Increasing Risk and Increasing Informativeness: Equivalence Theorems

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When considering problems of sequential decision making under uncertainty, two of the most interesting questions are: How does the value of the optimal decision variable change with an increase in risk? How does the value of the optimal decision variable change with a more informative signal? In this paper we show that, if the payoff function is linear in the random variable, then one model can simultaneously answer both questions. This result holds for the reaction functions and equilibria of non-cooperative games, as well as for single decision makers, with virtually no restrictions on the payoff functions. This is useful because otherwise it is very difficult to get at general results on the impact of learning. Furthermore, we clarify why the impacts of risk and a more informative signal are different when the payoff function is non-linear in the random variable. It is because the directional impacts of informativeness are independent of risk attitude; the impacts of risk are not.

Key Words: Decision Analysis, Theory: Comparative statics of risk and learning; Games, Stochastic: Impact of learning on equilibrium; Natural Resources, Energy: Climate Change Policy.

1. INTRODUCTION

Many choices are made under uncertainty – about the weather, costs, or a competitor's behavior. Very often choices can be revised or updated after learning takes place – a farmer can harvest his crop early if the weather calls for it; a firm can scale back production if costs soar. This process, often called sequential decision making under uncertainty, has very different implications than decision making under certainty or simple uncertainty. Two of the interesting questions when analyzing problems of sequential decision making under uncertainty come under the heading of what economists call comparative statics or what operation researchers call sensitivity analysis or perturbation analysis. In general, we are interested in predicting the response of some optimizer to a change in the economic environment. In this paper we analyze the response of decision makers to a change in an uncertain environment. Namely, what is the impact on the optimal decision variable of increasing uncertainty? Similarly, what is the impact of expecting to learn more, in the sense of receiving better information before later decisions must be made? There are examples in the literature where the comparative statics of increasing learning and increasing uncertainty are the same and examples where they are different.

In this paper I show that the comparative statics of risk and of learning are qualitatively the same when the payoff function is linear in the random variable. In fact, more generally, I show that the structure of decision problems with increasing risk is identical to the structure of decision problems with increasing informativeness. This allows us to apply results from the literature on choice under risk to questions of informativeness. Furthermore, I show that the assumption of linearity is not as restrictive as it appears. The comparative statics of learning are invariant to the curvature of the payoff function around the random variable. Said another way, risk attitude is not important when considering the effects of learning. Thus, the results in this paper apply directly to any payoff function that is separable in the random variable.

Thus, one simple model will allow us to assess the impact on decision variables (and on the value of the payoff) of both increasing risk and increasing informativeness. Often, understanding the effects of learning is more policy relevant than understanding the effects of increasing risk. For example, if an international body decides to significantly increase research on climate change, hence increasing the amount nations expect to learn, should nations increase or decrease current emissions? Analyzing the value of information is also important. For example, manufacturers and retailers are assessing the value of Electronic Data Interchange for improving supply chain management and reducing costs (See Lee, So, & Tang 2000, Gavirneni, Kapuscinski, & Tayur 1999). On the other hand, it is easier to model and determine the effects of an increase in risk than the effects of an increase in informativeness. Thus, the results in this paper allow us to get at the comparative statics of learning using simple, well-understood methods.

We stress that the effect of *increasing* risk (in the Rothschild-Stiglitz 1970 sense) is the same as the effect of *increasing* informativeness (in the Blackwell 1951 sense). This may appear surprising. We are accustomed to considering risk a bad thing and information a good thing. When, however, sequential decisions are made under uncertainty increased risk often increases utility, even for risk averters. For example, think of a stock option with strike price 1. All rational decision makers will prefer an option on a stock with a 50-50 chance of payoff 0 or 2 to an option on a stock with a 50-50 chance of payoff 1/2 or 3/2, even though the first stock is riskier. The second period decision − whether to exercise the option or not − induces a preference for risk.

I present two different, but related decision problems. Both are 2-period problems with uncertainty in the state of nature that affects the payoff. Decisions are made in both periods. The first problem involves partial learning. After the first period, but before the second, a signal is received, giving the decision maker information about the value of the random variable. We would like to know how the decision variables and the value of the payoff change with a more informative signal.

The second, related problem is identical to that above, except there is perfect learning: the value of the random variable is known before the second period decision is made. We can analyze how the decision variables and the value of the payoff change with increased risk. The objective of this paper is to answer the question, when are the effects of increasing informativeness the same as the effects of increasing risk? Epstein (1980) has shown by example that the directional effect on the first period decision variable is not always the same. We show, however, that in both the single decision maker case and the multiple decision makers case the effects are identical when the payoff function is linear in the random variable. In fact, we can apply these results to determine the comparative statics of informativeness whenever the payoff function is separable in the random variable.

In addition to showing when increasing risk and increasing informativeness have equivalent results, this paper clarifies why they have different results in many cases. It is because the value and effect of information is independent of the curvature of the payoff function around the random variable. In other words, it is perfectly general to model risk neutral payoffs when considering the qualitative effects of learning.

I illustrate the theorems by presenting three applications. The first application, to climate change, was the original inspiration for this work. I present a simple model of climate change as a non-cooperative game. Players balance the private benefits of energy use and creation (which results in harmful emissions being released into the atmosphere) against the uncertain damages caused by the stock of emissions. Using my theorems I am able to draw insights about the role of learning in climate change policy. The second application builds on Epstein's presentation of a three-period consumption-savings problem. Applying my methods allows a new interpretation of the comparative statics of risk and learning. In a multi-period problem the concavity of the utility function represents both risk aversion and elasticity of substitution between periods. Both of these roles are important when analyzing the effect of risk on consumption; only the elasticity of substitution is important in analyzing the effects of information. For the third application, I look at investment under uncertainty. I consider the impacts of uncertainty and learning on investment when there are strategic interactions.

Previous work on the comparative statics of learning has employed methods that are either more difficult or less general than those in this paper. Epstein (1980) provides a method for comparing first period decisions under different amounts of informativeness. The method requires determining whether a function is convex in a vector (or a cumulative distribution function in the continuous case). The results in this paper allow us to collapse the vector or function into a one-dimensional variable, greatly easing calculations. Epstein's method has not been generalized to non-cooperative games, perhaps because of the difficulty of working in vector or function spaces. Some papers have analyzed the effects of information in games using less general methods. Vives (1989) uses a mean-variance framework to determine how information affects the value of both flexibility and strategic commitment. Gal-Or (1985) uses a similar framework to conclude that firms will not share information on uncertain demand. Sakai (1985) and Ulph and Ulph (1996) calculate the value of perfect information in non-cooperative games. The equivalence theorems in this paper allow us to use general information structures and calculate comparative statics for *incremental* increases in informativeness.

More recently, Athey & Levin (2000) analyze information preferences and information demand for different classes of decision makers. A signal is more informative in the Blackwell sense if *all* decision makers prefer it. By contrast, Athey and Levin provide conditions under which all decision makers in a certain class (such as those with payoff functions that are supermodular or have concave incremental returns, for example) will prefer an information structure. They use these results to determine the demand for information in a variety of monotone decision problems. The class of decision makers that I consider (those with payoff functions that are separable in the random variable) is disjoint from the classes that they consider, and thus both results are useful in different situations. Using methods similar to Athey & Levin, Persico (2000) analyzes the class of payoffs with the single-crossing property and concludes that the demand for more accurate information is higher for participants in first-price auctions than in second-price auctions. These class-based definitions have not been extended, however, for comparative statics analysis on first period decisions.

The rest of the paper is organized as follows. In the next section I present the basic model and definitions of risk and informativeness. Section 3 contains the results for the case of a single decision maker, including the insight that risk attitude is irrelevant for the comparative statics of information. Section 4 extends the results to a 2-person non-cooperative game, with two possibly correlated random variables. Section 5 illustrates how the main ideas of the paper can be applied to the value of information. Section 6 summarizes the results and provides some new directions for research.

2. MODEL AND DEFINITIONS

2.1. Sequential Decision Making Under Uncertainty

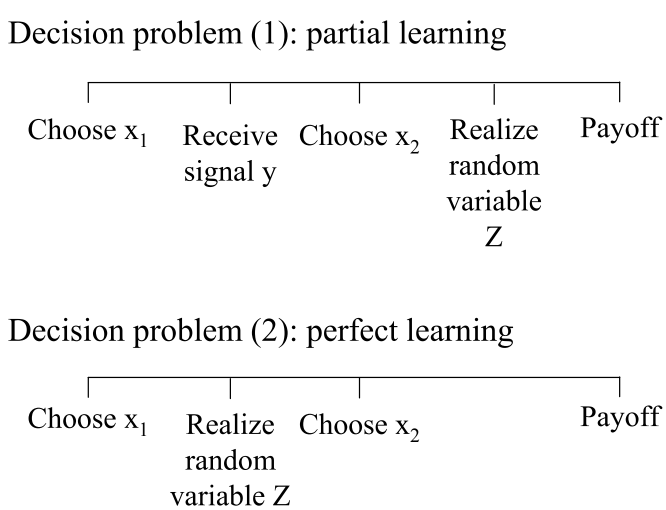

FIG. 1 Two related decision problems.

The impact of risk on the optimal decision is better understood than the impact of informativeness. I aim to use our understanding of the impact of risk in order to gain insights about the impact of informativeness or learning. I do this by showing

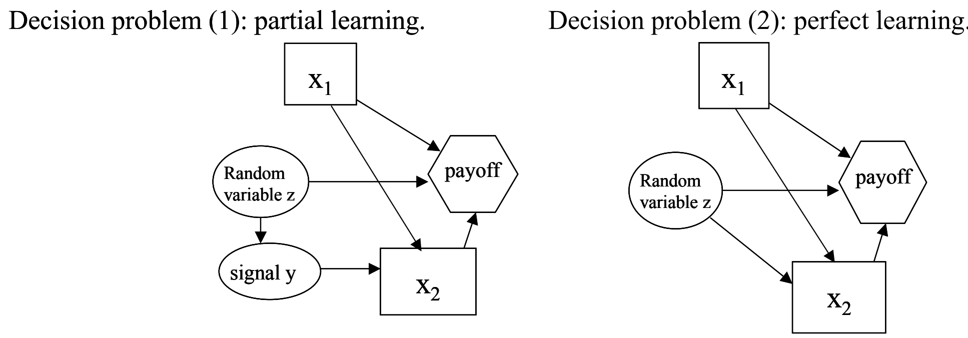

FIG. 2 Decision Diagram of two related decision problems. Square boxes represent decisions, ovals represent uncertainties, hexagons are payoffs, and arrows represent influences and information.

a parallel between two similar, but distinct decision problems. I show that the qualitative comparative statics of informativeness in one problem are the same as the qualitative comparative statics of risk in the other. Figure 1 illustrates the time paths of the two problems while figure 2 illustrates the decision diagrams. In this section I introduce the decision problems in abstract form. The first decision problem is more general, representing a 2-period decision problem with learning, as follows

$$\max_{x_1} \mathrm{E}_Y \max_{x_2} \mathrm{E}_{Z|Y} U(x_1, x_2, Z) \tag{1}$$

x_1, $x_2 \in \mathbb{R}$ are the first and second period decision variables, U is a payoff function (perhaps reflecting a Von Neumann-Morgenstern utility function, for example) and Z is a random variable that affects the payoff. E is the expectation operator. We write E_Z to mean the expected value over Z. Y is a random variable that is (possibly) relevant to Z, and thus provides information about the value of Z. Y is observed before x_2 is chosen. I will refer to (1) as the *learning problem.* We may ask, what is the effect on the optimal first period decision if we expect to receive a more informative or more accurate signal Y? For example, should the optimal carbon tax increase or decrease if we expect to learn more about climate change in the near future?

