An Exact Projection Method

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Perspectives on incompressible flows.
Comparison of different computational strategies

CSCAMM   April 7 - April 11, 2003

Projection Methods

Method for dealing with the incompressibility constraint.

(1) Solve momentum equation for velocity

(2) Impose incompressibility on that velocity

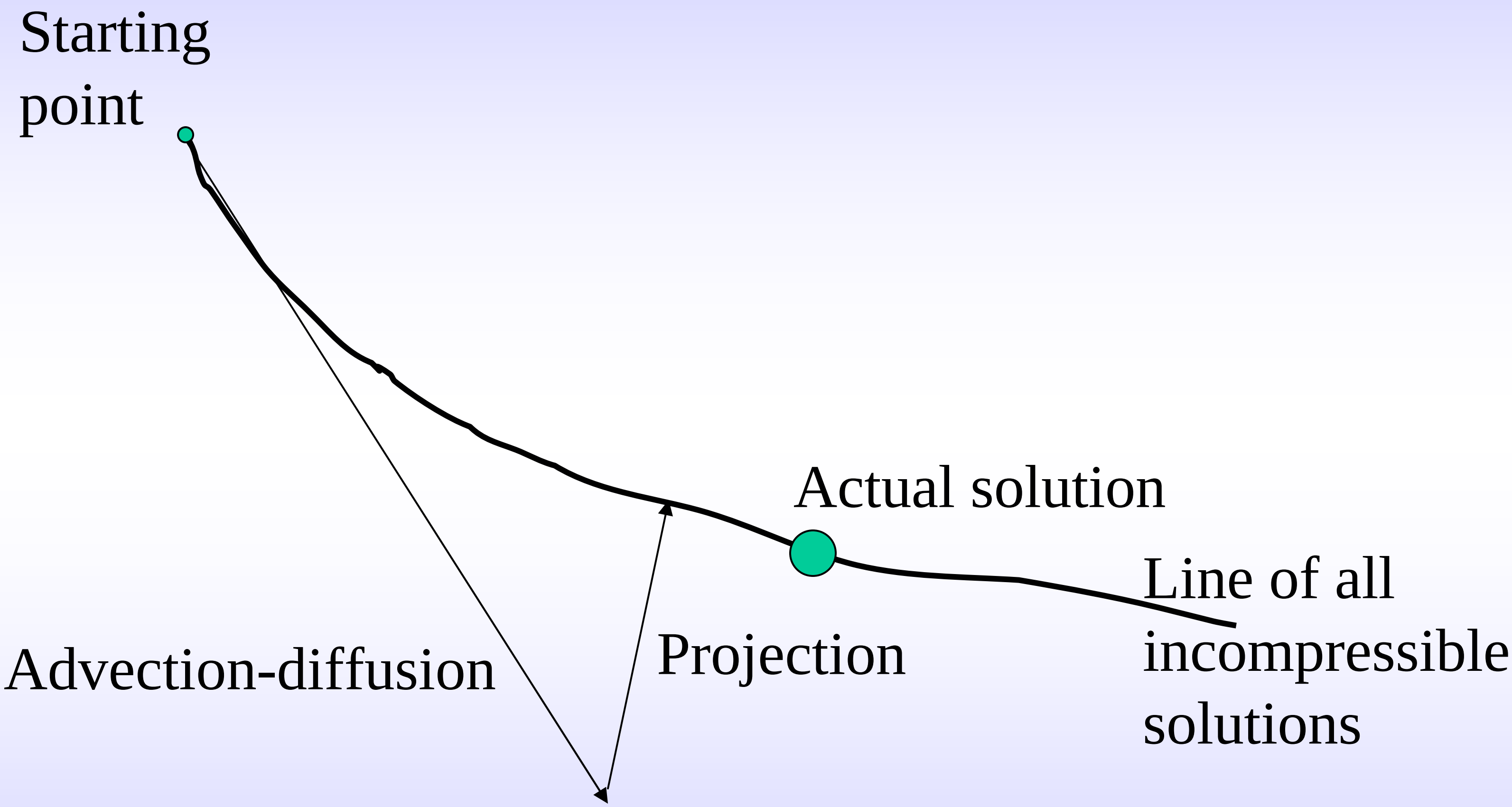
Pro:

- **Speed**
- **Simplicity**
- **Exact constraint satisfaction**

Con:

- **Boundary conditions**
- **Time accuracy**
- **Numerical precision**
- **Pressure issues**

Projection Methods



Time Splitting /Semi-Discrete Approach

$$(1) \quad \frac{\hat{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\Delta t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p^n + \mu \nabla^2 \tfrac{1}{2}(\hat{\mathbf{u}}^{n+1} + \mathbf{u}^n)$$

$$(2) \quad \frac{\mathbf{u}^{n+1} - \hat{\mathbf{u}}^{n+1}}{\Delta t} = -\nabla \tfrac{1}{2}(p^{n+1} - p^n)$$

$$(3) \quad \nabla^2 \tfrac{\Delta t}{2}(p^{n+1} - p^n) = \nabla \cdot \hat{\mathbf{u}}^{n+1}$$

- **Solve in the order (1), (3), then (2).**
- **Splits problem into Advection-Diffusion eqn. & Poisson eqn.**
- **Boundary conditions on $\hat{\mathbf{u}}^{n+1}$ are very difficult to specify.**
- **First order accurate in time, or neutrally stable.**

Fully Discrete Approach

Navier-Stokes $$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \frac{1}{2}(p^{n+1} - p^n) \end{bmatrix} = \begin{bmatrix} \mathbf{r}^n \\ 0 \end{bmatrix} + \begin{bmatrix} bc \\ bc \end{bmatrix}$$

Projection = Approximate Block LU factorization

L ***U*** ***error***

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ D & -(\Delta t)DG \end{bmatrix} \begin{bmatrix} I & (\Delta t)G \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & (I - \Delta t A)G \\ 0 & 0 \end{bmatrix}$$

- **Splitting error only in the momentum equation**
- **Continuity constraint is satisfied to machine precision.**

Fully Discrete Implementation

Solution of Block LU factorization

$$\begin{bmatrix} A & 0 \\ D & -(\Delta t)DG \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}^{n+1} \\ \phi \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{n} \\ 0 \end{bmatrix} + \begin{bmatrix} bc \\ bc \end{bmatrix}$$

$$\begin{bmatrix} I & (\Delta t)G \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \frac{1}{2}(p^{n+1} - p^{n}) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{u}}^{n+1} \\ \phi \end{bmatrix}$$

$$A\hat{\mathbf{u}}^{n+1} = \mathbf{r}^{n} + bc$$

$$DG(\tfrac{\Delta t}{2}(p^{n+1} - p^{n})) = D\hat{\mathbf{u}}^{n+1} - bc$$

$$\mathbf{u}^{n+1} = \hat{\mathbf{u}}^{n+1} - G(\tfrac{\Delta t}{2}(p^{n+1} - p^{n}))$$

- **No issues with intermediate Boundary Conditions.**
- **Completely describes the actual numerical method.**

Projection Error

- Projection error dominates near boundaries.
- Projection error is usually first order.
- Second order splitting is possible - but has other problems.
- Higher order approximation to A^{-1}? Possible but expensive.
- Projection error is very significant for implicit convection, moving meshes, or low Reynolds numbers.

Can the projection error be eliminated completely?

Without sacrificing speed and simplicity.

Exact Projection

Navier-Stokes
$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \frac{1}{2}(p^{n+1} - p^n) \end{bmatrix} = \begin{bmatrix} \mathbf{r}^n \\ 0 \end{bmatrix} + \begin{bmatrix} bc \\ bc \end{bmatrix}$$

Method 1: Just solve the full system.

- **Large and coupled.**
- **Scaling issues (units).**
- **Singular (zero eigenvalue)**
- **Indefinite (both + and - eigenvalues)**
- **Iterative convergence of Poisson, cost of Advect-Diffusion**
- **Constraint not exactly satisfied (with iterative solution)**

Exact Projection

Navier-Stokes $$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \frac{1}{2}(p^{n+1} - p^n) \end{bmatrix} = \begin{bmatrix} \mathbf{r}^n \\ 0 \end{bmatrix} + \begin{bmatrix} bc \\ bc \end{bmatrix}$$

Method 2: Construct the discrete velocity $\mathbf{u}^{n+1}$ such that the discrete constraint is automatically satisfied.

D is a rectangular matrix (more columns than rows).

