Bounds for Signal to Noise Ratio in Analog to Information Converter Systems

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We wish to characterize the SNR of the analog-to-information system using known analysis of CS performance. We present a theorem for sparse signals, i.e. for $\mathbf{x} \in \ell_0$.

Theorem 1. *Let $\mathbf{x} \in \mathbb{R}^n$ be an K-sparse signal, i.e. $\|\mathbf{x}\|_0 = K$, and let $\mathbf{y} = \Phi\mathbf{x}$ represent an Analog to Information Converter (AIC) measurement setup, where we label reconstruction from the measurements $\mathbf{y}$ as $\mathbf{x}^\sharp$ with AIC reconstruction using Basis Pursuit with Denoising (BPDN) [1]. If Φ holds the K-Modified Restricted Isometry Property (MRIP) with constant C_K and if $C_{3K} + 3C_{4K} < 6$, then the SNR of the AIC system obeys the lower bound*

$$SNR_{AIC} := 20 \log \left(\frac{\|\mathbf{x}\|_2}{\|\mathbf{x}^\sharp - \mathbf{x}\|_2} \right) \geq SNR_{ADC} - 20 \log(C_K C_{1,K})$$

where SNR_{ADC} is the SNR of the (low-rate) ADC that acquires the measurement vector $\mathbf{y}$:

$$SNR_{ADC} = 20 \log \left(\frac{\|\mathbf{y}\|_2}{\epsilon} \right), \tag{1}$$

and $C_{1,K}$ is a constant depending only on K.

The theorem states that the SNR of the AIC setup will be equal to the SNR of the ADC that acquires the measurements, with a bounded decay due to the CS compression/reconstruction process. The term $\mathrm{SNR}_{\mathrm{ADC}}$ will account for all the noise sources of the measurement acquisition hardware, including quantization, jitter, etc. The condition on the MRIP constants holds for random gaussian matrices when the number of rows M is large enough, i.e. when $M \geq CK \log(N/K)$ with C a constant [2]. Specifically, the constant C_K can be made arbitrarily close to 1 by increasing the value of M . For deterministic matrices, the constants can be calculated in combinatorial time. Numerical results can be derived for gaussian random matrices; for example, if M is large enough so that $C_S = 1.25$, then $C_{1,K} = 12.77$ [3] and $\mathrm{SNR}_{\mathrm{AIC}} \geq \mathrm{SNR}_{\mathrm{ADC}} - 23$ dB.

Proof. From [3], we have the following bound on the quality of reconstruction from BPDN that

$$\|\mathbf{x}^\sharp - \mathbf{x}\|_2 \leq C_{1,K}\epsilon + C_{2,K}\frac{\|\mathbf{x}-\mathbf{x}_K\|_1}{\sqrt{K}}$$

where $\mathbf{x}_K$ is the best K-term approximation of $\mathbf{x}$ and both C's are constants which depend on K. Substituting this expression we find

$$\mathrm{SNR}_{\mathrm{AIC}} \geq -20\log\left(\frac{C_{1,K}\epsilon}{\|\mathbf{x}\|_2} + C_{2,K}\frac{\|\mathbf{x}-\mathbf{x}_K\|_1}{\|\mathbf{x}\|_2\sqrt{K}}\right). \tag{2}$$

For $\mathbf{x}$ strictly sparse, there is no error resulting from an K-term approximation, so we have

$$\mathrm{SNR}_{\mathrm{AIC}} \geq -20\log\left(\frac{C_{1,K}\epsilon}{\|\mathbf{x}\|_2}\right).$$

We use the K-MRIP to achieve a bound on the system SNR. K-MRIP is defined as the inequality

$$C_K^{-1}\|\mathbf{x}\|_2 \leq \|\mathbf{y}\|_2 \leq C_K\|\mathbf{x}\|_2$$

for all $\mathbf{x}$ with $\|\mathbf{x}\|_0 \leq K$, and from this we get

$$\frac{C_{1,K}\epsilon}{\|\mathbf{x}\|_2} \leq \frac{C_K C_{1,K}\epsilon}{\|\mathbf{y}\|_2}$$

and so

$$\mathrm{SNR}_{\mathrm{AIC}} \geq -20\log\left(\frac{C_K C_{1,K}\epsilon}{\|\mathbf{y}\|_2}\right)$$

and finally putting the expression in a meaningful form

$$\mathrm{SNR}_{\mathrm{AIC}} \geq 20\log\left(\frac{\|\mathbf{y}\|_2}{\epsilon}\right) - 20\log(C_K C_{1,K}) = \mathrm{SNR}_{\mathrm{ADC}} - \log(C_K C_{1,K})$$

□

This lower bound on the AIC SNR is given by the SNR of the sampling system minus a constant amount due to CS. Since $C_K C_{1,K} \geq 1$, there will be some SNR loss as expected.

References

[1] S. Chen, D. Donoho, and M. Saunders, "Atomic decomposition by basis pursuit," *SIAM J. on Sci. Comp.*, vol. 20, no. 1, pp. 33–61, 1998.

[2] E. Candès and T. Tao, "Decoding by linear programming," *IEEE Trans. Inform. Theory*, vol. 51, pp. 4203–4215, Dec. 2005.

[3] E. Candès, J. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Communications on Pure and Applied Mathematics*, vol. 59, no. 8, pp. 1207–1223, Aug. 2006.