Recovery of Frequency-Sparse Signals from Compressive Measurements

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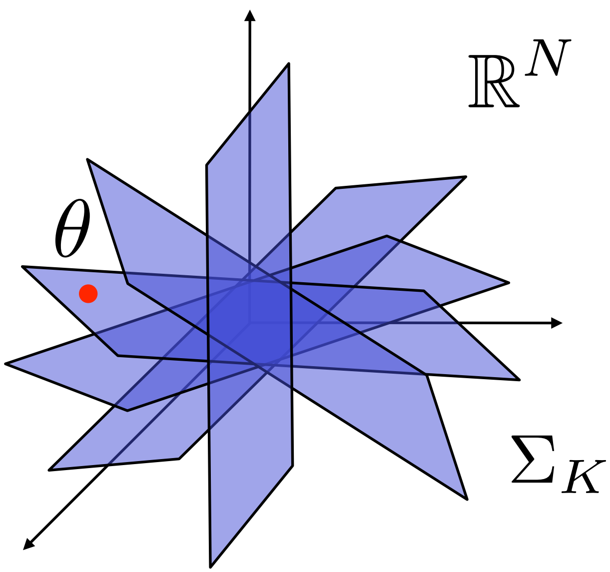
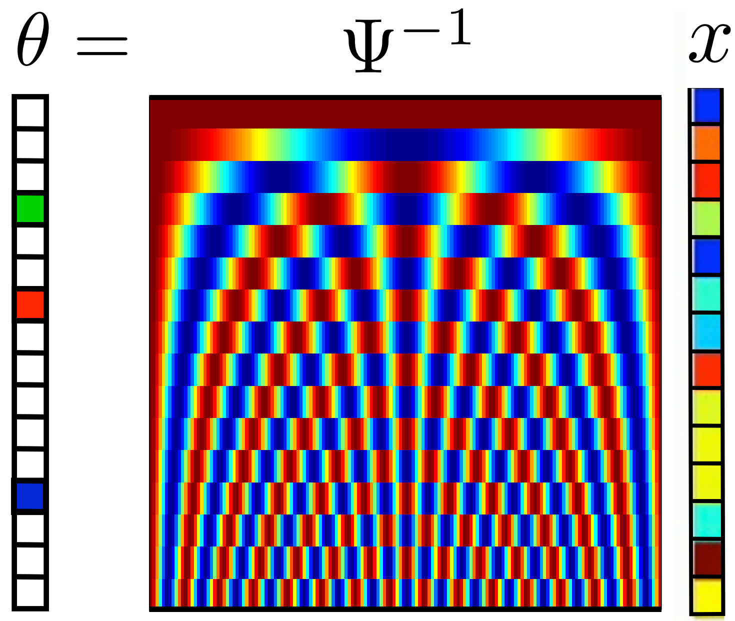
Concise Signal Structure

- **Sparse** signal: only K out of N coefficients nonzero
  - model: union of K-dimensional subspaces aligned w/ coordinate axes

$$x = \sum_n \psi_n \theta_n$$

$$x = \Psi \theta$$





$N \times 1$

sparse signal

K

nonzero entries

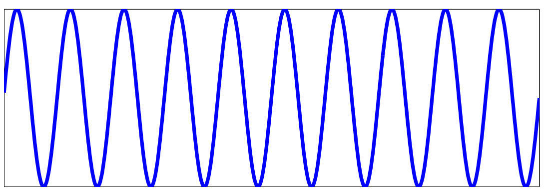
DFT Basis
for frequency-sparse signals

Frequency-Sparse Signals and the DFT Basis

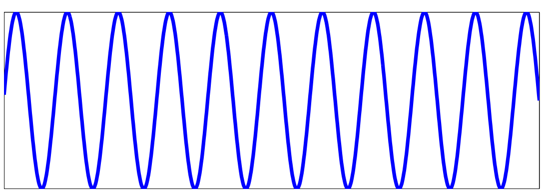
$$x = \sum_{k=1}^{K} a_k e(f_k) \qquad X(\omega) = \sum_{k=1}^{K} a_k \, \delta(\omega - \omega_k)$$

$$e(f) = \left[1 e^{j2\pi f/N} \;\; e^{j2\pi 2f/N} \;\; \dots \;\; e^{j2\pi(N-1)f/N} \right]$$