Now consider a special case of (1) when there is perfect learning before the second period: Z is known before x_2 is chosen. This related problem is:

$$\max_{x_1} \mathrm{E}_Z \max_{x_2} U(x_1, x_2, Z) \tag{2}$$

I will to refer to (2) as the *risk problem*. Regarding the *risk problem*, I ask what is the effect on the optimal first period decision if I vary the amount of uncertainty around the random variable Z? This question has been asked often in the economics literature. One example of an often recurring question is, do we expect savings to increase or decrease as the interest rate becomes more risky? What I will show in the next section is that when the payoff function is linear in the random variable Z, then the optimal first period decision in problem (1) is increasing in informativeness if and only if the optimal first period decision in (2) is increasing in risk.

The broad intuition of this result is as follows: if in problem (1) we expect to have more information before we choose x_2 then we will want to choose x_1 in such a way to leave ourselves more flexibility to react to what is learned. Similarly, the more prior risk we face in problem (2), the more flexibility we would like when choosing x_2. Hence, we might expect an increase in informativeness and an increase in uncertainty to have similar effects on x_1. This is only true, however, when the utility is linear in the random variable. In the next section we discuss why it does not hold true in general.

2.2. Informativeness and Risk

A "signal" in our framework is a way of gathering information. One signal is more informative than another if, *a priori*, it is a better way of gathering information. Say we are interested in tomorrow's weather. We can gather information from a number of different sources – we may look out the window, read the barometer, or check the weather report. Each of these alternatives is associated with a distinct information structure.

In order to make the idea of informativeness precise, we need define what is meant by "information structure". Let Y and Z be random variables with Joint Cumulative Distribution Function $F_{YZ}(y, z) = \Pr[Y \leq y, Z \leq z]$. Each Y is associated with a particular joint CDF F which defines the *information structure.* The information structure tells us everything we need to know about how much Y tells us about Z. The posterior probability distribution of Z conditional on Y is $F_{Z|Y}(z|y) = \Pr[Z \leq z | Y = y]$. For the rest of the paper I will simply refer to Y as an information structure.

In order to compare two information structures they must be *comparable* in the following sense: Y and Y' are *comparable* if they have the same priors for Z, that is if $F_Z = F'_Z$, where $F_Z = \mathrm{E}_Y\left[F_{Z|Y}(z|y)\right]$ is the marginal distribution of Z. The reason for this restriction is that our prior probability distribution should not be affected by which information structure we choose. The probability of a hot and humid day tomorrow is not affected by the decision to look out the window or read the weather report. Hence, we only compare information structures with the same priors.

Y is more informative than Y' if *every* decision maker is (weakly) better off expecting to receive a message from information structure Y rather than from information structure Y'. For example, if weather is the random variable, then observing the temperature and humidity would be more informative than observing temperature alone. Every decision maker will be better off (or at least no worse off) observing both variables. More informative still would be a weather report in which an expert combined a number of different signals.

For the rest of the paper I will use the notation for continuous distributions, but all results hold for discrete distributions. If the proof for finite distributions is not a trivial extension I include it or reference it.

Definition 1. Let Y and Y' be comparable information structures. Then we say that Y **is more informative than** Y' if

$$\mathrm{E}_Y \max_{x_2} \mathrm{E}_{Z|Y} U(x_1, x_2, Z) \geq \mathrm{E}_{Y'} \max_{x_2} \mathrm{E}_{Z|Y'} U(x_1, x_2, Z)$$

for all x_1 and U for which the maximum is defined.

Now, recall the Rothschild-Stiglitz (Rothschild and Stiglitz, 1970) definition of increasing risk:

Definition 2. We say that Z is **riskier** (or **more uncertain** or **more variable**) than Z' iff $\mathrm{E}_Z U(Z) \leq \mathrm{E}_{Z'} U(Z')$ for all concave U, which is true iff $\mathrm{E}_Z U(Z) \geq \mathrm{E}_{Z'} U(Z')$ for all convex U.

This says that Z is riskier than Z' if all risk averters prefer Z to Z'. Since $U(Z) = Z$ is both concave and convex, Definition 2 implies that Z and Z' can only be compared if they have the same mean, hence the term "mean-preserving spread." Definition 2 is similar – but not identical – to the definition of second order stochastic dominance (SOSD). Z' SOSD Z iff $EU(Z') \geq EU(Z)$ for all *increasing*, concave functions U.

We are interested in the comparative statics of risk and information. Thus, we need to be precise about what it means to "increase in risk" or "increase in informativeness." Throughout the paper we will use the term "increasing" to mean non-decreasing, and will say "strictly increasing" when that is what we mean. Let X be a decision parameterized by a random variable. Then we say X is **increasing in risk** if for any two random variables Z and Z', Z riskier than Z' implies that $X(Z) \geq X(Z')$. Similarly, if X is a decision parameterized by an information structure, then we say that X is **increasing in informativeness** if for any two comparable information structures Y and Y', Y more informative than Y' implies that $X(Y) \geq X(Y')$.

The equivalence theorems in this paper are useful because it is easier to determine the comparative statics of risk than the comparative statics of informativeness. Below we state a theorem with necessary and sufficient conditions for the first period decision variable to increase in risk. This theorem appeared in Rothschild & Stiglitz (1971) for payoff functions assumed to be differentiable in the decision variable. Theorem 1 below generalizes the result to any payoff function for which a maximum exists. Alternatively, the theorem follows from Corollary 1 in Athey (2000) .

Define

$$V(x_1, Z) \equiv \max_{x_2} U(x_1, x_2, Z) \tag{3}$$

$$x_1^{**} = \arg\max_{x_1} \mathrm{E}_Z \max_{x_2} U(x_1, x_2, Z) = \arg\max_{x_1} \mathrm{E}_Z V(x_1, Z)$$

$$x_1^{**\prime} = \arg\max_{x_1} \mathrm{E}_{Z'} \max_{x_2} U(x_1, x_2, Z') = \arg\max_{x_1} \mathrm{E}_{Z'} V(x_1, Z')$$

Theorem 1. *Assume that Z is riskier than Z'. If $V(x_H, Z) - V(x_L, Z)$ is convex (concave) in Z for all $x_H > x_L$ then $x_1^{**} \geq (\leq) x_1^{**\prime}$. If $V(x_H, Z) - V(x_L, Z)$ is neither convex nor concave, then the sign of $x_1^{**} - x_1^{**\prime}$ is ambiguous in the following sense: there exist random variables Z, Z', and Z'' such that Z' and Z'' are each less risky than Z and such that $x_1^{**\prime} \leq x_1^{**} \leq x_1^{**\prime\prime}$.*

See Appendix A for proof.

Below is the key theorem that connects informativeness and risk. It says that the possible posterior distributions, $F_{Z|Y}$, induced by the more informative information structure will be more variable, or *riskier* in a generalized Rothschild-Stiglitz sense than the posteriors induced by the less informative information structure. To understand this, think of the extreme case in which Y' is independent from Z and therefore provides no information. In this case the posterior $F'_{Z|Y}$ will always be equal to the prior for Z regardless of the realization of Y': there will be no variability in the posteriors. Define the set $\mathcal{F} = \{\theta : \mathbb{R} \to [0,1]\}$.

Theorem 2. *Y is more informative than Y' iff*

$$\int_Y \rho\left(F_{Z|Y}\right) dF_Y \geq \int_{Y'} \rho\left(F'_{Z|Y'}\right) dF'_{Y'} \quad \forall \text{ convex } \rho : \mathcal{F} \to \mathbb{R}$$

See appendix B for proof[1]. Notice that the inequality in Theorem 2 is parallel to the inequality in definition 2. The difference is that the convex functions ρ are defined on a function space and not on the real line. Thus it can be considered a generalization of the Rothschild-Stiglitz definition of risk.

2.2.1. Informativeness Lemma

The intuition given above regarding the variability of the posteriors also explains why an increase in informativeness will increase the variability or risk of $\mathrm{E}[Z|Y]$. Think of the two extreme cases. If Y is independent from Z, and therefore provides no information, then $\mathrm{E}[Z|Y] = \mathrm{E}[Z]$ for all possible realizations of Y : the random variable $\mathrm{E}[Z|Y]$ will have no variability. On the other hand, if Y provides perfect information about Z then $\mathrm{E}[Z|Y] = Z$, i.e. the variability of $\mathrm{E}[Z|Y]$ will be exactly equal to the variability of Z. In fact, Lemma 1 is more general than this, showing that the conditional expectation is more variable for any function of Z.

LEMMA 1. *If Y is more informative than Y' then $\mathrm{E}[g(Z)|Y]$ is riskier than $\mathrm{E}[g(Z)|Y']$ where g is any arbitrary function $g : \mathbb{R} \to \mathbb{R}$*

Proof. For any convex $\sigma : \mathbb{R} \to \mathbb{R}$ we can define $\rho_\sigma : \mathcal{F} \to \mathbb{R}$ such that

$$\rho_\sigma \left(F_{Z|Y}\right) \equiv \sigma \left(\int_Z g(z)\, dF_{Z|Y} \right) = \sigma \left(\mathrm{E}[g(Z)|y]\right)$$

Since the expectation is linear in $F_{Z|Y}$ this implies that $\rho_\sigma\left(F_{Z|Y}\right)$ is convex in $F_{Z|Y}$. Therefore Theorem 2 tells us that

$$\int_Y \sigma\left(\mathrm{E}[g(Z)|y]\right) dF_Y \geq \int_{Y'} \sigma\left(\mathrm{E}[g(Z)|y']\right) dF'_Y \quad \forall \text{ convex } \sigma : \mathbb{R} \to \mathbb{R}$$

which implies that $\mathrm{E}[g(Z)|Y]$ is riskier than $\mathrm{E}[g(Z)|Y']$. ∎

Now we have everything we need to proceed to the equivalence theorems.

3. SINGLE DECISION MAKER

There is an isomorphism between traditional models of optimal decision making under risk and models of optimal decision making under risk with imperfect learning. Up until now, this parallel has not been exploited. There is a great potential to apply many of the results from the literature on risk aversion and optimal decision making under risk to get insights about learning (See for example Pratt 1964, Meyer & Ormiston 1985, Jewitt 1987, Kimball 1990,,,). This paper concentrates on the special case of a linear payoff function, but I will start by pointing out the more general parallel.