It therefore has a right null-space C, such that DC = 0

Let $\mathbf{u}^{n+1} = C\, s^{n+1}$ also note that $C^T G = 0$

so $C^T A C\, s^{n+1} = C^T(r^n + bc)$

Exact Projection Assessment

(1) $C^TAC\, s^{n+1} = C^T(r^n + bc)$

(2) $\mathbf{u}^{n+1} = C\, s^{n+1}$

This approach requires:

C and C^T to be sparse and easy to construct.

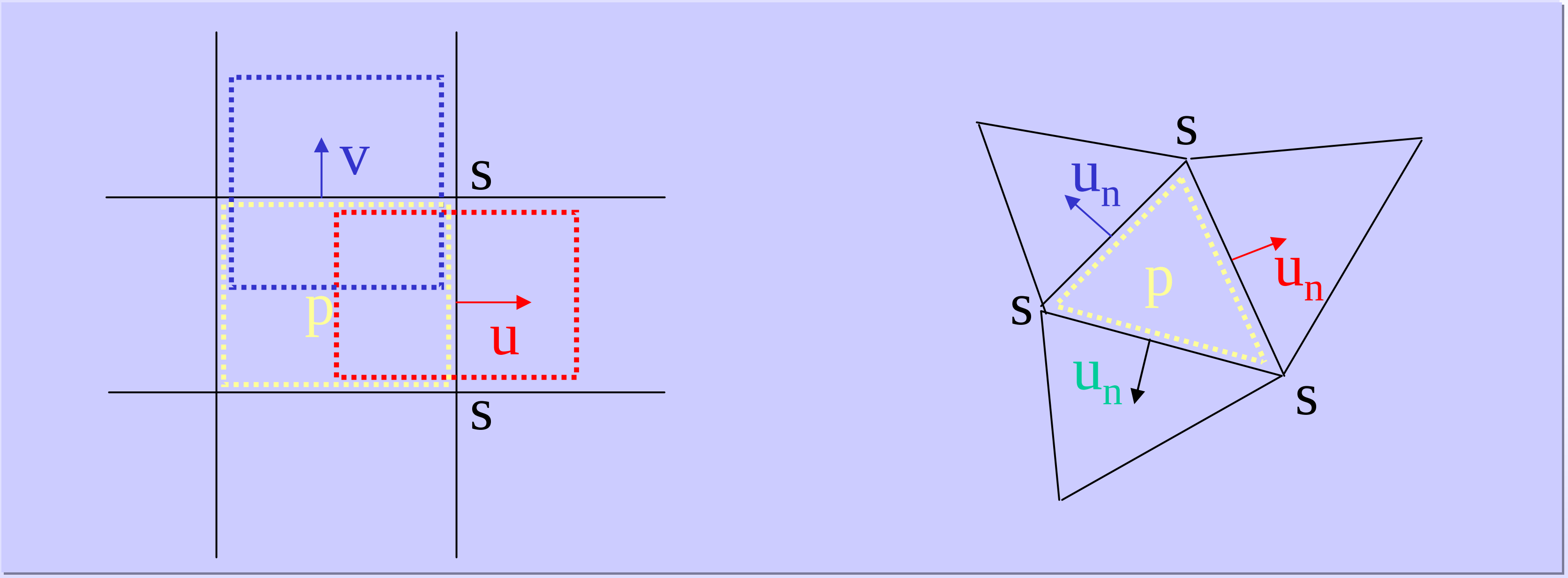
This approach *does not* require:

boundary conditions for the discrete streamfunction

significantly more unknowns for 3D problems

Unstructured Staggered Mesh

- Primary variable: velocity normal to cell faces

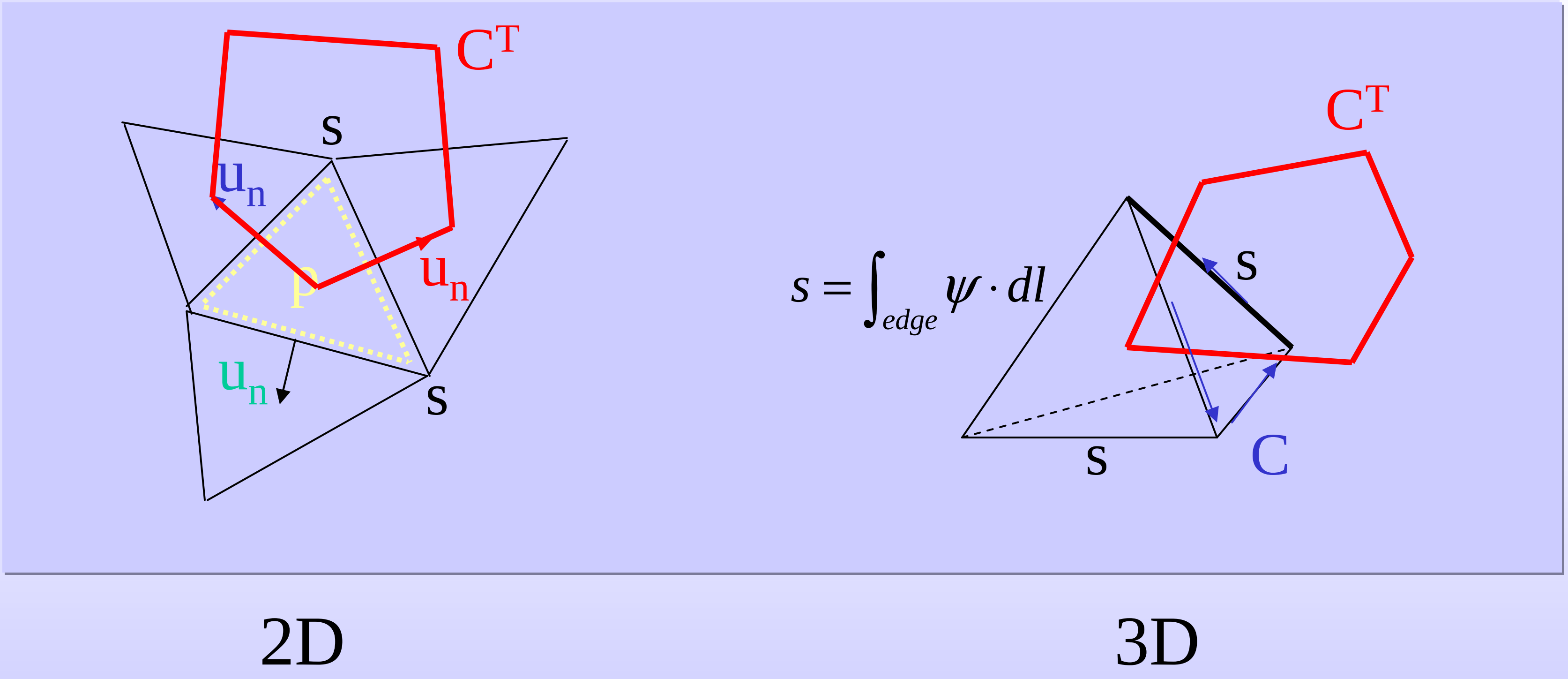


Cartesian

Unstructured

Construction of the Curl Operator

- C circuit around a face
- C^T circuit around an edge



Elliptic Solver Cost Assessment

- Roughly the same number of unknowns in both elliptic eqns.
- Proposed elliptic equation is more complex (2x cost/ iter)
- Proposed elliptic equation can have a much lower convergence tolerance, and the constraint is always satisfied

Classic projection methods require convergence of Poisson equation to nearly machine precision. Divergence condition is then somewhat satisfied. Not close to machine precision due to large condition numbers.

