

Increasing Risk and Increasing Informativeness: Equivalence Theorems

Erin Baker
Mechanical and Industrial Engineering
University of Massachusetts
220 Engineering Laboratory
Amherst, MA 01003
E-mail:edbaker@ecs.umass.edu

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When considering problems of sequential decision making under uncertainty, two of the most interesting questions are: How does the value of the optimal decision variable change with an increase in risk? How does the value of the optimal decision variable change with a more informative signal? In this paper we show that, if the payoff function is separable in the random variable, then one model can simultaneously answer both questions. This result holds for the reaction functions and equilibria of non-cooperative games, as well as for single decision makers, with virtually no restrictions on the payoff functions. This is useful because otherwise it is very difficult to get at general results on the impact of learning. Furthermore, we clarify why the impacts of risk and a more informative signal are different when the payoff function is non-linear in the random variable. It is because the directional impacts of informativeness are independent of risk attitude; the impacts of risk are not.

Key Words: Decision Analysis, Theory: Comparative statics of risk and learning; Games, Stochastic: Impact of learning on equilibrium; Natural Resources, Energy: Climate Change Policy.

1. INTRODUCTION

Many choices are made under uncertainty – about the weather, costs, or a competitor’s behavior. Very often choices can be revised or updated after learning takes place – a farmer can harvest his crop early if the weather calls for it; a firm can scale back production if costs soar. This process of sequential decision making under uncertainty has different implications than one-shot decisions made under uncertainty, particularly when it comes to the comparative statics or sensitivity analysis of the random structure. We consider the impact on optimal decision variables of an increase in risk (or uncertainty – we use the terms synonymously) and of a more informative signal. It has been noticed before that there is a parallel structure to risk and informativeness, however this parallel has not been fully exploited.

In this paper we show that the comparative statics of risk and of learning (i.e. informativeness) are qualitatively the same when the payoff function is separable in the random variable. Often, understanding the effects of learning is more policy relevant than understanding the effects of increasing risk. For example, if an international body decides to significantly increase research on climate change, hence increasing the amount nations expect to learn, should nations increase or decrease current emissions? Analyzing the value of information is also important. For example, manufacturers and retailers are assessing the value of Electronic Data Interchange for improving supply chain management and reducing costs (See Lee, So, & Tang 2000, Gavirneni, Kapuscinski, & Tayur 1999). On the other hand, it is easier to model and determine the effects of an increase in risk than the effects of an increase in informativeness. Thus, the results in this paper allows one to get at the comparative statics of learning using simple, well-understood methods.

We present two different, but related decision problems. Both are 2-period problems with uncertainty in the state of nature that affects the payoff. Decisions are made in both periods. The first problem involves partial learning. After the first period, but before the second, a signal is received, giving the decision maker information about the value of the random variable. It is of interest how the decision variables and the value of the payoff change with a more informative signal, in the Blackwell (1951) sense. The second problem is identical, except there is perfect learning: the value of the random variable is known before the second period decision is made. Many papers analyze how the optimal decision and the value of the payoff change with increased risk, in the Rothschild-Stiglitz (1970) sense. The objective of this paper is to show when the effects of increasing informativeness are the same as the effects of increasing risk. Epstein (1980) has shown by example that the directional effect on the first period decision variable is not always the same. We show, however, for both single and multiple decision makers, that these effects are identical when the payoff function is separable in the random variable.

In addition to showing when increasing risk and increasing informativeness have equivalent results, this

paper clarifies why they have different results in many cases. It is because the value and effect of information is independent of the curvature of the payoff function around the random variable. In other words, it is perfectly general to model risk neutral payoffs when considering the qualitative effects of learning.

Three examples illustrate the usefulness of the theorems. The first is a simple model of climate change as a non-cooperative game. Application of the theorems illuminate the role of learning in climate change policy. The second example builds on Epstein's presentation of a three-period consumption-savings problem. Applying our methods illustrates that risk aversion and the intertemporal elasticity of substitution are both important when analyzing the effect of risk on consumption; only the elasticity of substitution is important in analyzing the effects of information. In the third example, we consider the impacts of uncertainty and learning on investment when there are strategic interactions.

Previous work on the comparative statics of learning has employed methods that are either more difficult or less general than those in this paper. Epstein (1980) provides a method for comparing first period decisions under different amounts of informativeness. The method requires determining whether a function is convex in a vector (or a cumulative distribution function in the continuous case). The results in this paper allow us to collapse the vector or function into a one-dimensional variable, greatly easing calculations. Epstein's method has not been generalized to non-cooperative games, perhaps because of the difficulty of working in vector or function spaces. Some papers have analyzed the effects of information in games using less general methods. Vives (1989) and Gal-Or (1985) use mean-variance frameworks to analyze the impacts of information in oligopolies. Sakai (1985) and Ulph and Ulph (1996) calculate the value of perfect information in non-cooperative games. The equivalence theorems in this paper allow us to use general information structures and calculate comparative statics for *incremental* increases in informativeness.

More recently, Athey & Levin (2000) analyze information preferences and information demand for different classes of decision makers. A signal is more informative in the Blackwell sense if *all* decision makers prefer it. By contrast, Athey and Levin provide conditions under which all decision makers in a certain class (such as those with payoff functions that are supermodular) will prefer a signal. They use these results to determine the demand for information in a variety of monotone decision problems. The class of decision makers that we consider (those with payoff functions that are separable in the random variable) is not contained in any of the classes that they consider, and thus both results are useful in different situations. Moreover, these class-based definitions have not been extended for comparative statics analysis on sequential decision problems.

The rest of the paper is organized as follows. In the next section we present the basic model and definitions of risk and informativeness. Section 3 contains the results for the case of a single decision maker, including

the insight that risk attitude is irrelevant for the comparative statics of information. Section 4 extends the results to a 2-person non-cooperative game, with two possibly correlated random variables. Section 5 illustrates how the main ideas of the paper can be applied to the value of information. Section 6 summarizes the results.