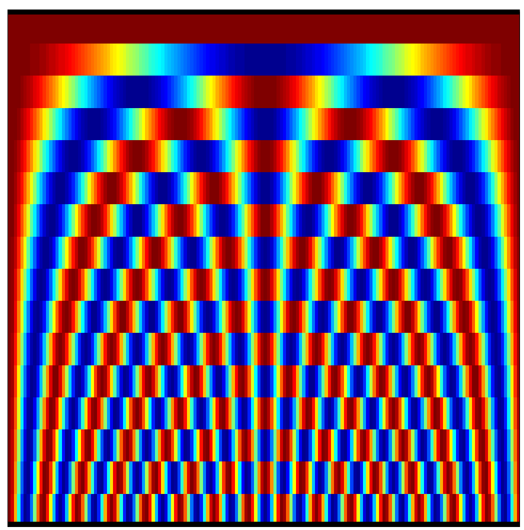
$$\theta = \Psi^{-1} x$$



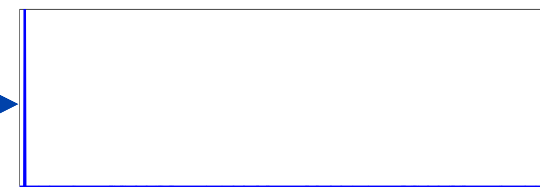
$x[n] = \sin\left(\frac{2\pi n}{N} \times 10\right)$



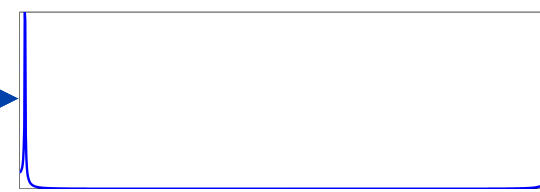
$x[n] = \sin\left(\frac{2\pi n}{N} \times 10.5\right)$