3.1. An Isomorphism

Note that we can define a new function

$$V\left(x_1, x_2, F_{Z|Y}\right) \equiv \mathrm{E}_{Z|Y}\, U\left(x_1, x_2, Z\right)$$

Where $V : \mathbb{R} \times \mathbb{R} \times \mathcal{F} \to \mathbb{R}$ is defined on a function space rather than the real line for it's third argument. Now we rewrite the *learning problem* (1) in terms of V and repeat the *risk problem* (2) for comparison.

$$\max_{x_1} \mathrm{E}_Y \max_{x_2} V\left(x_1, x_2, F_{Z|Y}\right) \tag{4}$$

$$\max_{x_1} \mathrm{E}_Z \max_{x_2} U\left(x_1, x_2, Z\right) \tag{5}$$

The two problems are structurally identical, with the (random) posterior probability distribution $F_{Z|Y}$ in (4) playing the role of the random variable Z in (5). As we saw from Theorem 2, the posterior probability distribution $F_{Z|Y}$ gets riskier in a generalized Rothschild-Stiglitz sense as Y gets more informative. Hence the parallel. Once we notice this parallel, then Epstein's Theorem 1 follows directly from the Rothschild-Stiglitz Theorem, presented as Theorem 1 in this paper. The difficultly lies in generalizing results on the real line into results on a vector or function space.

3.2. Problem Definition: Linear Payoff Function

If U is linear in the random variable Z then it is true that $\mathrm{E}\left[U\left(Z\right)\right] = U\left(\mathrm{E}\left[Z\right]\right).$ Therefore the *learning problem* (1) can be rewritten as

$$\max_{x_1} \mathrm{E}_Y \max_{x_2} U\left(x_1, x_2, \mathrm{E}_{Z|Y}\, Z\right) \tag{6}$$

Recall the risk problem (2):

$$\max_{x_1} \mathrm{E}_Z \max_{x_2} U(x_1, x_2, Z) \tag{7}$$

It is apparent that the random variable $\mathrm{E}_{Z|Y} Z$ plays the same role in (6) as Z plays in (7). As I've shown in Lemma 1 the more informative Y is, the more variability in the possible values of $\mathrm{E}_{Z|Y} Z$. Thus, an increase in risk around Z will have an identical effect on problem (7) as an increase in the informativeness of Y will have on problem (6).

3.3. Equivalence Theorems

3.3.1. Linear Payoff Functions

THEOREM 3. *Let x_1^* solve (1) and x_1^{**} solve (2). Assume that U is linear in Z. Then x_1^* is increasing (decreasing) in informativeness if and only if x_1^{**} is increasing (decreasing) in uncertainty. The effect of increasing informativeness on x_1^* is ambiguous if and only if the effect of increasing risk on x_1^{**} is ambiguous.*

Proof. We will write

$$x_1^*(Y) = \arg\max_{x_1} \mathrm{E}_Y \max_{x_2} U(x_1, x_2, \mathrm{E}[Z|Y])$$

and

$$x_1^{**}(Z) = \arg\max_{x_1} \mathrm{E}_Z \max_{x_2} U(x_1, x_2, Z)$$

where x_1^* is a function of the information structure rather than a particular realization of Y and x_1^{**} is a function of the distribution of Z. Assume x_1^{**} is increasing in uncertainty. This means that

- Z riskier than $Z' \Rightarrow x_1^{**}(Z) \geq x_1^{**}(Z')$.
- But since $\mathrm{E}[Z|Y]$ plays the same role in (6) as Z plays in (7), the above is equivalent to saying that $\mathrm{E}[Z|Y]$ riskier than $\mathrm{E}[Z|Y'] \Rightarrow x_1^*(Y) \geq x_1^*(Y')$.

- Lemma 1 tells us that if Y is more informative than Y' then $\mathrm{E}[Z|Y]$ is riskier than $\mathrm{E}[Z|Y']$.
- Therefore if Y is more informative than Y' then $x_1^*(Y) \geq x_1^*(Y') : x_1^*$ is increasing in informativeness.

The proof for the decreasing case uses the same logic with the opposite inequalities. See Appendix C for the proof of the converse and for the last statement in the theorem. ∎

We stress that the assumptions required for Theorem 3 are very weak. There is no requirement that the payoff function U be differentiable or even continuous. Furthermore, the maximizers x_1^* and x_1^{**} may represent sets. With any reasonable definition of "x_1^* increasing", the logic of the proof still holds. For example, see Topkis (1998) p. 32 for the definition of **induced set ordering.**

Theorem 3 tells us that we need only solve one problem to get two results. If we analyze problem (7) and find that the first period decision variable is increasing in risk, we can immediately say that the first period decision variable in problem (6) is increasing in the informativeness of Y.

Application – Climate Change I Climate change epitomizes a sequential decision problem under uncertainty. While it has come to be generally accepted that the stock of carbon emissions due to human activities is having an effect on the climate, there are huge uncertainties about how emissions today will come back to haunt us in the future. Yet, we expect to learn more about the relationship between emissions and damages as time goes on.

I model climate change as a two-period problem (see the top diagram in figure 3). The first period represents now, when there is great uncertainty. The second period is some time in the future, when the world is assumed to have learned the relationship between emissions and future damages. The decision maker chooses current emissions under uncertainty, then learns all there is to know about the relationship between emissions and climate damages, then chooses 2nd period emissions based on what was learned. We can ask, what is the impact on optimal current emissions of an increase in risk?

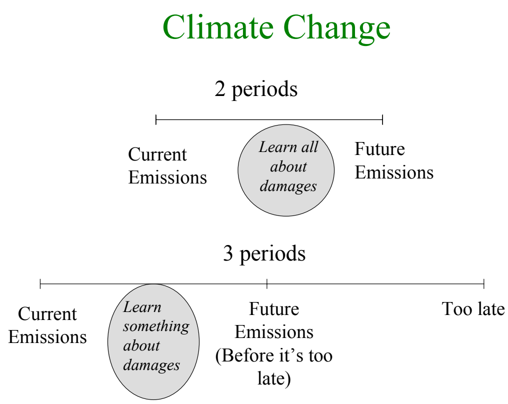

FIG. 3 Two ways of thinking about climate change.

But there is another model we are interested in, with an implied 3rd period (see the lower diagram in figure 3). In this model the decision maker chooses emissions under uncertainty, then gets a signal about the relationship between emissions and damages, then makes future decisions based on the signal. It is possible that the decision maker will learn more after the 2nd period, but it will be too late to have a significant impact on climate damages. This model can give us insights into the real-world situation, where policy makers continue to learn and make decisions through time, each subsequent decision perhaps having less impact than the one before. In this model we can ask, what is the impact on current emissions of learning more before it is too late, of moving knowledge from the later period to the earlier period?

Let the payoff function be

$$b(x_1) + \delta \left[b\left(x_2\right) - ZD(s) \right] \tag{8}$$

Where x_1, x_2 are 1st and 2nd period emissions, s is the total stock of emissions, δ is the discount factor, $b\left(\cdot\right)$ measures the benefits from emissions, $D\left(\cdot\right)$ measures the damages caused by the stock of emissions in the atmosphere, and Z is a random variable. In the absence of climate change, carbon emissions would have no economic value one way or the other. But, carbon emissions are a side-effect of energy use and production, which are associated with costs and benefits. Thus, we can model the net benefits of energy use as a function of carbon emissions. We assume that we know the shape of the damages from climate change (i.e. $D(s)$ is a deterministic function of the stock of emissions), but we are uncertain about the magnitude. Thus we multiply the deterministic damage function with a stochastic shift parameter, Z.

Using this model, Baker (2002a) shows that, if emissions are not constrained to be non-negative, then optimal emissions increase with risk and, therefore, with informativeness.

3.3.2. Non-linear Payoff Functions

What if the payoff is *not* linear in the random variable? First notice that if Y is more informative than Y' for the random variable Z, then Y is more informative than Y' for all functions $g(Z)$. This follows directly from Definition 1, since the set of all $U(\cdot,\cdot,g(Z))$ is a subset of all $U(\cdot,\cdot,Z)$. From this fact we get the following corollary to Theorem 3.

COROLLARY 1. *Let x_1^* solve (1) and x_1^{**} solve (2). Assume that U is separable in some function $g(Z)$ in the following sense: $U(x_1,x_2,Z) = U_1(x_1,x_2) + g(Z)\,U_2(x_1,x_2)$. Then x_1^* is increasing (decreasing) in informativeness if x_1^{**} is increasing (decreasing) in uncertainty around $g(Z)$.*

Corollary 1 tell us that the functional form of $g(\cdot)$ is not important when considering the effects of more information. It is, however, important when considering the effects of increasing risk. The reason is that if Y is increasing in informativeness for Z then it is equally increasing in informativeness for $g(Z)$. The analogous statement is not true for increasing risk. The effect of increasing risk on $g(Z)$ is ambiguous: it depends on the functional form of $g(\cdot)$. $g(Z)$ may actually decrease in risk while Z is increasing in risk. More importantly, an increase in risk around Z will change the mean of $g(Z)$.

We claim that Corollary 1 implies that risk attitude is not important when looking at the qualitative effects of information. If U is separable in $g(Z)$ then the risk attitude is embodied in the curvature of $g(\cdot)$. The curvature in the decision variables x_1 and x_2 do not effect risk attitude, since they are not uncertain. The qualitative effects of changes in information (on both the expected value of the payoff and on x_1) are independent of the functional form of $g(Z)$. Therefore, in the separable case, risk attitude is irrelevant. If U is not separable then it is impossible to separate out the effects of risk attitude from the effects of parameters on x_1 and x_2 or the effect of the interrelationships between all three variables. The following example further clarifies this idea.

Application: A Consumption-Savings Problem An individual has a given wealth w_1 which she wishes to allocate between consumption in three periods. What she does not consume in any period she invests in a single asset. Investment in period 1 yields a sure gross return r and investment in period 2 yields a random gross return Z. The consumer maximizes expected utility, with a payoff function:

$$u(w_1 - s_1) + \delta u(rs_1 - s_2) + \delta^2 u(s_2 Z) \tag{9}$$

where w_1 is initial wealth, s_1 and s_2 are the savings in periods 1 and 2, r and Z are the first and (uncertain) second period savings rates, and δ is a discount factor. Let

$$u(w) = \frac{w^{1-\alpha}}{1-\alpha} \quad \alpha \neq 1$$

so that utility has constant relative risk aversion coefficient α. In this case the payoff function (9) is separable in a function of Z, namely $g(Z) = Z^{1-\alpha}$ if $\alpha \neq 1$; $g(Z) = \log Z$ if $\alpha = 1$. Corollary 1 says that the functional form of $g(\cdot)$ is irrelevant to determine the effects of more information, so we can replace $g(Z)$ with a random variable t and analyze the effects of increasing risk on

$$\max_{s_1} u(w_1 - s_1) + \mathrm{E}_Z \max_{s_2} \delta u(rs_1 - s_2) + \delta^2 u(s_2)\, t$$

Clearly this payoff is linear in t and therefore risk neutral. Yet, as we analyze the problem below we show that the coefficient α is still important to the problem.