2. MODEL AND DEFINITIONS

2.1. Sequential Decision Making Under Uncertainty

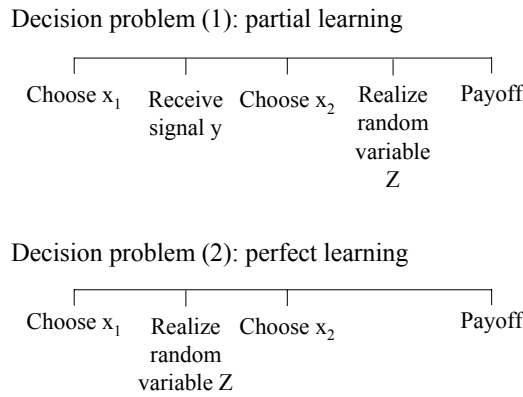
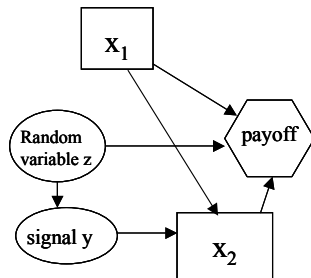


FIG. 1 Two related decision problems.

In this section we introduce the two related decision problems upon which the results are based. Figure 1 illustrates the time paths of the two problems while figure 2 illustrates the decision diagrams. The first decision problem is more general, representing a 2-period decision problem with learning, as follows

$$\max_{x_1} E_Y \max_{x_2 \in C(x_1)} E_{Z|Y} U(x_1, x_2, Z) \quad (1)$$

Decision problem (1): partial learning.



Decision problem (2): perfect learning.

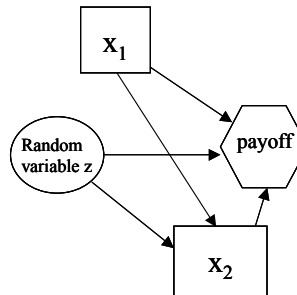


FIG. 2 Decision Diagram of two related decision problems. Square boxes represent decisions, ovals represent uncertainties, hexagons are payoffs, and arrows represent influences and information.

$x_1, x_2 \in \mathbb{R}$ are the first and second period decision variables, U is a payoff function, $C(x_1) \subseteq \mathbb{R}$ represents the choice set for x_2 , which may or may not be constrained by x_1 . Y and Z are measurable random variables defined on a given probability space (Ω, \mathcal{A}, P) defined by the sample space Ω , a σ -field \mathcal{A} of events of Ω , and a probability measure P . Thus Y may provide information about Z . Define the set $A_y \equiv \{\omega | Y(\omega) = y\}$ then $F_{Z|y}(z) = P(Z(\omega) \leq z | \omega \in A_y)$ is well defined for any event A_y with positive probability¹. Thus, $F_{Z|Y}$ can be considered an infinite dimensional random variable with distribution function F_Y . E is the expectation operator, where E_Z means the expected value over Z . We refer to (1) as the *learning problem*. The question of interest is, what is the effect on the optimal first period decision if the decision maker expects to receive a more informative signal Y ?

Now consider a special case of (1) when there is perfect learning before the second period: Z is known before x_2 is chosen. This related problem is:

$$\max_{x_1} E_Z \max_{x_2 \in C(x_1)} U(x_1, x_2, Z) \quad (2)$$

We refer to (2) as the *risk problem*. Regarding the *risk problem*, the question is, what is the effect on the optimal first period decision if the uncertainty around the random variable Z is varied? In section 3 we show that when the payoff function is separable in the random variable Z , then the optimal first period decision in problem (1) is increasing in informativeness if and only if the optimal first period decision in (2) is increasing in risk.

The broad intuition of this result is as follows: if in problem (1) one expects to have more information before choosing x_2 then it is preferable to choose x_1 in such a way to increase the ability to react to what is learned. Similarly, the more prior risk faced in problem (2), the more valuable is the ability to react in choosing x_2 . Hence, we might expect an increase in informativeness and an increase in uncertainty to have similar effects on x_1 . This is only true, however, when the payoff function is separable in the random variable. In section 3 we discuss why it does not hold true in general.

2.2. Informativeness and Risk

A *signal* is a way of gathering information, often called an experiment. One signal is more informative than another if *every* decision maker is (weakly) better off expecting to receive a signal Y rather than a signal Y' .

DEFINITION 1. Let Y and Y' be random variables defined on the same probability space as Z . Then we

say that Y is **more informative than** Y' if

$$\mathbb{E}_Y \max_{x_2 \in C(x_1)} \mathbb{E}_{Z|Y} U(x_1, x_2, Z) \geq \mathbb{E}_{Y'} \max_{x_2 \in C(x_1)} \mathbb{E}_{Z|Y'} U(x_1, x_2, Z)$$

for all x_1 , U , and C for which the maximum is defined.

Notice that if Y is more informative than Y' for the random variable Z , then Y is more informative than Y' for all functions $g(Z)$. This follows from the definition directly since the set of all $U(\cdot, \cdot, g(Z))$ is a subset of all $U(\cdot, \cdot, Z)$. Now, recall the Rothschild-Stiglitz definition of increasing risk, often called a mean-preserving spread²:

DEFINITION 2. Z is **riskier** (or **more uncertain** or **more variable**) than Z' iff $\mathbb{E}_Z U(Z) \geq \mathbb{E}_{Z'} U(Z')$ for all convex U .

The following Lemma says that we need only check a subset of all convex functions to determine whether Z is riskier than Z' . See Athey (2000), Gollier (2001), or Shaked and Shanthikumar (1994) for proof.

LEMMA 1. Z is riskier than Z' if and only if $\mathbb{E}_Z U(Z) \geq \mathbb{E}_{Z'} U(Z')$ for all $U \in B$, where $B \equiv \{u|u(z) = -z\} \cup \{u|u(z) = \max[z - a, 0] \ a \in \mathbb{R}\}$

We need to be precise about what it means to “increase in risk” or “increase in informativeness.” Throughout the paper we will use the term “increasing” to mean non-decreasing, and will say “strictly increasing” when that is what we mean. Let $x, x', \tilde{x}, \tilde{x}'$ be the optimal values of x_1 in (2) given random variables $Z, Z', \tilde{Z}, \tilde{Z}'$ respectively. Then we say x_1 is **increasing in risk** if for any two random variables Z and Z' , Z riskier than Z' implies that $x \geq x'$; and x_1 is **ambiguous in risk** if there exists random variables Z riskier than Z' and \tilde{Z} riskier than \tilde{Z}' such that $x > x'$ but $\tilde{x} < \tilde{x}'$. Similarly, let $x, x', \tilde{x}, \tilde{x}'$ be the optimal value of x_1 in (1) given signals $Y, Y', \tilde{Y}, \tilde{Y}'$. Then we say x_1 is **increasing in informativeness** if, for any underlying random variable Z , Y more informative than Y' for Z implies that $x \geq x'$; and x is **ambiguous in informativeness** if there exists random variables $Z, Y, Y', \tilde{Y}, \tilde{Y}'$ more informative than Y' for Z , and $\tilde{Z}, \tilde{Y}, \tilde{Y}'$, \tilde{Y} more informative than \tilde{Y}' for \tilde{Z} where $x > x'$ but $\tilde{x} < \tilde{x}'$.