$N = 1024$

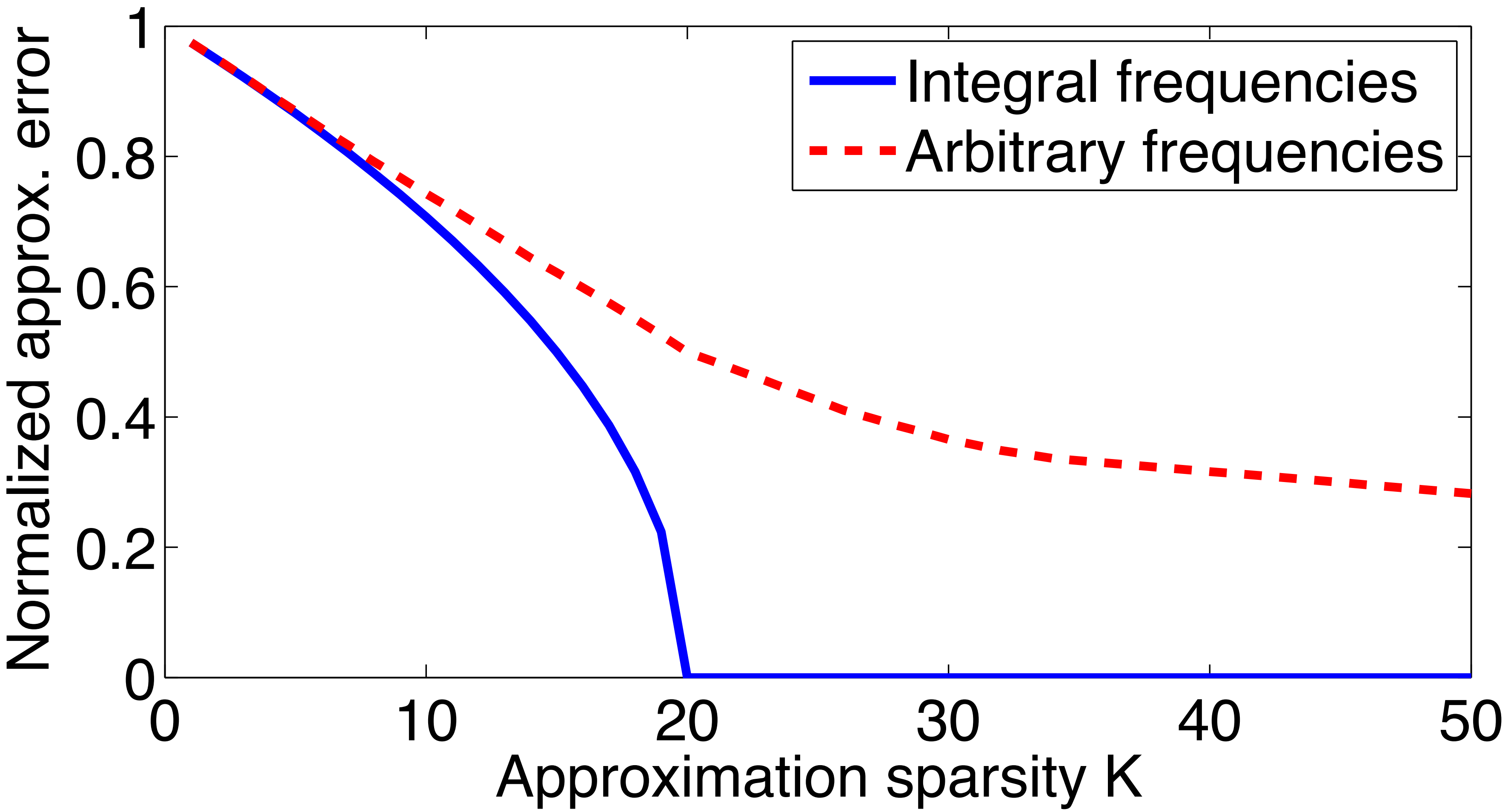


$\|\theta\|_0 = 2, \|\theta - \theta_2\|_2 = 0$