Optimal savings s_1 is increasing in informativeness if and only if s_1 is increasing in the risk of t. Let

$$V(s_1, t) = \max_{s_2} \delta u(rs_1 - s_2) + \delta^2 u(s_2) t$$

then

$$s_2^* = \arg\max_{s_2} \delta u(rs_1 - s_2) + \delta^2 u(s_2) t = \frac{rs_1}{1 + t^{-\frac{1}{\alpha}}}$$

and

$$V_{s_1}(s_1, t) = r\delta u'(rs_1 - s_2^*) = r\delta (rs_1)^{-\alpha} \left(1 - \frac{1}{1 + t^{-\frac{1}{\alpha}}}\right)^{-\alpha} = C\left(1 + t^{\frac{1}{\alpha}}\right)^{\alpha} \quad (10)$$

where V_{s_1} indicates the partial derivative of V by s_1. Thus V_{s_1} is convex in t if $\alpha < 1$ and concave in t if $\alpha > 1$. Savings is increasing in informativeness for large α and decreasing in informativeness for small α. Epstein points out that (9) is ordinally equivalent to a CES utility function with elasticity of substitution (between consumption in different periods) of $\frac{1}{\alpha}$. Hence, if α is low – implying that elasticity of substitution is high – then we expect more savings in period one in order to take advantage of the better information. If elasticity of substitution is low, then the income effect dominates, and we expect the consumer to consume more in the first period in expectation of more income later.

In order to check the effect of increasing risk we substitute $Z^{1-\alpha}$ for t in equation (10).

$$V_{s_1}(s_1, Z) = C\left(1 + Z^{\frac{1-\alpha}{\alpha}}\right)^{\alpha}$$

We find that the effect of increasing risk is almost the opposite of the effect of increasing informativeness: V_{s_1} is convex if $\alpha > 1$ and concave if $\frac{1}{2} < \alpha < 1$, but it is neither convex nor concave for $\alpha < \frac{1}{2}$. Savings is increasing in risk for large α, decreasing in risk for medium α, and indeterminate for small α.

Why are the results for increasing risk opposite from those for increasing informativeness? When considering the effects of informativeness, α only plays the role of elasticity-of-substitution parameter. When risk aversion is present, however, α has two roles. As α increases, the elasticity of substitution decreases *and* risk aversion increases. Because of risk aversion, the expected utility of third period consumption will get smaller as risk increases. If α is large, the elasticity of substitution between periods is low, implying that optimally, consumption will stay at similar levels in each period. Therefore, second period savings s_2 will increase (in expectation) in order to assure enough consumption in the third period. First period savings will increase in order to help offset the lower expected second period consumption. Conversely, if α is small, the elasticity of substitution between periods is high, so s_2 will reduce (in expectation) in order to avoid the low-value third period. First period savings will decrease in order to offset the higher expected second period consumption. Finally, as α goes to 0, risk aversion becomes less prominent, and the response to an increase in risk becomes indeterminate.

Comparing the results from risk and informativeness gives us new insight into how consumption changes with risk. It depends both on the elasticity of substitution between periods an on the level of risk aversion. These two things interact to determine the level of consumption.

3.4. Summary

Hart (1942) states in his early essay on sequential decision making under uncertainty that "the central problems of uncertainty can be posed and largely solved under the assumption of risk neutrality."[2] Corollary 1 indicates that this is precisely true for questions of informativeness, but not necessarily for questions of riskiness.

4. MULTIPLE DECISION MAKERS

We have shown that the comparative statics of risk and information are qualitatively the same in a single decision maker context. In this section, we show that this is true in a 2-person non-cooperative game as well. We set up a 2-person decision problem with two (possibly correlated) random variables, then discuss what it means for a signal to be more informative when there are two players and two random variables. We first show that the reaction functions are increasing in informativeness if they are increasing in risk. We then go on to apply the same logic we have used throughout the paper to the equilibria of the game, showing that equilibrium actions are increasing in informativeness if they are increasing in risk.

4.1. Problem Definition

Consider the problems analogous to (1) and (2) but in a strategic framework. Assume there are 2 players, X and W, with strategy sets (x_1, x_2) and (w_1, w_2), $x_1, x_2.w_1, w_2 \in A$, A an ordered set; and payoff functions $U^x(x_1, x_2, w_1, w_2; Z^x)$ and $U^w(x_1, x_2, w_1, w_2; Z^w)$. (In this formulation we are restricting the strategies to pure strategies. However, x_i, w_i could be interpreted as mixed strategies, with each x_i representing a probability distribution over some set of actions. Then we could define $x_i > x_i'$ if, say, $\mathrm{E}\, x_i > \mathrm{E}\, x_i'$.) Z^x and Z^w are (possibly dependent) random variables. The information structure is represented by the joint distribution of Z^x, Z^w, and $Y : F_{Z^x Z^w Y}$. The prior joint probability distribution of Z^x and Z^w is given by the marginal distribution $F_{Z^x Z^w}$. Each message $Y = y$ will yield a posterior distribution $F_{Z^x Z^w|y}$. For two information structures F and F' to be comparable they must have the same priors: $F_{Z^x Z^w} = F'_{Z^x Z^w}$. There is no private information. The solution concept is Sub-game Perfect Nash Equilibrium.

First consider the problem with learning analogous to (1). Each player observes the first period actions x_1 and w_1 and receives a public signal y before choosing 2nd period actions. The set of equilibrium points of the second period sub-game is determined by the first period actions and the signal: $(x_2^*, w_2^*) = (x_2^*(x_1, w_1, \mathrm{E}[Z^w|Y], \mathrm{E}[Z^x|Y]), w_2^*(x_1, w_1, \mathrm{E}[Z^w|Y], \mathrm{E}[Z^x|Y]))$.

Taking the second period decision rules x_2^* and w_2^* as given, the first-period, 2-person *learning game* analogous to (1) can be written as

$$\max_{x_1} \mathrm{E}_Y\, U^x(x_1, x_2^*, w_1, w_2^*; \mathrm{E}[Z^x|Y]) \tag{11}$$

Where player X takes w_1 as fixed. Player W simultaneously solves

$$\max_{w_1} \mathrm{E}_Y\, U^w\left(x_1, x_2^*, w_1, w_2^*; \mathrm{E}\left[Z^w|Y\right]\right)$$

The *risk game* analogous to (2) is similar, with the random variables Z^x and Z^w in the place of the conditional expectations. The second period decision rules are determined by the first period actions and the realization of the random variable: $(x_2^{**}, w_2^{**}) = (x_2^{**}(x_1, w_1, z^w, z^x), w_2^*(x_1, w_1, z^w, z^x))$. The first period game is as follows:

$$\max_{x_1} \mathrm{E}_{Z^x Z^w}\, U^x\left(x_1, x_2^{**}, w_1, w_2^{**}; Z^x\right) \tag{12}$$

Where player X takes w_1 as fixed. Player W simultaneously solves

$$\max_{w_1} \mathrm{E}_{Z^x Z^w}\, U^w\left(x_1, x_2^{**}, w_1, w_2^{**}; Z^w\right)$$

What we show in the next sections is that an increase in risk (around Z^x say) will has same effect on X's first period actions in problem (12) as an increase in informativeness has on X's first period actions in problem (11). But first we discuss what "more informative" means when there are two players and two random variables.

4.2. Informativeness

What does it mean for Y to be increasing in informativeness in the multiple decision maker case? First consider problems where there is only one underlying random variable, for example competitors facing an uncertain price. Then an increase in informativeness has unambiguous impacts. Next, consider problems where the two random variables are independent. Then the public signal Y can range in informativeness for each signal. Comparing 2 signals Y and Y' we may find that Y is more informative than Y' for Z^w but less informative for Z^x. This ambiguity makes it hard to come up with interesting results. So we define the notion of a signal being more informative for one random variable, say Z^x, while holding the informativeness relative to the other random variable constant. We will define Y to be more informative than Y' for ε but not for τ if Y' has the same information relevant to τ as Y does. See Baker (2002b) for formal definitions, and the restatement and proof of Lemma 1.

Below is an example of two information structures, where one is more informative than the other for one variable, but not the other.

EXAMPLE 1. Let Z and ε be independent random variables; Y and Y' are information structures relevant to Z and ε. Let

$$Z = \left\{ \begin{array}{cc} 0 & p = \frac{1}{2} \\ 1 & p = \frac{1}{2} \end{array} \right\}$$

$$\varepsilon = \left\{ \begin{array}{cc} 0 & p = \frac{1}{2} \\ 1 & p = \frac{1}{2} \end{array} \right\}$$

$$\begin{aligned} Y' &= Z + \varepsilon \\ Y &= Z(Z + \varepsilon) \end{aligned}$$

Then Y is more informative than Y' for Z but not for ε.

$$\mathrm{E}[Z|Y'] = \left\{ \begin{array}{cc} 0 & p = \frac{1}{4} \\ \frac{1}{2} & p = \frac{1}{2} \\ 1 & p = \frac{1}{4} \end{array} \right\}$$

$$\mathrm{E}[Z|Y] = \left\{ \begin{array}{cc} 0 & p = \frac{1}{2} \\ 1 & p = \frac{1}{2} \end{array} \right\}$$

$$\mathrm{E}[\varepsilon|Y] \overset{\circ}{=} \mathrm{E}[\varepsilon|Y'] = \left\{ \begin{array}{cc} 0 & p = \frac{1}{4} \\ \frac{1}{2} & p = \frac{1}{2} \\ 1 & p = \frac{1}{4} \end{array} \right\}$$

Where "$\overset{\circ}{=}$" means "equal in distribution." $\mathrm{E}[Z|Y]$ is riskier than $\mathrm{E}[Z|Y']$; $\mathrm{E}[\varepsilon|Y]$

and $\mathrm{E}[\varepsilon|Y']$ are equal in distribution, and therefore have the same risk.

Finally, we can consider problems where the variables are imperfectly correlated, such as in climate change where damages are likely to be correlated, but not perfectly. We can get unambiguous results if we assume a special relationship between the two variables as follows:

$$Z^w = CZ^x + \varepsilon + K \tag{13}$$

with C and K constants, ε a random variable independent from Z^x. In this case we can again consider the impact of a signal increasing in informativeness for Z^x, but not for ε.

4.3. Equivalence Theorems

In this section I state first the theorem that holds for reaction functions. I then go on to show that the concept works for any specified set of equilibria as well. I assume that the random variables have the special relationship mentioned in (13). This in fact contains the cases where the random variables are perfectly correlated (i.e. $\varepsilon \equiv 0$), and independence ($C = 0$). For simplicity, I only present the case of perfect correlation in section 4.3.2.

4.3.1. Reaction Functions

THEOREM 4. *Let x_1^* solve (11) and x_1^{**} solve (12). Assume that U is linear in Z, and that $Z^w = CZ^x + \varepsilon + K$ for some constants C, K and independent random variable ε. Then x_1^* is increasing (decreasing) in informativeness for Z^x if x_1^{**} is increasing (decreasing) in uncertainty around Z^x; x_1^* is increasing (decreasing) in informativeness for ε if x_1^{**} is increasing (decreasing) in uncertainty around ε.*

Proof. The proof follows the logic of Theorem 3. ∎

4.3.2. Equilibria

Theorem 4 is a comparative statics result for the reaction functions of a game, but the logic follows through just as well for the equilibrium points of a game. We will show that if any specified equilibrium is increasing in risk in the *risk game*, then the associated equilibrium is increasing in informativeness in the *learning game*. By a specified equilibrium, we mean an equilibrium that can be described, such as the highest, the lowest, or the symmetric equilibrium. For example, we will show that if the highest equilibrium in the risk game is increasing in risk then the highest equilibria in the learning game is increasing in informativeness.