The equivalence theorems in this paper are useful because it is easier to determine the comparative statics of risk than the comparative statics of informativeness. In particular, the well-known theorem of Rothschild & Stiglitz (1971) (which we will denote RS for the rest of the paper) gives conditions for x to increase in risk³.

2.2.1. Informativeness Lemma

We apply the logic of Blackwell's Informativeness Theorem (see Blackwell 1950, 1951) to show that an increase in informativeness will increase the risk of the conditional expectation, $E[Z|Y]$. For intuition consider the two extreme cases. If Y is independent from Z then $E[Z|Y] = E[Z]$ for all possible realizations of Y : the random variable $E[Z|Y]$ is constant. On the other hand, if Y provides perfect information about Z , that implies that $Z = h(Y)$ for some function h , thus $E[Z|Y] = E[h(Y)|Y] = h(Y) = Z$, i.e. the variability of $E[Z|Y]$ will be exactly equal to the variability of Z . In fact, Lemma 2 is more general than this, showing that the conditional expectation is riskier for any function of Z .

LEMMA 2. *If Y is more informative than Y' then $E[g(Z)|Y]$ is riskier than $E[g(Z)|Y']$ where g is any function $g: \mathbb{R} \rightarrow \mathbb{R}$ for which the expectations are well defined.*

Proof. For any $g(\cdot)$ for which the expectations are well defined and any $a \in \mathbb{R}$ we can define a payoff function $U_a(x, z) \equiv x[g(z) - a]$. Note that

$$\max_{x \in [0,1]} E_{Z|y} U_a(x, z) = \max \{E[g(z)|y] - a, 0\} \quad (3)$$

Combining (3) with Definition 1 implies that if Y is more informative than Y' then

$$E_Y \max \{E[g(z)|y] - a, 0\} \geq E_{Y'} \max \{E[g(z)|y'] - a, 0\} \quad (4)$$

Additionally

$$E_Y E[g(z)|y] = E_{Y'} E[g(z)|y'] \quad (5)$$

thus Lemma 1 implies that $E[g(z)|Y]$ is riskier than $E[g(z)|Y']$. ■

3. EQUIVALENCE THEOREM: SINGLE DECISION MAKER

Let U be separable in the random variable Z in the following sense: $U(x_1, x_2, Z) = U_1(x_1, x_2) + g(Z)U_2(x_1, x_2)$ for some function $g(\cdot)$. Then U can be rewritten as $\tilde{U}(x_1, x_2, g)$, \tilde{U} linear in g , a random variable defined by Z . Note that linear functions are a subset of separable functions. Then the *learning problem* (1) and the *risk problem* (2) can be rewritten as

$$\max_{x_1} E_Y \max_{x_2 \in C(x_1)} \tilde{U}(x_1, x_2, E_{Z|Y} g) \quad (6)$$

$$\max_{x_1} E_Z \max_{x_2 \in C(x_1)} \tilde{U}(x_1, x_2, g) \quad (7)$$

THEOREM 1. Let x_1^* solve (1) and x_1^{**} solve (2), and assume that both maxima are well-defined and unique. Assume that \tilde{U} is linear in g . Then x_1^* is increasing (decreasing) in informativeness if x_1^{**} is increasing (decreasing) in uncertainty around g . If g is invertible then the converse holds and the effect of increasing informativeness on x_1^* is ambiguous if and only if the effect of increasing risk on x_1^{**} is ambiguous.

Proof. Define $x_1^*(Y)$ and $x_1^{**}(Z)$ as the arg max of (6) and (7), where they are functions of the distributions of Y and Z respectively, as opposed to a particular realization of the variables. Assume x_1^{**} is increasing in uncertainty around g . This means that

- g riskier than $g' \Rightarrow x_1^{**}(Z) \geq x_1^{**}(Z')$.
- But since $E[g|Y]$ plays the same role in (6) as g plays in (7), the above is equivalent to saying that $E[g|Y]$ riskier than $E[g|Y'] \Rightarrow x_1^*(Y) \geq x_1^*(Y')$.
- Lemma 2 tells us that if Y is more informative than Y' then $E[g|Y]$ is riskier than $E[g|Y']$.
- Therefore if Y is more informative than Y' then $x_1^*(Y) \geq x_1^*(Y') : x_1^*$ is increasing in informativeness.

The proof for the decreasing case uses the same logic with the opposite inequalities. See Appendix A for the proof of the converse and for the last statement in the theorem. ■

We stress that the assumptions required for Theorem 1 are very weak. There is no requirement that the payoff function U be differentiable or even continuous. The assumption of unique solutions can be relaxed by considering only the highest or lowest maximizer, for example. Furthermore, the decision variables need not be constrained to the real line: the first statement in Theorem 1 holds true if $x_1, x_2 \in K$ a compact subset of any normed linear space.

The example below illustrates the usefulness of the theorem. In all examples, we present the relevant problem assuming perfect learning before period 2, analogous to the *risk problem* (2). We then use the results from this paper to determine the comparative statics of learning in the more general problem, analogous to the *learning problem* (1). We do not explicitly restate the problem in terms of partial learning, but assume that a possibly imperfect signal is received before the second period.

EXAMPLE 1. Climate Change I

Climate change epitomizes a sequential decision problem under uncertainty. While it has come to be generally accepted that the stock of carbon emissions due to human activities is having an effect on the climate, there are huge uncertainties about how emissions today will impact humans in the future. Yet, we expect to learn more about the relationship between emissions and damages as time goes on.