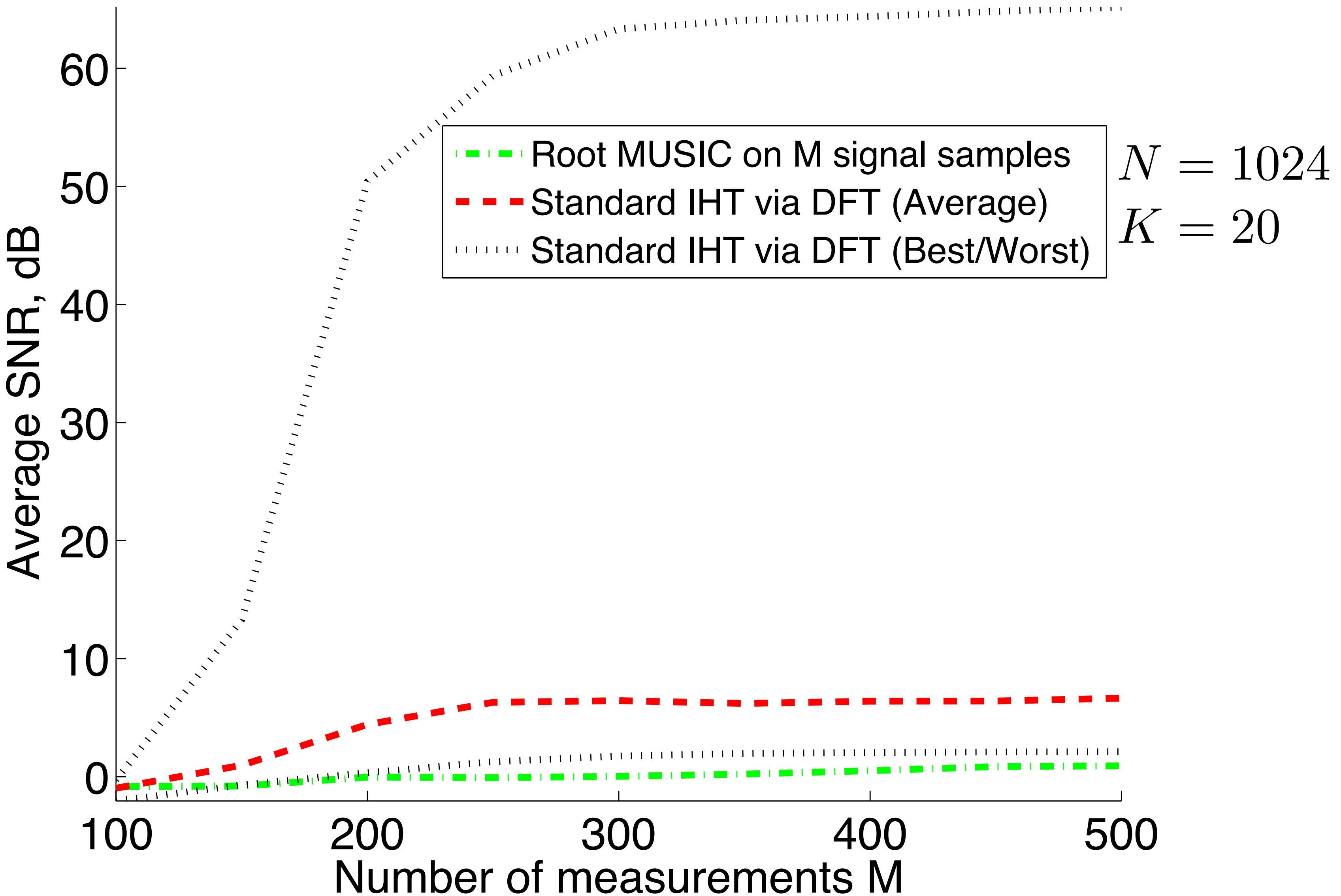
$\|\theta\|_0 = 1024,$
$\|\theta - \theta_2\|_2 = 0.76\|\theta\|_2$