We will define $X(Z,C)$ to be an *equilibrium point specified by condition C.* In order to do this we need to define the reaction functions of each game. Assume for simplicity that $Z^x = Z^w = Z$. Let the second period actions x_2^*, w_2^*, x_2^{**}, w_2^{**} be defined as in Section 4.1. Define the reaction functions of the *risk game* (12) to be player X's optimal response to action w when the random variable is Z and similarly for player W:

$$\begin{aligned} R_x(w,Z) &\equiv \arg\max_x \mathrm{E}_Z\, U^x\left(x, x_2^{**}, w, w_2^{**}; Z\right) \\ R_w(x,Z) &\equiv \arg\max_w \mathrm{E}_Z\, U^w\left(x, x_2^{**}, w, w_2^{**}; Z\right) \end{aligned}$$

Let

$$X(Z) \in \{x | R_x\left(R_w(x,Z), Z\right) = x\}$$

For each $X(Z)$ there exists a $W(Z)$ such that

$$\left(X(Z), W(Z)\right) = \left(R_x(w',Z), R_w(x',Z)\right)$$

Hence $X(Z)$ is an equilibrium first period action for X, given the random variable Z.

$X(Z,C)$ is an *equilibrium point specified by condition C* if it satisfies some condition C. Formally, $x(Z,C) = \{x | x \in X(Z), x \text{ satisfies } C\}$. Some examples of conditions follow:

- $x = \liminf \{x | R_x (R_w (x, Z), Z) = x\}$

- $x = \limsup \{x | R_x (R_w (x, Z), Z) = x\}$

- $x = W(Z)$

Now consider the reaction functions of the learning game (11). When the payoff is linear in the random variable it is apparent that $\mathrm{E}[Z|Y]$ is sufficient for the information structure Y. Thus we can write the reaction function as follows:

$$\begin{aligned} \tilde{R}_x (w, Y) &\equiv \arg\max_x \mathrm{E}_Y\, U^x (x, x_2^*, w, w_2^*; \mathrm{E}[Z|Y]) \\ &= \arg\max_x \mathrm{E}_{\mathrm{E}[Z|Y]}\, U^x (x, x_2^*, w, w_2^*; \mathrm{E}[Z|Y]) = R_x (w, \mathrm{E}[Z|Y]) \end{aligned}$$

Similarly the *equilibrium point specified by condition* C in the learning game (11) given information structure Y is equivalent to

$$X(\mathrm{E}[Z|Y], C) = \{x | x \in X(\mathrm{E}[Z|Y]), x \text{ satisfies condition } C\}$$

THEOREM 5. *Let $Z^x = Z^w = Z$. Let C be a condition which specifies an equilibrium. The equilibrium specified by condition C in the learning game (11) is increasing (decreasing) in informativeness if the equilibrium specified by condition C in the risk game (12) is increasing (decreasing) in risk.*

Proof. Assume that $X(Z, C)$ is increasing in risk. Then, since $\mathrm{E}[Z|Y]$ is a random variable with the same mean as Z, $X(\mathrm{E}[Z|Y], C)$ is also increasing in risk. By Lemma 1, if Y is increasing in informativeness then $\mathrm{E}[Z|Y]$ is increasing in riskiness. Therefore if $X(Z, C)$ is increasing with risk then $X(\mathrm{E}[Z|Y], C)$ is increasing with informativeness. This argument can be repeated for the decreasing case. ∎

By applying the above Theorem twice, we can conclude that if the lowest and highest equilibria are increasing in risk, then they are increasing in informativeness and thus we can get at "how the bounds on behavior ... change with changing parameters"(Milgrom and Roberts, 1994).

Theorem 4 can be extended beyond the linear assumption in two special cases: when two random variables Z^x, Z^w are independent and when $Z^x = Z^w = Z$. The theorems are stated in appendix E. The proofs are straight forward applications of the logic of Theorem 4.

4.3.3. Application: Climate Change II

Another prominent factor in the climate change problem (beyond uncertainty and learning) is that it involves a number of nations making independent decisions about climate policy. It is evident that nations are considering the behavior of other nations when making policies. For example, the United States has refused to join the Kyoto protocol citing as one of the reasons the failure of the protocol to include any participation by developing countries. On the other hand, it has been argued that Europe has agreed to the protocol specifically because the U.S. has dropped out: it turns out that without the U.S. the cost of the protocol to Europe, given the hot air[3] in the former Soviet Union, is close to zero. Kyoto is not a comprehensive worldwide agreement and there is evidence that nations are making climate policy strategically. Therefore, it is important to analyze climate policy as a non-cooperative game. In particular, I am interested in the impact of uncertainty and learning in a non-cooperative game.

I model climate change as a 2-player, 2-period non-cooperative game. Each player represents a nation, or perhaps a coalition of nations. Each player is trying to balance the benefits from emissions against the uncertain damages from the stock of emissions (see payoff function (8)) . In addition to being uncertain about the overall magnitude of climate damages, there is uncertainty about how damages will be distributed. Hence let Z^i = magnitude of the damages to player i, where $i = X, W$. We investigate how the results change depending on the correlation of Z^x and Z^w.

Given Theorem 4 above, we can analyze the risk game, and then use those results to draw conclusions about the learning game. Hence, we will assume that there is perfect learning about the variables Z^x and Z^w before second period decisions are

Correlation	**Impact on Emissions**	**Conditions**
Perfect	Increase	none
Independence – own uncertainty	Increase	$Z^x < -2\gamma\frac{b''}{D''}$
Independence – other's uncertainty	Decrease	none
Perfect negative	Decrease	$D'' < -\frac{b''}{2}$

TABLE 1
Results by level of correlation and by conditions

made. Thus, equilibrium second period emissions will be functions of first period emissions and the realizations of the random variables, z^x and z^w. So the first order condition for the maximization of X's payoff function (taking w_1 as given) is

$$b'(x_1) = \delta \,\mathrm{E}\left\{Z^x D'\left(s\left(x_1 + w_1, Z^x, Z^w\right)\right)\left[\gamma + \frac{\partial w_2}{\partial x_1}\right]\right\} \qquad (14)$$

This is a standard condition: the marginal benefits of first period emissions must equal the expected marginal damages from first period emissions. The term on the right-hand-side in square brackets is a "strategic" term. It indicates that player X is considering the impact of his first period emissions on Player W's second period emissions.

What happens to first period emissions x_1 as risk increases? Results from the stochastic dominance literature show that if the expression inside the expectation is concave in a random variable, then expected value will decrease with risk around that random variable. If expected damages decrease in risk, then first period emissions will increase with risk, and vice-versa. Baker (2002b) shows that whether emissions increase or decrease in risk (and therefore informativeness) depends on the level of correlation between the random variables Z^x and Z^w. The results are summarized in Table 1. When damages are perfectly correlated across nations then equilibrium emissions increase as nations expect to learn more about climate change. The intuition for this is the same as for a single decision maker. If the nations learn that damages are worse than expected, then both nations will reduce emissions accordingly, causing the stock of emissions to fall. Where these results get interesting is in the third row: when damages are independent across nations, each nation will reduce first period emissions as they expect more to be learned about the other nation's damages. By first thinking in terms of increased risk and the relationship with convexity, we can gain some insight as to why this is true. As the other player learns about his actual damages, the more he will respond to what he learns, by reducing emissions an appropriate amount below the no-damage optimal. But, he will reduce emissions at a slower and slower rate as he learns damages are worse. This is because the marginal cost of decreasing damages is increasing. Hence, the expression inside the expectation is decreasing and convex in the random variable Z^w. Thus, expected damages are increasing in learning. For example, consider a simple case where Z was equal to 2 or 0 each with probability 1/2. If player W doesn't learn anything before the second period, he will choose a level of emissions consistent with $Z = 1$. If, on the other hand, he learns the exact value of the damages, he will respond accordingly. If $Z = 0$ he will leave emissions at a business-as-usual level. If $Z = 2$ then he will reduce emissions. But, because the marginal cost of decreasing emissions is increasing, emissions when $Z = 2$ will not be twice as low as emissions when $Z = 1$. Thus, on average, emissions will be higher when learning takes place.

This is a new insight to the literature. We were able to get this intuition because of Theorem 5. Without the theorem, analyzing the most general effects of informativeness in a non-cooperative game had proved intractable. The relevance of these results is that it puts a brake on the idea that the more we expect to learn, the less we need to control emissions now. This is only true when the damages are perfectly correlated. In a world where damages are imperfectly or negatively correlated, learning can come back to haunt us.

4.4. Summary

Increasing risk and informativeness have the same directional impacts on the decision variables if the payoff functions are linear in the random variables. The effect of increasing informativeness is independent of risk attitude, while the effect of increasing risk is not. These results hold true even when comparing equilibria of non-cooperative games. The added complexity of this section comes from the possibly complex relationship between the two random variables. If the random variables are dependent, then it is hard to separate the effect of informativeness on one from the other. Nevertheless, we have shown in the two most commonly modeled cases (independence and a single common random variable) that the theorems from section 3 hold exactly. Additionally, with Theorem 4 we have provided a nice way to analyze the relationship between correlation and learning

5. VALUE OF INFORMATION

The paper so far focuses on the effects of better information on first period actions. But the essential finding of this paper – that risk and the expectation of learning have the same effects if the payoff is linear in the random variable – can be easily extended to analyze the value of information. In the literature, the value of information typically means the value of *perfect* information: the difference in the value of the payoff when perfect information is available versus when no information is available. But, in practice, most signals are not perfect, therefore it is important to know whether imperfect information has a positive value. Furthermore, we need to know the marginal value of information in order to determine the demand for information.

If the value of a decision problem is increasing in informativeness, then every incremental improvement in informativeness has a non-negative value – i.e. the marginal value of information is non-negative. If, on the other hand, the value of a decision problem is ambiguous in informativeness, then the marginal value of information will be negative in some regions. This will be true even if the value of perfect information is strictly positive.

5.1. Single Decision Maker: Compare Values of Information

In a single decision-maker framework, information always has a non-negative value[4]. This is a trivial consequence of Definition 1. The intuitive reason that this is true is that a single decision maker can always choose to disregard information.

We may be interested, however, in comparing the value of information to two individual decision makers. For example we may want to know if a monopolist values information at the same rate as a social planner, or compare the value of information for two classes of utility or profit functions. The following corollary shows how we can compare the (marginal) value of information between two decision makers.