Consider the following model:

$$\max_{x_1} b(x_1) + \delta \mathbb{E} \left[\max_{x_2 \in C(x_1)} b(x_2) - ZD(s) \right] \quad (8)$$

Where x_1 , x_2 are 1st and 2nd period emissions, $s = x_1 + x_2$ is the total stock of emissions, δ is the discount factor, $b(\cdot)$ measures the benefits from emissions, $D(\cdot)$ measures the damages caused by the stock of emissions in the atmosphere, and Z is a random variable. Assume that b and D are quadratic. If emissions are not restricted to be non-negative, then an application of RS shows that optimal first period emissions are increasing in risk. An application of Theorem 1 to this result immediately implies that optimal first period emissions in the *learning problem* are increasing in the amount that we expect to learn. However, if we constrain $x_2 \geq 0$, the condition for a monotone increase in risk is gone: the marginal second period payoff in problem (8) is neither convex nor concave in the random variable Z . Thus, Theorem 1 implies that first period optimal emissions in the *learning problem* will decrease for some increases in informativeness around some underlying random variables Z .⁴

3.1. Risk versus Informativeness

If the original payoff function U is linear in the random variable, then increasing risk and increasing informativeness have identical impacts. If, on the other hand, the original payoff U is only separable in the random variable (i.e. linear in a function g), then increasing informativeness will not have the same impact as increasing risk around the original random variable Z . One way to interpret this fact is that the functional form of $g(\cdot)$ is *not important* when considering the effects of more information. It is, however, important when considering the effects of increasing risk. The reason is that if Y is increasing in informativeness for Z then it is equally increasing in informativeness for $g(Z)$. The analogous statement is not true for increasing risk. In particular, if g is non-linear then an increase in risk around Z will change the mean of $g(Z)$.

This suggests that risk attitude is not important when looking at the qualitative effects of information. If U is separable in $g(Z)$ then risk attitude is embodied in the curvature of $g(\cdot)$. The curvature in the decision variables x_1 and x_2 do not effect risk attitude, since they are not uncertain. The qualitative effects of changes in information (on both the expected value of the payoff and on x_1) are independent of the functional form of $g(Z)$. Therefore, in the separable case, risk attitude is irrelevant. If U is not separable then it is impossible to separate out the effects of risk attitude from other effects, most prominently preferences toward smoothing consumption over time⁵. The following example (based on Epstein 1980) further clarifies this idea.

EXAMPLE 2. A Consumption-Savings Problem

An individual has a given wealth w_1 which she wishes to allocate between consumption in three periods.

What she does not consume in any period she invests in a single asset. Investment in period 1 yields a sure gross return r and investment in period 2 yields a random gross return Z . The consumer maximizes expected utility, with a payoff function:

$$u(w_1 - s_1) + \delta u(rs_1 - s_2) + \delta^2 u(s_2 Z) \quad (9)$$

where w_1 is initial wealth, s_1 and s_2 are the savings in periods 1 and 2, r and Z are the first and (uncertain) second period savings rates, and δ is a discount factor. Let

$$u(w) = \frac{w^{1-\alpha}}{1-\alpha} \quad \alpha \neq 1$$

so that utility has constant relative risk aversion coefficient α . In this case the payoff function (9) is separable in a function of Z , namely $g(Z) = Z^{1-\alpha}$ if $\alpha \neq 1$; $g(Z) = \log Z$ for the limit case where $\alpha = 1$. The functional form of $g(\cdot)$ is irrelevant to determine the effects of more information, so we simply analyze the effects of increasing risk on

$$\max_{s_1} u(w_1 - s_1) + \mathbb{E}_Z \max_{s_2} \delta u(rs_1 - s_2) + \delta^2 u(s_2) g$$

Clearly this payoff is linear in g and therefore risk neutral. Yet, in the analysis below we show that the coefficient α is still important to the problem.

Optimal savings s_1 is increasing in informativeness (assuming an imperfect signal is received before the period two decision) if and only if s_1 is increasing in the risk of g . Let

$$V(s_1, g) = \max_{s_2} \delta u(rs_1 - s_2) + \delta^2 u(s_2) g$$

then

$$s_2^* = \arg \max_{s_2} \delta u(rs_1 - s_2) + \delta^2 u(s_2) g = \frac{rs_1}{1 + g^{-\frac{1}{\alpha}}}$$

and

$$V_{s_1}(s_1, g) = r\delta u'(rs_1 - s_2^*) = r\delta (rs_1)^{-\alpha} \left(1 - \frac{1}{1 + g^{-\frac{1}{\alpha}}}\right)^{-\alpha} \quad (10)$$

where V_{s_1} indicates the partial derivative of V by s_1 . Thus V_{s_1} is convex in g if $\alpha < 1$ and concave in g if $\alpha > 1$. Savings is increasing in informativeness for large α and decreasing in informativeness for small α . Epstein points out that (9) is ordinally equivalent to a CES utility function with intertemporal elasticity of substitution of $\frac{1}{\alpha}$. Hence, if α is low – implying that elasticity of substitution is high – then we expect more savings in period one in order to take advantage of the better information. If elasticity of substitution is

low, then the income effect dominates, and we expect the consumer to consume more in the first period in expectation of more income later.

In order to check the effect of increasing risk we substitute $Z^{1-\alpha}$ for g in equation (10).

$$V_{s_1}(s_1, Z) = C \left(1 + Z^{\frac{1-\alpha}{\alpha}} \right)^\alpha$$

We find that the effect of increasing risk is almost the opposite of the effect of increasing informativeness: V_{s_1} is convex if $\alpha > 1$ and concave if $\frac{1}{2} < \alpha < 1$, but it is neither convex nor concave for $\alpha < \frac{1}{2}$. Savings is increasing in risk for large α , decreasing in risk for medium α , and indeterminate for small α .

Why are the results for increasing risk opposite from those for increasing informativeness? When considering the effects of informativeness, α only plays the role of elasticity-of-substitution parameter. When risk aversion is present, however, α has two roles. As α increases, the elasticity of substitution decreases *and* risk aversion increases. Because of risk aversion, the expected utility of third period consumption will get smaller as risk increases. If α is large, the elasticity of substitution between periods is low, implying a preference for smoothed out consumption. Therefore, second period savings s_2 will increase (in expectation) in order to assure enough consumption in the third period. First period savings will increase in order to help offset the lower expected second period consumption. Conversely, if α is small, the elasticity of substitution between periods is high, so s_2 will reduce (in expectation) in order to avoid the low-value third period. First period savings will decrease in order to offset the higher expected second period consumption. Finally, as α goes to 0, risk aversion becomes less prominent, and the response to an increase in risk becomes indeterminate.

Comparing the results from risk and informativeness gives new insight into how consumption changes with risk. It depends on the interaction of the intertemporal elasticity of substitution and the level of risk aversion. This suggests the importance of analyzing consumption problems in a framework that disentangles risk and time, such as those discussed in Epstein and Zin (1989).

