \mathbf{n} A_f$$

Time Derivative in N.S.

$$\mathbf{X}^T \tfrac{\partial}{\partial t}(V^{-1}\mathbf{X}U_f)$$

$$\sum_{\substack{cell \\ faces}} \mathbf{n}_f A_f \mathbf{X}^T \tfrac{\partial}{\partial t}(V^{-1}\mathbf{X}U_f) = \tfrac{\partial}{\partial t}(\mathbf{X}U_f)\Big|_{cell} = \tfrac{\partial}{\partial t}(V\overline{\mathbf{v}_c})\Big|_{cell}$$

Conservation Properties

- Voronoi Method

$$V\overline{\mathbf{v}_c} = \sum_{\substack{cell\\faces}} L_{fc} U_f \mathbf{n}_f$$

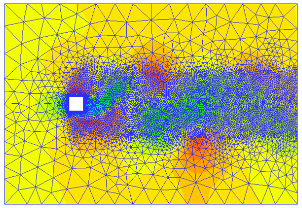
  - Conserves KE
  - Rotational Form -- Conserves Vorticity
  - Divergence Form – Conserves Momentum
  - Cartesian Mesh – Conserves Both

- Staggered Mesh Method

$$V\overline{\mathbf{v}}_c = \sum_{\substack{cell\\faces}} \{\mathbf{x}_f^{CG} - \mathbf{x}_c^{CG}\} U_f$$

  - Conserves KE
  - Divergence Form – Conserves Momentum

Incompressible Flow



- **Define a discrete vector potential** $U_f = \mathbf{C} s_e$
  - **So** $\mathbf{D} U_f = 0$ **always.**
  - **C^T of the momentum equation eliminates pressure (except on the boundaries where it is an explicit BC)**
- **Resulting system is:**
  - **Symmetric pos def (rather than indefinite)**
  - **Exactly incompressible**
  - **Fewer unknowns**

Conclusions

- **Physical PDE systems can be discretized (made finite) <u>exactly</u>. Only constitutive equations require numerical (and physical) approximation.**
- **Vector reconstruction is useful for:**
  convection, adaptation, conservation, Hodge star construction, nonlinear material properties.
- **Hodge star operators have internal structure that is useful and related to interpolation/integration.**

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