Iterative method convergence error is all dilatational

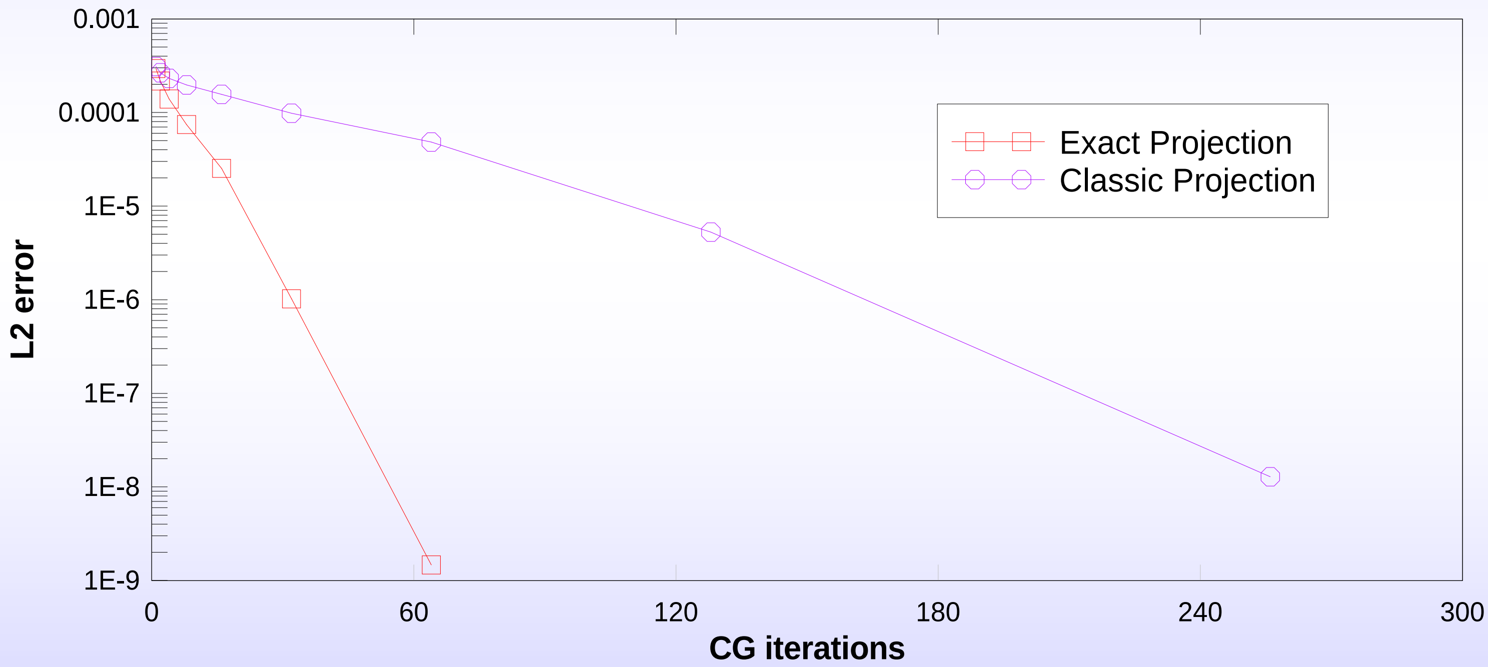
Proposed methods always satisfies continuity constraint.

Iterative method convergence error is all rotational

Iterative Error

2D Cavity Flow, Re = 100, one time step of Jacobi CG.

Error in velocity is with discretization error and splitting error removed.