4. MULTIPLE DECISION MAKERS

We have shown that the comparative statics of risk and information are qualitatively the same in a single decision maker context. In this section, we show that this is true in a 2-person non-cooperative game as well. We set up a 2-person decision problem with two (possibly correlated) random variables, then discuss what it means for a signal to be more informative when there are two players and two random variables. We first show that the reaction functions are increasing in informativeness if they are increasing in risk. We then go on to apply the same logic we have used throughout the paper to the equilibria of the game, showing that

equilibrium actions are increasing in informativeness if they are increasing in risk.

4.1. Problem Definition

Consider the problems analogous to (1) and (2) but in a strategic framework. Assume there are 2 players, X and W, with strategy sets (x_1, x_2) and (w_1, w_2) , $x_1, x_2, w_1, w_2 \in A$, A an ordered set; and payoff functions $U^x(x_1, x_2, w_1, w_2; Z^x)$ and $U^w(x_1, x_2, w_1, w_2; Z^w)$ ⁶. Z^x and Z^w and the signal Y are all measurable random variables defined over a given probability space (Ω, \mathcal{A}, P) . There is no private information. The solution concept is Sub-game Perfect Nash Equilibrium. In all discussion below we assume that the maxima are well-defined and an equilibrium exists.

First consider the problem with learning analogous to (1). Each player observes the first period actions x_1 and w_1 and receives a public signal y before choosing 2nd period actions. The set of equilibrium points of the second period sub-game is determined by the first period actions and the signal: $(x_2^*, w_2^*) = (x_2^*(x_1, w_1, E[Z^w|Y], E[Z^x|Y]), w_2^*(x_1, w_1, E[Z^w|Y], E[Z^x|Y]))$. The first period equilibria, therefore, will depend on the distribution of Y . The *risk game* analogous to (2) is defined in a similar way, with 2nd period equilibrium points (x_2^{**}, w_2^{**})

4.2. Informativeness

In order to discuss the impacts of a more informative signal Y , we have to consider what it means for a signal to increase in informativeness when there are multiple random variables. For example, consider problems where the two random variables are independent. Two signals Y and Y' may be such that Y is more informative than Y' for Z^w but less informative for Z^x . This ambiguity makes it hard to come up with interesting results. So we define the notion of a signal being more informative for one random variable, say Z^x , while holding the informativeness relative to the other random variable constant. We will define Y to be more informative than Y' for ε but not for τ if Y' has the same information relevant to τ as Y does.

DEFINITION 3. A message Y is **more informative than Y' for ε but not τ** if

$$\begin{aligned} E_Y \max_{x_2} E_{\varepsilon|Y} U^x(x_1, x_2, w_1, w_2; \tau, \varepsilon) &\geq E_{Y'} \max_{x_2} E_{\varepsilon|Y'} U^x(x_1, x_2, w_1, w_2; \tau, \varepsilon) \\ &\forall U, x_1, w_1, w_2, \tau \text{ for which the maximum exists.} \end{aligned}$$

but

$$\begin{aligned} E_Y \max_{x_2} E_{\tau|Y} U^x(x_1, x_2, w_1, w_2; \tau, \varepsilon) &= E_{Y'} \max_{x_2} E_{\tau|Y'} U^x(x_1, x_2, w_1, w_2; \tau, \varepsilon) \\ &\forall U, x_1, w_1, w_2, \varepsilon \text{ for which the maximum exists.} \end{aligned}$$

See Baker (2002) for the restatement and proof of Lemma 2.

4.3. Equivalence Theorems

Assume that the random variables are related as follows: $Z^w = CZ^x + \varepsilon + K$, with C and K constants, ε a random variable defined on Ω , independent from Z^x . This contains the special cases where the random variables are perfectly correlated (i.e. $\varepsilon \equiv 0$), or independent ($C = 0$).

4.3.1. Reaction Functions

THEOREM 2. *Let x_1^* and x_1^{**} be the optimal policies for player X (taking the other player's action as given) in the learning game and risk game, respectively. Assume that U is linear in Z , and that $Z^w = CZ^x + \varepsilon + K$ for some constants C, K and independent random variable ε . Then x_1^* is increasing (decreasing) (ambiguous) in informativeness for Z^x if x_1^{**} is increasing (decreasing) (ambiguous) in uncertainty around Z^x ; x_1^* is increasing (decreasing) (ambiguous) in informativeness for ε if x_1^{**} is increasing (decreasing) (ambiguous) in uncertainty around ε .*

Proof. The proof follows the logic of Theorem 1 on page 8 and in Appendix A. ■

Theorem 2 can be extended beyond the linear assumption in two special cases: when two random variables Z^x, Z^w are independent and when $Z^x = Z^w = Z$. The theorems are stated in appendix C. The proofs are straight forward applications of the logic of the paper.

4.3.2. Equilibria

Theorem 2 is a comparative statics result for the reaction functions of a game, but the logic follows through just as well for the equilibrium points of a game. We show that if any specified equilibrium is increasing in risk in the *risk game*, then the associated equilibrium is increasing in informativeness in the *learning game*. By a specified equilibrium, we mean an equilibrium that can be described, such as the highest, lowest, or symmetric equilibrium. In addition, while we present this theorem in terms of 2 players, with minor notational differences (See the author for details) it can be shown to hold for N players.

Assume for simplicity that $Z^x = Z^w = Z$. Let the second period actions $x_2^*, w_2^*, x_2^{**}, w_2^{**}$ be defined as in Section 4.1. Define the reaction functions of the *risk game* to be player X's optimal response to action w when the random variable is Z and similarly for player W:

$$\begin{aligned} R_x(w, Z) &\equiv \arg \max_x \mathbb{E}_Z U^x(x, x_2^{**}, w, w_2^{**}; Z) \\ R_w(x, Z) &\equiv \arg \max_w \mathbb{E}_Z U^w(x, x_2^{**}, w, w_2^{**}; Z) \end{aligned}$$

Define the set of equilibrium first period actions for player X given random variable Z :

$$X(Z) = \{x | R_x(R_w(x, Z), Z) = x\}$$

Let $x(Z, \mathcal{C}) \equiv \{x | x \in X(Z), x \text{ satisfies } \mathcal{C}\}$ for some condition \mathcal{C} . Some examples of conditions follow:

- $x = \liminf \{x | R_x(R_w(x, Z), Z) = x\}$
- $x = \limsup \{x | R_x(R_w(x, Z), Z) = x\}$
- $x \in W(Z)$

$x(Z, \mathcal{C})$ is an equilibrium action specified by condition \mathcal{C} . Now consider the reaction functions of the learning game. When the payoff is linear in the random variable we can write the reaction function as follows:

$$\begin{aligned} \tilde{R}_x(w, Y) &\equiv \arg \max_x E_Y U^x(x, x_2^*, w, w_2^*; E[Z|Y]) \\ &= \arg \max_x E_{E[Z|Y]} U^x(x, x_2^*, w, w_2^*; E[Z|Y]) = R_x(w, E[Z|Y]) \end{aligned}$$

Similarly the equilibrium point specified by condition \mathcal{C} in the learning game given signal Y is equivalent to

$$X(E[Z|Y], \mathcal{C}) = \{x | x \in X(E[Z|Y]), x \text{ satisfies condition } \mathcal{C}\}$$

THEOREM 3. *Let $Z^x = Z^w = Z$. Let \mathcal{C} be a condition which specifies an equilibrium. The equilibrium specified by condition \mathcal{C} in the learning game is increasing (decreasing)(ambiguous) in informativeness if the equilibrium specified by condition \mathcal{C} in the risk game is increasing (decreasing)(ambiguous) in risk.*

Proof. Assume that $X(Z, \mathcal{C})$ is increasing in risk. Then, since $E[Z|Y]$ is a random variable with the same mean as Z , $X(E[Z|Y], \mathcal{C})$ is also increasing in risk. By Lemma 2, if Y is increasing in informativeness then $E[Z|Y]$ is increasing in riskiness. Therefore if $X(Z, \mathcal{C})$ is increasing with risk then $X(E[Z|Y], \mathcal{C})$ is increasing with informativeness. This argument can be repeated for the decreasing case. The proof for the ambiguous case follows the argument in Appendix A. ■

By applying Theorem 3 twice, we can conclude that if the lowest and highest equilibria are increasing in risk, then they are increasing in informativeness and thus we can get at “how the bounds on behavior ... change with changing parameters” (Milgrom and Roberts, 1994).

EXAMPLE 3. Climate Change II

The difficulties imposed on the climate change problem by uncertainty and learning are exacerbated by the fact that climate change is a global problem: it involves a number of nations (or coalitions) making independent decisions about climate policy. Thus, it is judicious to analyze climate policy as a non-cooperative game under uncertainty and learning. In a model similar to that found in Example 1, but in a framework of 2 non-cooperative players, Baker (2003) uses Theorem 2 to provide general results on the impacts of informativeness, showing, for example, that first period equilibrium emissions will decrease with learning in a closed loop game when damages are negatively correlated across players. Without Theorems 2 and 3, analyzing the most general effects of informativeness in non-cooperative games such as in this example had proved intractable.

5. VALUE OF INFORMATION

The paper so far focuses on the effects of better information on first period actions. But the essential finding of this paper – that risk and the expectation of learning have the same effects if the payoff is separable in the random variable – can be easily extended to analyze the value of information. In the literature, the value of information typically means the value of *perfect* information. But, in practice, most signals are not perfect, therefore it is important to know whether imperfect information has a positive value. Furthermore, the marginal value of information is needed in order to determine the demand for information.

If the value of a decision problem is increasing in informativeness, then every incremental improvement in informativeness has a non-negative value – i.e. the marginal value of information is non-negative. If, on the other hand, the value of a decision problem is ambiguous in informativeness, then the marginal value of information will be negative for some prior probability distributions in some regions. This will be true even if the value of perfect information is strictly positive.

5.1. Single Decision Maker: Compare Values of Information

In a single decision-maker framework, information always has a non-negative value⁷. This is a trivial consequence of Definition 1. The intuitive reason that this is true is that a single decision maker can always choose to disregard information. We may be interested, however, in comparing the value of information to two individual decision makers. For example we may want to know if a monopolist values information at the same rate as a social planner, or compare the value of information for two classes of utility or profit functions. The following corollary shows how we can compare the (marginal) value of information between two decision makers.

COROLLARY 1. Let u^1 and u^2 be payoff functions that are linear in $g(Z)$. Let

$$\pi^1 = \max_{x_1^1} E_y \max_{x_2^1} E_{z|y} u^1(x_1^1, x_2^1, g(Z)) \text{ and } \pi^2 = \max_{x_1^2} E_y \max_{x_2^2} E_{z|y} u^2(x_1^2, x_2^2, g(Z))$$

and define

$$D(g) \equiv \max_{x_1^1} u^1(x_1^1, x_2^1, g) - \max_{x_2^2} u^2(x_1^2, x_2^2, g)$$

Then the value of information is greater for π^1 than π^2 if $D(g)$ is convex in g .

Proof. Theorem 7 from Appendix C implies that if $D(g)$ is convex in g then $\pi^1 - \pi^2$ is increasing in informativeness. This in turn implies that any increase in informativeness increases the value of π^1 more than π^2 , hence π^1 has a greater marginal value of information than π^2 . ■

5.2. Multiple Decision Makers

When there are multiple decision-makers then public information may have a negative value⁸. If, for example, a competitor is in a better position to react to new information, a firm may prefer to forgo information on, say, demand, in order to prevent its competitor learning the same information. When a signal is public, a decision maker can no longer choose to disregard it. The following theorem can be used to determine whether public information has a positive or negative value in a non-cooperative game. The proof, which is exactly analogous to earlier proofs, is omitted.

THEOREM 4. Under the assumptions of Theorems 2, 5, and 6, the value of the payoff in the learning game is increasing (decreasing) (ambiguous) in informativeness if the value of the payoff in the risk game is increasing (decreasing) (ambiguous) in risk.

EXAMPLE 4. Investment Under Uncertainty in a Duopoly

We consider the value of imperfect information about uncertain demand received by two Cournot competitors before the firms choose output quantity. The value of information is always positive for a monopolist, but may not be for competitors. To see why, consider the effect of better information on player 1. Better (public) information will have two effects. The first is to allow player 1 to increase production if demand is high and decrease it if demand is low. This effect will always increase the value of the payoff. The second effect is on the price. If demand is high then better information will lead to a lower price than no information, because both players will produce more. It is this second effect that may lead to a negative value of information.