COROLLARY 2. *Let u^1 and u^2 be payoff functions that are linear in $g(Z)$. Let*

$$\pi^1 = \max_{x_1^1} \mathrm{E}_y \max_{x_2^1} \mathrm{E}_{z|y}\, u^1\left(x_1^1, x_2^1, g(Z)\right) \text{ and } \pi^2 = \max_{x_1^2} \mathrm{E}_y \max_{x_2^2} \mathrm{E}_{z|y}\, u^2\left(x_1^2, x_2^2, g(Z)\right)$$

and define

$$D(g(Z)) \equiv \max_{x_2^1} u^1\left(x_1^1, x_2^1, g(Z)\right) - \max_{x_2^2} u^2\left(x_1^2, x_2^2, g(Z)\right)$$

Then the value of information is greater for π^1 than π^2 if $D(g(Z))$ is convex in $g(Z)$.

Proof. Theorem 9 from the appendix implies that if $D(g(Z))$ is convex in $g(Z)$ then $\pi^1 - \pi^2$ is increasing in informativeness. This in turn implies that any increase in informativeness increases the value of π^1 more than π^2, hence π^1 has a greater marginal value of information than π^2. ∎

5.1.1. Application: Climate Change III

We can investigate how the marginal cost of reducing emissions impacts the value of information on future damages. Let $b(\cdot)$ and $\widetilde{b}(\cdot)$ be two different quadratic benefit functions, with $b' > \widetilde{b}'$. First period emissions will be higher when the marginal cost of reducing emissions is greater: $x_1 > \widetilde{x}_1$. This in turn implies that damages will increase in the shift parameter Z more for the first case than the second. Thus

$$-Z\left[D(s^*) - D(\widetilde{s}^*)\right]$$

will be convex in Z. This tells us that a decision maker facing higher marginal costs will have a higher value of information. The intuition is that the more it costs to reduce emissions now, the more we would like to learn about actual damages in order to avoid the chance of reducing emissions too much.

In the single decision maker case (a one-world scenario) the value of information increases with increases in the marginal cost of reducing emissions. This conclusion provokes another interesting question: are R&D investments in low carbon technologies a substitute to research on climate damages?

5.2. Multiple Decision Makers

When there are multiple decision-makers then public information may have a negative value[5]. If, for example, a competitor is in a better position to react to new information, a firm may prefer to forgo information on, say, demand, in order to prevent its competitor learning the same information. When a signal is public, a decision maker can no longer choose to disregard it. The following theorem can be used to determine whether public information has a positive or negative value in a non-cooperative game. The proof, which is exactly analogous to earlier proofs, is omitted.

THEOREM 6. *Under the assumptions of Theorems 4, 7, and 8, the value of the payoff in the partial learning game is increasing is informativeness if the value of the payoff in the perfect learning game is increasing in risk.*

5.2.1. Application: Investment Under Uncertainty in a Duopoly

Two firms X and W are Cournot competitors. They face a linear demand curve

$$price = Z - \beta (x + w)$$

where Z is the unknown intercept, x and w are the quantity of the good produced by X and W and β is the slope. When the cost of production for X and W are

$c(x)$ and $\lambda(w)$ then the payoff functions are

$$x(Z - \beta(x + w)) - c(x)$$

and

$$w(Z - \beta(x + w)) - \lambda(w)$$

The payoff functions are linear in the random variable Z so Theorem 6 applies. Let

$$V^x(Z) = x^*(Z - \beta(x^* + w^*)) - c(x^*)$$

Where x^* and w^* are determined by solving the following first order conditions simultaneously:

$$Z - \beta w - 2\beta x - c'(x) = 0$$

and

$$Z - \beta x - 2\beta w - \lambda'(w) = 0$$

If V^x is convex then the value of information is positive, if it is concave then the value of information is negative, and if it is neither convex nor concave then the value of information depends on the specific information structure. Applying the envelope theorem we get

$$V_Z^x = x^*\left(1 - \beta\frac{\partial w}{\partial Z}\right)$$

and

$$V_{ZZ}^x = \frac{\partial x}{\partial Z}\left(1 - \beta\frac{\partial w}{\partial Z}\right) - x^*\beta\frac{\partial^2 w}{\partial Z^2} \tag{15}$$

The first term in (15) is always positive (see appendix F.1 for derivations). So if $\frac{\partial^2 w}{\partial Z^2} \leq 0$ the value of information is unambiguously positive. If, however, $\frac{\partial^2 w}{\partial Z^2} > 0$ then the value of information is ambiguous. This means that some information structures will have a strictly negative value. To see why, consider the effect of better information on player X. Better (public) information will have 2 effects. The first is to allow player X to increase production if demand is high and decrease it if demand is low. This effect will always increase the value of the payoff. The second effect is on the price. If demand is high then better information will lead to a lower price than no information, because both players will produce more. It is this second effect that may lead to a negative value of information. If $\frac{\partial^2 w}{\partial Z^2} > 0$ then player W is increasing production at a faster and faster rate as demand (Z) increases. He is more responsive to an increase in demand than a decrease in demand, causing price ($= Z - \beta(x^* + w^*)$) to be concave in Z. This means that an increase in informativeness will lead to a lower average price.

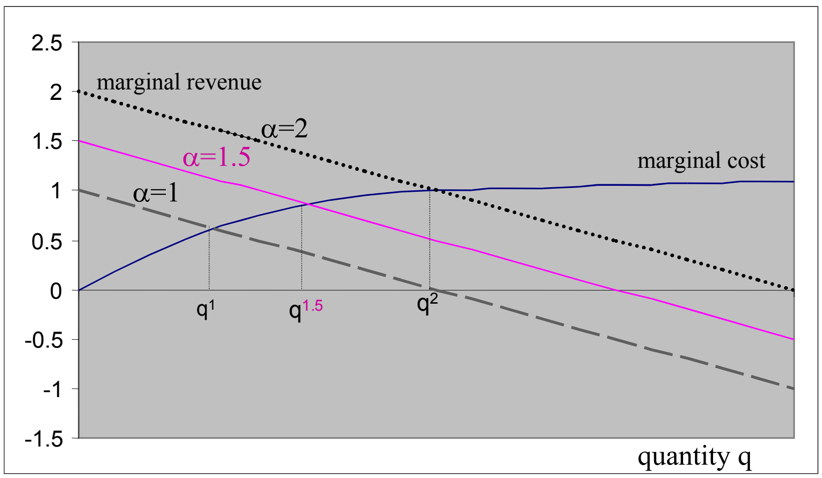

FIG. 4 Quantity is convex in the uncertain demand if marginal cost is concave.

The sign of $\frac{\partial^2 w}{\partial Z^2}$ is a function of the third derivatives of the cost functions. It is interesting to note that $\frac{\partial^2 w}{\partial Z^2} = 0$ when the third derivatives vanish, so information always has a positive value when costs are linear or quadratic. If the players are symmetric, then the value of information is ambiguous (to both players) if and only if marginal cost is concave. If the players are not symmetric, then the value of information is ambiguous to a player if her rival's marginal cost is concave and her marginal cost is "less concave" than her rival's (i.e. if $c''' > \lambda'''$). Why is the curvature of the marginal cost curve important? Figure 4 gives some intuition. If marginal cost is concave, then an increase in demand has a greater effect on the optimal quantity than a decrease in demand. Hence quantity is convex in demand.

A convex marginal cost has the opposite effect. So, if the marginal cost is concave then a mean preserving spread in demand (or, equivalently, better information about demand) increases the average quantity produced. If W's marginal cost is more concave than X's, then W will increase his expected quantity as information increases more than X will, causing price to decrease out of proportion to X's increase in quantity.

6. CONCLUSION

Uncertainty and learning are present in almost every real-world economic decision. This paper has the provocative conclusion that, in the absence of risk aversion, increasing risk and increasing informativeness have the same results. This conclusion holds for the reaction functions and equilibria of non-cooperative games, as well as for single decision makers, with virtually no restrictions on the payoff functions. It holds for first period decision variables and for the value of information, and can easily be shown to hold for the expected value of second period decision variables as well. The reason is that an increase in informativeness is equivalent to an increase in the variability of the prior probability distribution: more information implies more risk.

The second conclusion of this thesis is that the qualitative effects of learning appear to be independent of risk attitude. The curvature of the payoff function around the random variable is irrelevant as long as the payoff function is separable in the random variable. It appears that the central problems of uncertainty and learning can be posed and solved under assumptions of risk neutrality with no loss of generality.

This result is not only intrinsically interesting, but it provides a simple method for determining the comparative statics of learning. Modeling the impact of learning on decisions under uncertainty is useful because it more closely represents real-life problems. In general, most learning that takes place is imperfect. One field where this is particularly useful is environmental policy, such as climate change and biodiversity, where policies are made under scientific uncertainty.

Additionally, besides providing an analytically tractable method for analyzing comparative statics of informativeness, it provides us with essential insights into informativeness. A key role is played by the induced convexity of the payoff function. If the decision-maker is able to ameliorate bad outcomes (by reducing emissions or changing investment plans) then the first period decision will increase in informativeness. If, instead, bad outcomes are exacerbated by other's actions (or by an inability to act) the first period decision will decrease in informativeness.

The application of these theorems have given us new insight into climate change policies. Learning does induce a "go slow" policy for a single decision maker (in the absence of a constraint on non-negative emissions). Nevertheless, learning can have the opposite effect in a non-cooperative game, if the uncertain damages of climate change are independent or negatively correlated. In fact, in Baker (2002b) we discuss more thoroughly when learning will induce a "precautionary" policy, even when damages are positively correlated.

This analysis shines light on the precautionary savings problem, clarifying the dual role of the concavity of the utility function – it represents both risk aversion and the elasticity of substitution between periods. This helps to explain the results. Additionally, in our investment under uncertainty example we are able to show explicitly when information has a negative value and relate that to the curvature of the cost function.

A potential application of this work is to market design, for example the design of electricity markets. Markets are complex, and therefore are designed under a great deal of uncertainty. But learning takes place, from sources around the world as well as internally. This research may help us address the question of what is an optimal market design given uncertainty and learning. Also, the insights of this paper may be useful for expanding the real options framework when the underlying asset is not perfectly observed.

Another direction for future work is to apply the insights of this work to other results in the literature. A particularly interesting line of investigation is whether Kimball's prudence can be re-interpreted in terms of flexibility or the value of information. The work on game theory can be expanded, considering other forms of correlation, as well as applying the framework to models with private signals. That would allow a comparison of the effects of private and public information. Additionally, the insights derived from this model can be applied to more detailed climate change models, for example investigating the value of a technology policy as a hedge against uncertainty. Generalizing these results to other definitions of informativeness promises to provide further insights into the nature of information. Finally, it would be useful to derive more results on the comparative statics of risk, perhaps applying the strong results from lattice theory and supermodularity.

This paper has shed some light on decision making under uncertainty. Uncertainty and learning are very important aspects of many decisions, and most especially in environmental policy. Often the presence of uncertainty seems to lead to either policy-paralysis or policy-panic. Perhaps this work, in adding to the body of work on decision making under uncertainty, will encourage rational discussions on important issues that are fraught with uncertainty.