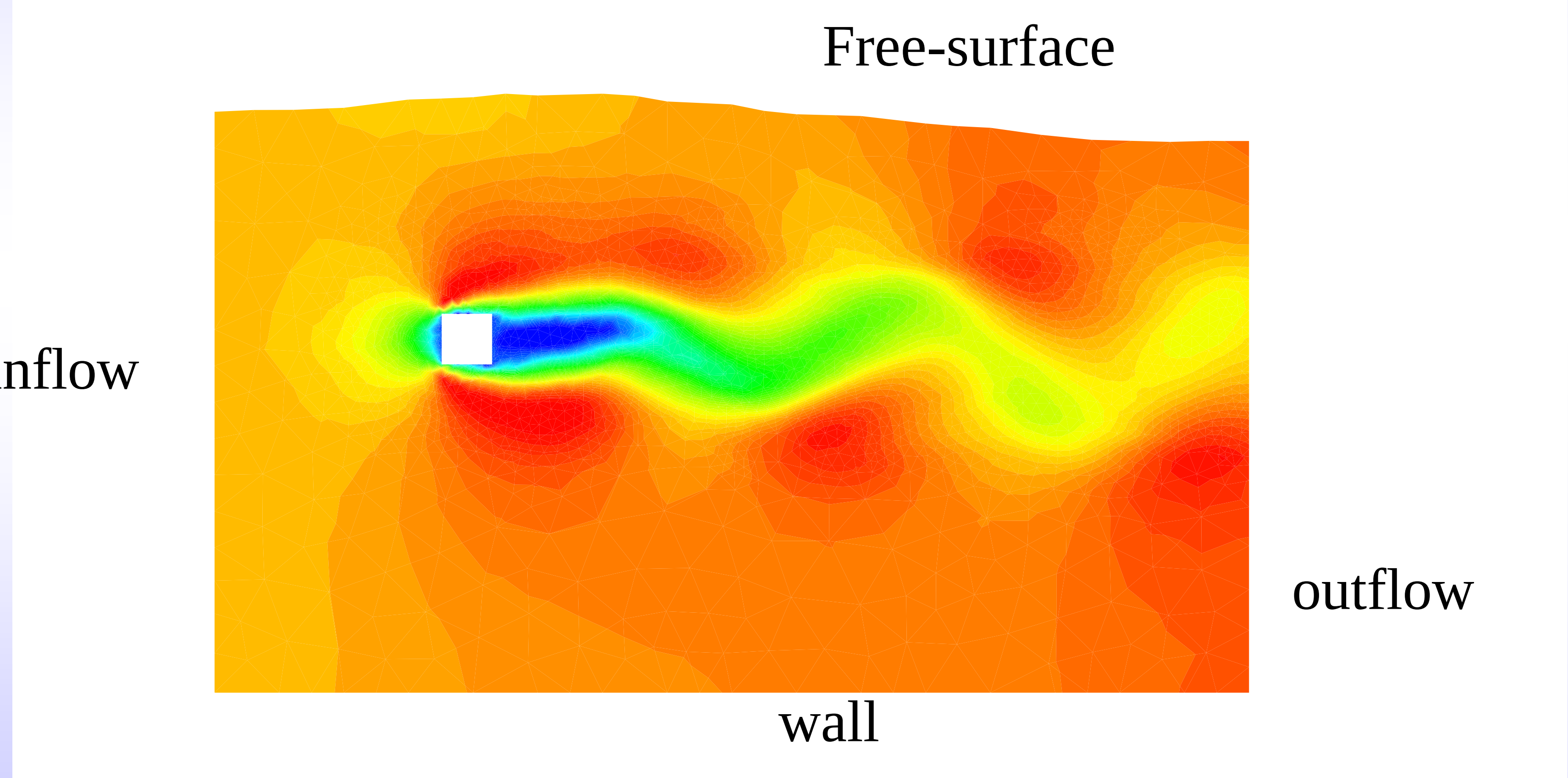


Dilatational error (mass creation) is more detrimental than rotational error (vorticity creation). (4x iterations)

Implementation Difficulty: 2D

Flow around a square cylinder with a free-surface.

Re = 1000, Fr = 0.75



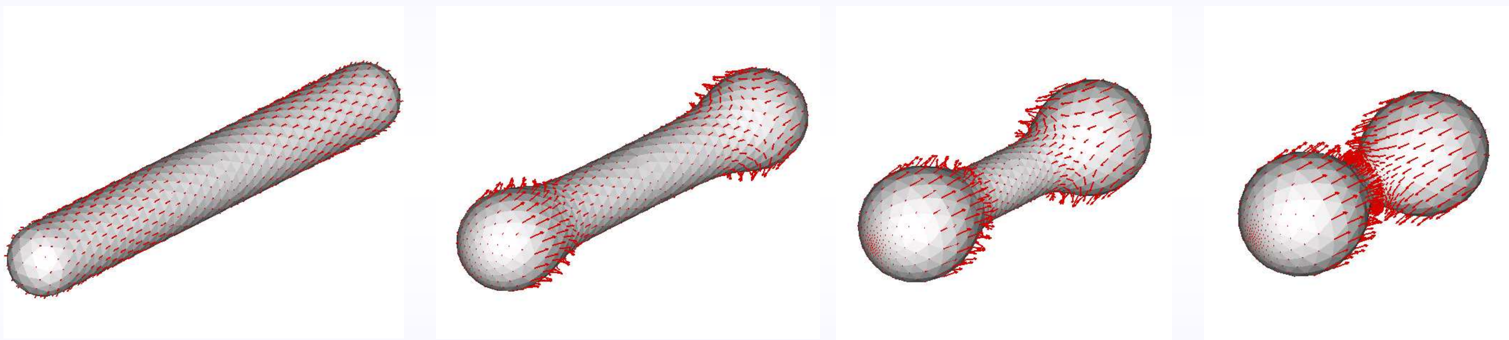
s^{n+1} varies temporally on the cylinder and along the free surface.

Implementation Difficulty: 3D

Collapse of a fluid ligament with surface tension.

Re = 3, We = 3

Video Clip



Exact mass conservation critical for correct evolution.

Summary

- Exact projection has no projection (splitting) error.
- It is effectively 2x faster than classic projection methods (when iterative solution methods are used).
- Boundary conditions on intermediate variables are not required.
- Pressure is not required. (except possibly as a B.C.)
- Not complex to implement. (but does require a sparse C.)

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