The players face a linear demand curve with unknown intercept Z and slope β . Firm i produces a

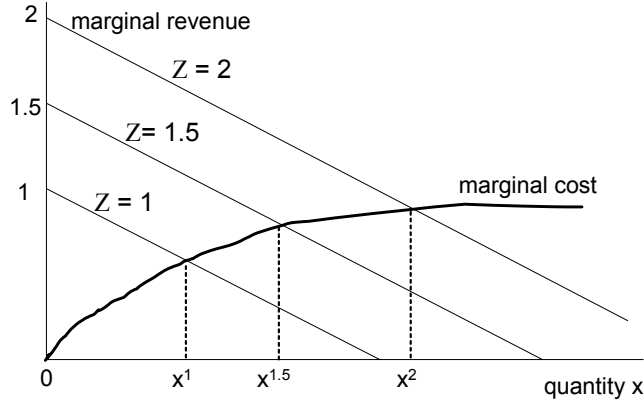


FIG. 3 The downward sloping lines represent marginal revenue for three different values of Z . Quantity x , is convex in the uncertain demand Z if marginal cost is concave.

quantity of x_i and has cost of production $c_i(x_i)$. The equilibrium profits for the firms are

$$x_i^*(Z - \beta(x_1^* + x_2^*)) - c_i(x_i^*) \quad (11)$$

where the equilibrium quantities $x_i^*, i = 1, 2$ are determined by solving the first order conditions simultaneously:

$$Z - \beta x_j - 2\beta x_i - c_i'(x_i) = 0 \quad (12)$$

Theorem 4 tells us that the the value of information is unambiguously positive if (11) is convex in the random variable Z and is ambiguous if (11) is neither convex nor concave. When the value of information is ambiguous it means that for some Z , some increases in informativeness will decrease expected profit. Note that profits (11) are always convex in Z for a monopolist; but the convexity of (11) in a game depends on the curvature of the marginal costs of the players (See Appendix C.1 for details). If the players are symmetric, then the value of information is ambiguous (to both players) if and only if marginal cost is concave. If the players are not symmetric, then the value of information is ambiguous to player i if her rival's marginal cost is concave and her marginal cost is "less concave" than her rival's (i.e. if $c_i''' > c_j'''$). If marginal cost is concave, then an increase in demand has a greater effect on the optimal quantity than a decrease in demand, hence quantity is convex in demand (See Figure 3). When quantity is convex in demand, a mean-preserving-spread in demand leads to an increase in the expected quantity produced. So, if Player 2's marginal cost is more concave than Player 1's, then Player 2 will increase his expected quantity as information increases more than Player 1 will, causing price to decrease out of proportion to Player 1's increase in quantity.

6. CONCLUSION

Uncertainty and learning are present in almost every real-world economic decision. This paper concludes that, in the absence of risk aversion, increasing risk and increasing informativeness have the same results. This conclusion holds for the reaction functions and equilibria of non-cooperative games, as well as for single decision makers, with virtually no restrictions on the payoff functions. It holds for first period decision variables and for the value of information, and can easily be shown to hold for the expected value of second period decision variables as well. The reason is that an increase in informativeness is equivalent to an increase in the variability of the prior probability distribution: more information implies more risk.

The second conclusion of this paper is that the qualitative effects of learning appear to be independent of risk attitude. The curvature of the payoff function around the random variable is irrelevant as long as the payoff function is separable in the random variable. It appears that the central problems of uncertainty and learning can be posed and solved under assumptions of risk neutrality with no loss of generality.

This result provides a simple method for determining the comparative statics of learning. Modeling the impact of learning on decisions under uncertainty is useful because it more closely represents real-life problems. In general, most learning that takes place is imperfect. One field where this is particularly useful is environmental policy, such as climate change and biodiversity, where policies are made under scientific uncertainty.

The applications of the theorems in this paper provide new insight into climate change policies. While learning does induce a "go slow" policy for a single decision maker (in the absence of a constraint on non-negative emissions), it can have the opposite effect in a non-cooperative game. In our analysis of the precautionary savings problem, we clarify the dual role of the concavity of the utility function – it represents both risk aversion and the elasticity of substitution between periods. Only the elasticity of substitution is important in the comparative statics of learning. Additionally, in the investment under uncertainty example we are able to show explicitly when information may have a negative value and relate that to the curvature of the cost function.

This paper has shed some light on decision making under uncertainty. Uncertainty and learning are very important aspects of many decisions, and most especially in environmental policy. Often the presence of uncertainty seems to lead to either policy-paralysis or policy-panic. Perhaps this work, in adding to the body of work on decision making under uncertainty, will encourage rational discussions on important issues that are fraught with uncertainty.

Proof. If $g(z)$ is invertible it means there is a one-to-one relationship between random variable g and random variable $z = g^{-1}(g)$, thus for simplicity we refer directly to random variable g throughout. Assume that x_1^* is ambiguous in risk. That means that there exist random variables g, g' and g'', g''' such that g is riskier than g' and g'' is riskier than g''' , but $x_1^{**}(g) > x_1^{**}(g')$ while $x_1^{**}(g'') < x_1^{**}(g''')$. Theorem 2.A.3 of Shaked and Shanthikumar (1994)⁹ states that there exists random variables $\hat{g}, \hat{g}', \hat{g}'', \hat{g}'''$ equal in distribution to g, g', g'', g''' (respectively) where $E[\hat{g}|\hat{g}'] = \hat{g}'$ and $E[\hat{g}''|\hat{g}'''] = \hat{g}'''$. Thus we can construct signals $Y = \hat{g}, Y' = \hat{g}'$, noting that 1) Y is more informative than Y' for \hat{g} (since it is perfectly informative) and 2) $E[\hat{g}|Y] = \hat{g}$ which is equal in distribution to g and $E[\hat{g}|Y'] = \hat{g}'$ which is equal in distribution to g' . Thus, $x_1^*(Y) = x_1^{**}(g) > x_1^{**}(g') = x_1^{**}(E[\hat{g}|Y']) = x_1^*(Y')$. Similarly we can construct signals $Y'' = \hat{g}'', Y''' = \hat{g}'''$, Y'' more informative than Y''' for \hat{g}'' but $x_1^*(Y'') < x_1^*(Y''')$. Thus, x_1^* is ambiguous in informativeness. The increasing, decreasing, and ambiguous cases together imply the converse. ■