APPENDIX A: PROOF OF ROTHSCHILD-STIGLITZ THEOREM

Proof. Assume that $V(x_H, Z) - V(x_L, Z)$ is convex. The expected value of a convex function increases in risk, so

$$\mathrm{E}_Z\left[V\left(x_1^{**'}, Z\right) - V(x, Z)\right] \geq \mathrm{E}_{Z'}\left[V\left(x_1^{**'}, Z'\right) - V(x, Z')\right] \qquad \forall x < x_1^{**'}$$

But, by optimality

$$\mathrm{E}_Z V\left(x_1^{**'}, Z\right) < \mathrm{E}_Z V\left(x_1^{**}, Z\right)$$

implying that

$$\mathrm{E}_Z\left[V\left(x_1^{**'}, Z\right) - V\left(x_1^{**}, Z\right)\right] \leq 0 \leq \mathrm{E}_{Z'}\left[V\left(x_1^{**'}, Z'\right) - V\left(x_1^{**}, Z'\right)\right]$$

which implies $x_1^{**\prime} \leq x_1^{**}$.

Conversely, assume that $V(x_H, Z) - V(x_L, Z)$ is concave. Then

$$\mathrm{E}_Z\left[V\left(x_1^{**}, Z\right) - V(x, Z)\right] \leq \mathrm{E}_{Z'}\left[V\left(x_1^{**}, Z'\right) - V\left(x, Z'\right)\right] \qquad \forall x < x_1^{**}$$

But, optimality implies that

$$\mathrm{E}_Z\left[V\left(x_1^{**}, Z\right) - V\left(x_1^{**\prime}, Z\right)\right] \geq 0 \geq \mathrm{E}_{Z'}\left[V\left(x_1^{**}, Z'\right) - V\left(x_1^{**\prime}, Z'\right)\right] \qquad \forall x < x_1^{**}$$

which implies $x_1^{**\prime} \geq x_1^{**}$. For the proof of the last statement in the theorem see (Athey, 2000). ∎

APPENDIX B: BLACKWELL'S THEOREM WITH CONTINUOUS SIGNALS AND CONTINUOUS RANDOM VARIABLE.

Theorem 2

$$\mathrm{E}_Y \max_{x_2} \mathrm{E}_{Z|Y}\, U(x_1, x_2, Z) \quad \geq \quad \mathrm{E}_{Y'} \max_{x_2} \mathrm{E}_{Z|Y'}\, U(x_1, x_2, Z) \qquad \text{(I)}$$
$$\forall U, x_1 \text{ for which the maximum exists}$$

iff

$$\int_Y \rho\left(F_{Z|Y}\right) dF_Y \geq \int_{Y'} \rho\left(F'_{Z|Y}\right) dF'_Y \quad \forall \text{ convex } \rho : \mathcal{F} \to \mathbb{R} \qquad \text{(C)}$$

Where $F = \{\theta : \mathbb{R} \to [0,1]\}$

Proof. Assume that **C** is true. Note that

$$V\left(x_1, F_{Z|Y}\right) \equiv \max_{x_2} \int_Z U(x_1, x_2, Z)\, dF_{Z|Y}$$

is convex in the posterior (since the integral is linear in F and the maximum of the sum of functions is less than or equal to the sum of the maxima). Therefore $\mathbf{C} \Rightarrow \mathbf{I}$ directly.

Assume conversely that **I** is true. Let A be an arbitrary finite set of integrable functions $\{a_i : Z \to \mathbb{R}; i = 1, .., n; z \in Z \subset \mathbb{R}\}$ Define $L_i : \mathcal{F} \to \mathbb{R}$

$$L_i(F) \equiv \int_Z a_i dF$$

Then $L_i(cF + k) = cL_i(F) + K$ for any constants c, and k therefore L_i is linear in F. In fact

$$\{L_i(F) | \forall a_i \text{ integrable}\} = \{\text{All linear functions of } F\}$$

Let

$$L(F) = \max_i L_i(F)$$

L is the maxima of a finite number of linear functions and therefore is convex. For any $a_i(z)$ define

$$u(i, z) \equiv a_i(z)$$

Then **I** implies that

$$\begin{aligned} \int_Y L\left(F_{Z|Y}\right) dF_Y &= \int_Y \max_i \int_Z a_i dF_{Z|Y} dF_Y = \int_Y \max_i \int_Z u(i, z) dF_{Z|Y} dF_Y \\ &\geq \int_Y \max_i \int_Z u(i, z) dF'_{Z|Y} dF'_Y = \int_Y L\left(F'_{Z|Y}\right) dF'_Y \end{aligned}$$

i.e.

$$\mathbf{I} \Rightarrow \int_Y L\left(F_{Z|Y}\right) dF_Y \geq \int_Y L\left(F'_{Z|Y}\right) dF'_Y$$

for all L that are the maxima of a finite number of linear functions, therefore it is true for all convex L. ∎

APPENDIX C: THE PROOF OF THEOREM 3

Let x_1^* solve (1) and x_1^{**} solve (2). Let U be linear in $g(Z)$. Then x_1^* is increasing (decreasing) in informativeness if and only if x_1^{**} is increasing (decreasing) in uncertainty around $g(Z)$. The effect of increasing information on x_1^* is ambiguous *if and only if the effect of increasing risk around $g(Z)$ on x_1^{**} is ambiguous.*

Proof. First we provide some definitions

$$J(x_1, F) \equiv \max_{x_2} \mathrm{E}[U(x_1, x_2, Z)] = \max_{x_2} \int_Z U(x_1, x_2, Z)\, dF$$

$$V(x_1, Z) \equiv \max_{x_2} U(x_1, x_2, Z) = U(x_1, x_2^*(x_1, Z), Z) \tag{16}$$

$$x_1^* \equiv \arg\max_{x_1} \mathrm{E}_Y\, J\left(x_1, F_{z|y}\right)$$

$$x_1^{*\prime} \equiv \arg\max_{x_1} \mathrm{E}_Y\, J\left(x_1, F'_{z|y}\right)$$

I will show that the following 4 statements are equivalent

- x_1^* is increasing (decreasing)(ambiguous) in informativeness
- $J(x_H, F) - J(x_L, F)$ is convex (concave)(neither convex nor concave) in F
- $V(x_H, Z) - V(x_L, Z)$ is convex (concave)(neither convex nor concave) in Z
- x_1^{**} is increasing (decreasing)(ambiguous) in risk.

1. *x_1^* is increasing in informativeness implies that $J(x_H, F) - J(x_L, F)$ is convex in F*[6]. Proof by contradiction. If x_1^* is increasing in informativeness then $x_1^* \geq x_1^{*'}$. Note that by optimality the following is true for any $x \neq x_1^*$:

$$\mathrm{E}_Y\, J\left(x_1^*, F_{Z|Y}\right) > \mathrm{E}_Y\, J\left(x, F_{Z|Y}\right)$$

Assume that $J(x_H, F) - J(x_L, F)$ is concave. Then $-[J(x_H, F) - J(x_L, F)]$ is convex, and Theorem 2 tells us that

$$\mathrm{E}_Y\left[J\left(x_1^*, F_{Z|Y}\right) - J\left(x_1^{*'}, F_{Z|Y}\right)\right] \leq \mathrm{E}_{Y'}\left[J\left(x_1^*, F'_{Z|Y}\right) - J\left(x_1^{*'}, F'_{Z|Y}\right)\right]$$

for F more informative than F'. But by optimality the left-hand-side is positive and the right-hand-side is negative. Hence a contradiction and we conclude that $J(x_H, F) - J(x_L, F)$ is not concave.

Assume that $J(x_H, F) - J(x_L, F)$ is neither convex nor concave. Then, by Theorem 2, there exists some F' that is less informative than F and for which it is true that

$$\mathrm{E}_Y\left[J\left(x_1^*, F_{Z|Y}\right) - J\left(x_1^{*'}, F_{Z|Y}\right)\right] \leq \mathrm{E}_{Y'}\left[J\left(x_1^*, F'_{Z|Y}\right) - J\left(x_1^{*'}, F'_{Z|Y}\right)\right]$$

again giving a contradiction. Therefore $J(x_H, F) - J(x_L, F)$ must be convex.

2. *$J(x_H, F) - J(x_L, F)$ is convex in F implies $V(x_H, Z) - V(x_L, Z)$ is convex in Z.* Since U is linear in Z, J can be written as

$$\begin{aligned} J\left(x_1, F_{Z|Y}\right) &\equiv \max_{x_2} \mathrm{E}_{Z|Y} U(x_1, x_2, Z) \\ &= \max_{x_2} U(x_1, x_2, \mathrm{E}[Z|Y]) = U(x_1, x_2^*(x_1, \mathrm{E}[Z|Y]), \mathrm{E}[Z|Y]) \end{aligned} \tag{17}$$

Recall the definition of V

$$V(x_1, Z) \equiv \max_{x_2} U(x_1, x_2, Z) = U(x_1, x_2^*(x_1, Z), Z)$$

Since $\mathrm{E}[Z|Y]$ is linear in the probability distribution $F_{Z|Y}$,

$$V(x_H, Z) - V(x_L, Z) = U(x_H, x_2^*(x_H, Z), Z) - U(x_L, x_2^*(x_L, Z), Z)$$

is convex in Z iff

$$\begin{aligned} & U(x_H, x_2^*(x_H, \mathrm{E}[Z|Y]), \mathrm{E}[Z|Y]) - U(x_L, x_2^*(x_L, \mathrm{E}[Z|Y]), \mathrm{E}[Z|Y]) \\ = \; & J\left(x_H, F_{Z|Y}\right) - J\left(x_L, F_{Z|Y}\right) \end{aligned}$$

is convex in $F_{Z|Y}$.

3. *$V(x_H, Z) - V(x_L, Z)$ is convex in Z implies x_1^{**} is increasing in risk.* This follows directly from Theorem 1.

4. *x_1^{**} is increasing in risk implies x_1^* is increasing in informativeness.*

From the proof of Theorem 3 on page 14.

The proof for the decreasing case is exactly analogous. The increasing and decreasing cases together imply the last statement in the theorem, by exhausting the possibilities.

∎

APPENDIX D: LEMMA 13

Let Z and ε be independent random variables. If Y is more informative than Y' for Z but not for ε then $\mathrm{E}[g(Z)|Y]$ is riskier than $\mathrm{E}[g(Z)|Y']$ while $\mathrm{E}[h(\varepsilon)|Y]$ and $\mathrm{E}[h(\varepsilon)|Y']$ are equal in distribution, and therefore have the same risk.

Proof. If Y is more informative than Y' for Z but not for ε then by definition

$$\mathrm{E}_Y \max_{x_2} \mathrm{E}_{Z|Y} U(x_1, x_2, w_1, w_2; \varepsilon, Z) \geq \mathrm{E}_{Y'} \max_{x_2} \mathrm{E}_{Z|Y'} U(x_1, x_2, w_1, w_2; \varepsilon, Z)$$

and

$$\mathrm{E}_Y \max_{x_2} \mathrm{E}_{\varepsilon|Y} U(x_1, x_2, w_1, w_2; \varepsilon, Z) = \mathrm{E}_{Y'} \max_{x_2} \mathrm{E}_{\varepsilon|Y'} U(x_1, x_2, w_1, w_2; \varepsilon, Z) \quad (18)$$

Lemma 1 tells us directly that $\mathrm{E}[g(Z)|Y]$ is riskier than $\mathrm{E}[g(Z)|Y']$. Applying Theorem 2 to (18) tells us that

$$\int \rho\left(F_{\varepsilon|y}\right) dF_y \geq \int \rho\left(F'_{\varepsilon|y}\right) dF'_y \quad \forall \text{ convex } \rho : S^m \to \mathbb{R}$$

and

$$\int \rho\left(F_{\varepsilon|y}\right) dF_y \leq \int \rho\left(F'_{\varepsilon|y}\right) dF'_y \quad \forall \text{ convex } \rho : S^m \to \mathbb{R}$$

Therefore

$$\int \rho\left(F_{\varepsilon|y}\right) dF_y = \int \rho\left(F'_{\varepsilon|y}\right) dF'_y \quad \forall \text{ convex } \rho : S^m \to \mathbb{R}$$

Applying the logic of Lemma 1 we can conclude that

$$\int \sigma\left(\mathrm{E}\left[h\left(\varepsilon\right)|Y\right]\right) dF_y = \int \sigma\left(\mathrm{E}\left[h\left(\varepsilon\right)|Y'\right]\right) dF'_y \quad \forall \text{ convex } \sigma : \mathbb{R} \to \mathbb{R} \tag{19}$$

The stochastic dominance literature (Rothschild and Stiglitz, 1970) have shown that, for any two random variables t, t' with distributions F, F' if $\mathrm{E}\,\sigma(t) \geq \mathrm{E}\,\sigma(t')$ for all convex σ, then

$$\int_{\underline{y}}^{y} \left[F'(z) - F(z)\right] dz \leq 0 \quad \forall y \ (\underline{y} \text{ is the minimum value of } y) \tag{20}$$

Let G and G' be the distributions of $\mathrm{E}\left[h(\varepsilon)|Y\right]$ and $\mathrm{E}\left[h(\varepsilon)|Y'\right]$. Then applying (20) to (19) twice implies that

$$\int_{\underline{y}}^{y} \left[G'(z) - G(z)\right] dz \leq 0$$

and

$$\int_{\underline{y}}^{y} \left[G'(z) - G(z)\right] dz \geq 0$$

for all y. Therefore, $\mathrm{E}\left[h(\varepsilon)|Y\right]$ and $\mathrm{E}\left[h(\varepsilon)|Y'\right]$ are equal in distribution. ∎

APPENDIX E: EXTENSIONS OF THEOREM 4

THEOREM 7. *Assume Z^x, Z^w are independent random variables, and that U^x is linear in some function $g^x(Z^x)$; U^w is linear in some function $g^w(Z^w)$. Let x_1^* and x_1^{**} be defined as above. Then x_1^* is increasing (decreasing) in informativeness for Z^x if x_1^{**} is increasing (decreasing) in uncertainty around $g^x(Z^x)$; x_1^* is increasing (decreasing) in informativeness for Z^w if x_1^{**} is increasing (decreasing) in uncertainty around $g^w(Z^w)$.*

THEOREM 8. *Assume $Z^x = Z^w = Z$, and that U^x and U^w are linear in some function $g(Z)$. Let x_1^* and x_1^{**} be defined as above. Then x_1^* is increasing (decreasing) in informativeness for Z if x_1^{**} is increasing (decreasing) in uncertainty* *around* $g(Z)$.

APPENDIX F: VALUE OF INFORMATION

Define

$$V \equiv \sum_i c_i \max_{x_1^i} \mathrm{E}_y \max_{x_2^i} \mathrm{E}_{z|y}\, u^i\left(x_1^i, x_2^i, g(Z)\right)$$

and

$$\overline{V} \equiv \sum_i c_i \max_{x_1^i} \mathrm{E}_Z \max_{x_2^i} u^i\left(x_1^i, x_2^i, g(Z)\right)$$

for any constants c_i.

Theorem 9. *Let V and $\overline{V}$ be defined as above, assuming each u^i is linear in $g(Z)$. Then V is increasing (decreasing) in informativeness if $\bar{V}$ is increasing (decreasing) in risk around $g(Z)$. Furthermore, $\bar{V}$ is increasing in risk around $g(Z)$ if and only if $\sum_i c_i \max_{x_2^i} u^i\left(x_1^i, x_2^i, g(Z)\right)$ is convex (concave) in $g(Z)$.*

Proof. 1. *$\bar{V}$ increasing in risk around $g(Z)$ implies V increasing in informativeness.* Recall

$$V \equiv \sum_i c_i \max_{x_1^i} \mathrm{E}_y \max_{x_2^i} \mathrm{E}_{z|y}\, u^i\left(x_1^i, x_2^i, g(Z)\right) = \sum_i c_i \max_{x_1^i} \mathrm{E}_y \max_{x_2^i} u^i\left(x_1^i, x_2^i, \mathrm{E}\left[g(Z)\,|y\right]\right)$$

and

$$\overline{V} \equiv \sum_i c_i \max_{x_1^i} \mathrm{E}_Z \max_{x_2^i} u^i\left(x_1^i, x_2^i, g(Z)\right)$$

- $g(Z)$ riskier than $g(Z') \Rightarrow \bar{V}(g(Z)) \geq \bar{V}(g(Z'))$.
- But since $\mathrm{E}\left[g(Z)\,|Y\right]$ plays the same role in V as $g(Z)$ plays in $\bar{V}$, the above is equivalent to saying that $\mathrm{E}\left[g(Z)\,|Y\right]$ riskier than $\mathrm{E}\left[g(Z)\,|Y'\right] \Rightarrow V(Y) \geq V(Y')$.
- Lemma 1 tells us that if Y is more informative than Y' then $\mathrm{E}\left[g(Z)\,|Y\right]$ is riskier than $\mathrm{E}\left[g(Z)\,|Y'\right]$.

- Therefore if Y is more informative than Y' then $V(Y) \geq V(Y')$: V is increasing in informativeness.

2. *$\bar{V}$ increasing (decreasing) in risk around $g(Z)$ if $\sum_i c_i \max_{x_2^i} u^i\left(x_1^i, x_2^{i*}\left(x_1^i, g(Z)\right), g(Z)\right)$ is convex (concave) in $g(Z)$ where*

$$x_2^{i*} = \arg\max \max_{x_2^i} u^i\left(x_1^i, x_2^i, g(Z)\right)$$

Rewrite

$$\begin{aligned}
\overline{V} &\equiv \sum_i c_i \max_{x_1^i} \mathrm{E}_Z \max_{x_2^i} u^i\left(x_1^i, x_2^i, g(Z)\right) \\
&= \sum_i c_i \max_{x_1^i} \mathrm{E}_Z\, u^i\left(x_1^i, x_2^{i*}\left(x_1^i, g(Z)\right), g(Z)\right) \\
&= \sum_i c_i\, \mathrm{E}_Z\, u^i\left(x_1^{i*}(F), x_2^{i*}\left(x_1^i, g(Z)\right), g(Z)\right) \\
&= \mathrm{E}_Z \sum_i c_i u^i\left(x_1^{i*}(F), x_2^{i*}\left(x_1^i, g(Z)\right), g(Z)\right)
\end{aligned}$$

where F is the distribution function of Z and

$$x_1^{i*}(F) \equiv \arg\max_{x_1^i} \mathrm{E}_Z\, u^i\left(x_1^i, x_2^{i*}\left(x_1^i, g(Z)\right), g(Z)\right)$$

If $\sum_i c_i \max_{x_2^i} u^i\left(x_1^i, x_2^{i*}\left(x_1^i, g(Z)\right), g(Z)\right)$ is convex (concave) in $g(Z)$ for all x_1^i then it is convex (concave) for the optimal $x_1^{i*}(F)$, therefore Theorem 1 implies that $\bar{V}$ is increasing in risk. ∎

F.1. Duopolists

Recall

$$V^x(Z) \equiv x^*\left(Z - \beta\left(x^* + w^*\right)\right) - c\left(x^*\right)$$

Where x^* and w^* are determined by the first order conditions

$$Z - \beta w - 2\beta x - c'(x) = 0$$

and

$$Z - \beta x - 2\beta w - \lambda'(w) = 0$$

solved simultaneously. Applying the envelope theorem to the first order conditions we calculate the total effect of Z on x and w:

$$\begin{aligned}
\frac{\partial x}{\partial Z} &= \frac{\beta + \lambda''}{(2\beta + c'')\left(2\beta + \lambda''\right) - \beta^2} \\
\frac{\partial w}{\partial Z} &= \frac{\beta + c''}{(2\beta + c'')\left(2\beta + \lambda''\right) - \beta^2} \\
\frac{\partial^2 w}{\partial Z^2} &= \frac{\beta\left(2\beta + \lambda''\right)\left(\beta + \lambda''\right)c''' - (2\beta + c'')(\beta + c'')^2\lambda'''}{\left[(2\beta + c'')\left(2\beta + \lambda''\right) - \beta^2\right]^3}
\end{aligned}$$

Substituting these expressions into (15) gives

$$\begin{aligned}
V^x_{ZZ} &= \frac{\left(\beta + \lambda''\right)^2(2\beta + c'')}{\left((2\beta + c'')\left(2\beta + \lambda''\right) - \beta^2\right)^2} \\
&\quad - x^*\beta\frac{\beta\left(2\beta + \lambda''\right)\left(\beta + \lambda''\right)c''' - (2\beta + c'')(\beta + c'')^2\lambda'''}{\left[(2\beta + c'')\left(2\beta + \lambda''\right) - \beta^2\right]^3}
\end{aligned} \tag{21}$$

The second order conditions require that

$$2\beta + \lambda'' > 0 \text{ and } 2\beta + c'' > 0$$

and stability conditions say that

$$\left(2\beta + \lambda''\right)(2\beta + c'') - \beta^2 > 0$$

Therefore the first term in (21) is positive and the sign of the second term depends on c''' and λ'''.

APPENDIX G

Notes

[1]See Marschak & Miyasawa (1968) p. 164 for a statement and proof of the theorem in the finite case. The theorem seems to have originated with Bohnenblust, Shapley, and Sherman as referenced in Blackwell (1951) who provides a proof for a similar theorem with continuous signal and finite random variable. Strassen (1964) has shown in a very general case that the second part of Theorem 3 is equivalent to statistical sufficiency, but there does not exist a readily available proof in the literature for Theorem 3 using continuous random variables

[2]This quote was first referenced in Jones and Ostroy (1984).

[3]Kyoto targets for emission reductions were set before the fall of the Soviet Union and the collapse of industrial production in that region. Hence, the emissions targets for the former Soviet Union are currently far below actual emission rates. When pooled with Europe, the net reduction in emissions will be close to zero.

[4]This assumes that the signal itself does not directly effect the payoff function. See Sulganik and Silcha (1997) or Datta, Mirman and Schlee (2000) for a discussion of this case.

[5]See Hirshleifer (1971) for the most well-known example. Zhu (1999) presents an example in a strategic framework.

[6]The converse of this statement, with the assumption that the payoff function is concave and differentiable in x_1 is Epstein's Theorem 1 in (Epstein, 1980).

APPENDIX H

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