APPENDIX B: EXTENSIONS OF THEOREM 3

THEOREM 5. *Assume Z^x, Z^w are independent random variables, and that U^x is linear in some function $g^x(Z^x)$; U^w is linear in some function $g^w(Z^w)$ [for the "ambiguous" results to hold g must additionally be invertible]. Let x_1^* and x_1^{**} be defined as above. Then x_1^* is increasing (decreasing) (ambiguous) in informativeness for Z^x if x_1^{**} is increasing (decreasing) (ambiguous) in uncertainty around $g^x(Z^x)$; x_1^* is increasing (decreasing) (ambiguous) in informativeness for Z^w if x_1^{**} is increasing (decreasing) (ambiguous) in uncertainty around $g^w(Z^w)$.*

THEOREM 6. *Assume $Z^x = Z^w = Z$, and that U^x and U^w are linear in some function $g(Z)$ [for the "ambiguous" results to hold g must additionally be invertible]. Let x_1^* and x_1^{**} be defined as above. Then x_1^* is increasing (decreasing) (ambiguous) in informativeness for Z if x_1^{**} is increasing (decreasing) (ambiguous) in uncertainty around $g(Z)$.*

APPENDIX C: VALUE OF INFORMATION

Define

$$V \equiv \sum_i c_i \max_{x_1^i} E_Y \max_{x_2^i} E_{Z|y} u^i(x_1^i, x_2^i, g(Z))$$

and

$$\bar{V} \equiv \sum_i c_i \max_{x_1^i} E_Z \max_{x_2^i} u^i(x_1^i, x_2^i, g(Z))$$

for any constants c_i .

THEOREM 7. *Let V and \bar{V} be defined as above, assuming each u^i is linear in $g(Z)$. Then (i) V is increasing (decreasing) (ambiguous) in informativeness if \bar{V} is increasing (decreasing) (ambiguous) in risk around $g(Z)$, and (ii) \bar{V} is increasing (decreasing) in risk around $g(Z)$ if $\sum_i c_i \max_{x_2^i} u^i(x_1^i, x_2^i, g(Z))$ is convex (concave) in $g(Z)$ for all x_1^i .*

Proof. The proof of (i) follows the same logic as the proof of Theorem 1 on page 8. To prove (ii) we rewrite

$$\begin{aligned} \bar{V} &\equiv \sum_i c_i \max_{x_1^i} \mathbb{E}_Z \max_{x_2^i} u^i(x_1^i, x_2^i, g(Z)) \\ &= \sum_i c_i \max_{x_1^i} \mathbb{E}_Z u^i(x_1^i, x_2^{i*}(x_1^i, g(Z)), g(Z)) \\ &= \sum_i c_i \mathbb{E}_Z u^i(x_1^{i*}(F), x_2^{i*}(x_1^{i*}, g(Z)), g(Z)) \\ &= \mathbb{E}_Z \sum_i c_i u^i(x_1^{i*}(F), x_2^{i*}(x_1^{i*}, g(Z)), g(Z)) \end{aligned}$$

where F is the distribution function of Z and

$$x_1^{i*}(F) \equiv \arg \max_{x_1^i} \mathbb{E}_Z u^i(x_1^i, x_2^{i*}(x_1^i, g(Z)), g(Z))$$

If $\sum_i c_i u^i(x_1^i, x_2^{i*}(x_1^i, g(Z)), g(Z))$ is convex (concave) in $g(Z)$ for all x_1^i then it is convex (concave) for the optimal $x_1^{i*}(F)$, therefore, \bar{V} is the expected value of a convex (concave) function and thus increasing (decreasing) in risk by direct application of Definition 2. ■

C.1. Duopolists

Applying the envelope theorem and differentiating (11) twice we get

$$\frac{\partial x_i}{\partial Z} \left(1 - \beta \frac{\partial x_j}{\partial Z} \right) - x_i^* \beta \frac{\partial^2 x_j}{\partial Z^2} \quad (13)$$

To determine the sign of (13) we totally differentiate the first order conditions (12) and apply Cramer's rule to calculate the total effect of Z on x_i , then differentiate a second time:

$$\begin{aligned} \frac{\partial x_i}{\partial Z} &= \frac{\beta + c_j''}{(2\beta + c_i'')(2\beta + c_j'') - \beta^2} \\ \frac{\partial^2 x_i}{\partial Z^2} &= \frac{\beta(2\beta + c_i'')(2\beta + c_i''')c_j''' - (2\beta + c_j'')(2\beta + c_j''')c_i'''}{[(2\beta + c_i'')(2\beta + c_j'') - \beta^2]^3} \end{aligned}$$

Substituting these expressions into (13) gives

$$\frac{(\beta + c_j'')^2 (2\beta + c_i'')}{((2\beta + c_i'') (2\beta + c_j'') - \beta^2)^2} \tag{14}$$

$$-x_i^* \beta \frac{\beta (2\beta + c_j'') (\beta + c_j'') c_i''' - (2\beta + c_i'') (\beta + c_i'')^2 c_j'''}{[(2\beta + c_i'') (2\beta + c_j'') - \beta^2]^3}$$

The second order conditions require that $2\beta + c_i'' > 0$, $i = 1, 2$ and stability conditions say that $(2\beta + c_i'') (2\beta + c_j'') - \beta^2 > 0$. Therefore the first term in (14) is positive and the sign of the second term depends on c_i''' $i = 1, 2$.

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Notes

¹We adopt the convention that random variables need be defined with probability one only (see Galambos, 1995).

²Rothschild and Stiglitz defined risk in terms of concave functions: $E_Z U(Z) \leq E_{Z'} U(Z')$ for all concave U . The two definitions are equivalent.

³For a more general version of the theorem see Corollary 1 in Athey (2000) or Baker (2002).

⁴See Baker (Forthcoming)(Baker, ming) for more details.

⁵See Gollier (2001) for a discussion.

⁶In this formulation we are restricting the strategies to pure strategies. However, x_i, w_i could be interpreted as mixed strategies, with each x_i representing a probability distribution over some set of actions. Then we could define $x_i > x_i'$ if, say, $E x_i > E x_i'$.

⁷This assumes that the signal itself does not directly effect the payoff function. See Sulganik and Silcha (1997) or Datta, Mirman and Schlee (2000) for a discussion of this case.

⁸See Hirshleifer (1971) for the most well-known example. Zhu and Weyant (2003) presents an example in a strategic framework.

⁹See Theorem 1.5.20 of Muller and Stoyan (2002) for presentation of the proof.

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