Frequency-Sparse Signals and the DFT Basis

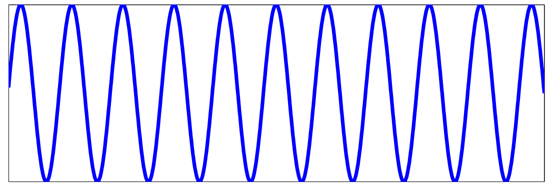


Signal is sum of 10 sinusoids

Compressive Sensing for Frequency-Sparse Signals



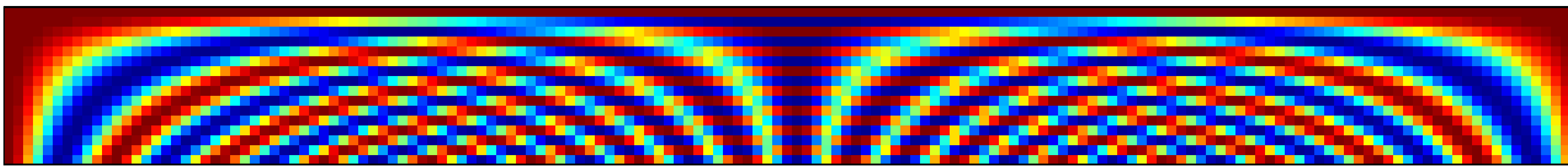
The Redundant DFT Frame



$x[n] = \sin\left(\frac{2\pi n}{N} \times 10.5\right)$

$N = 1024$

$\Psi(c),\ c = 10$



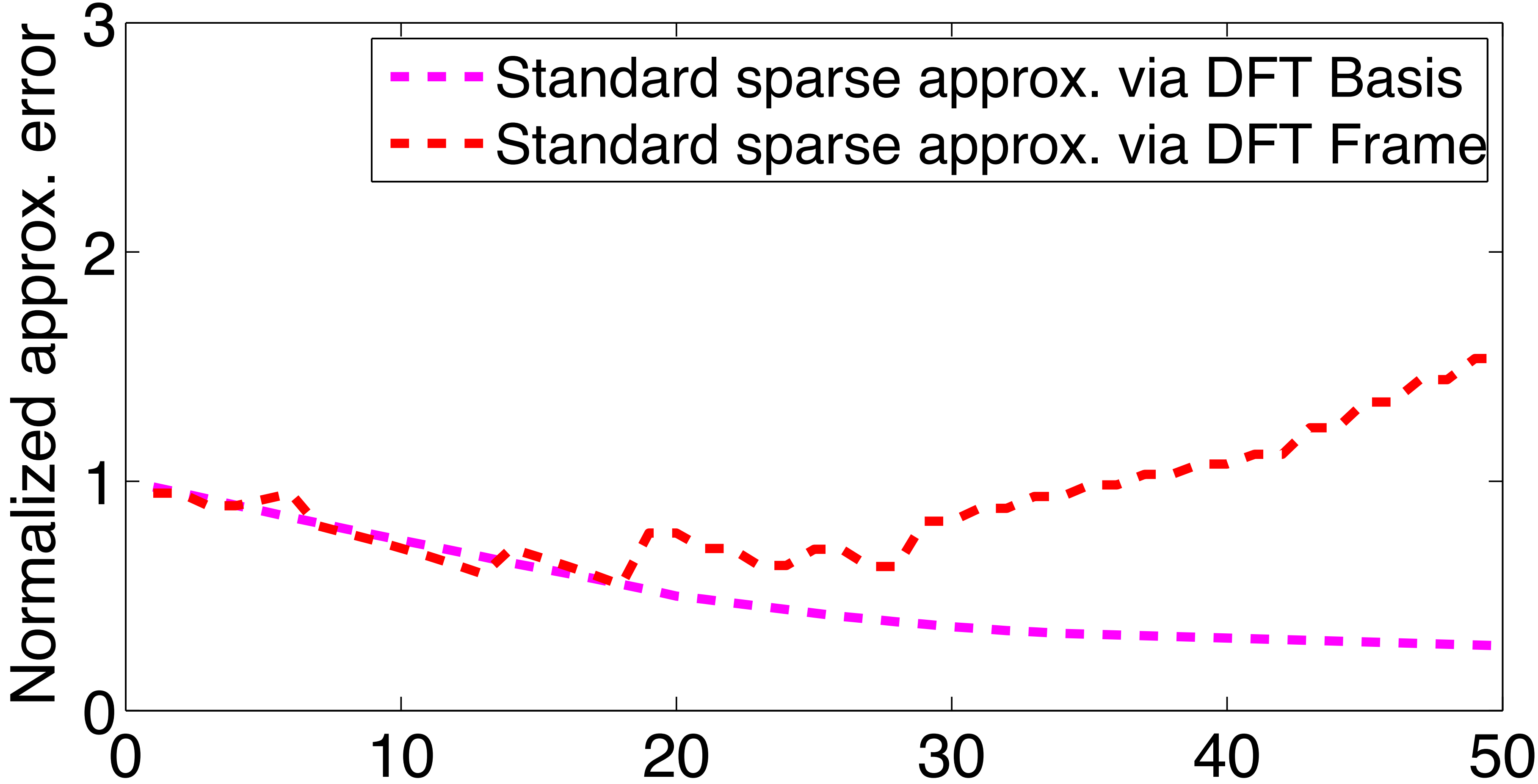
$x = \Psi(c)\theta$

$\|\theta\|_0 = 2, \|\theta - \theta_2\|_2 = 0$

$\mu(\Psi(c)) \approx 0.98$

Sparse approximation algorithms fail

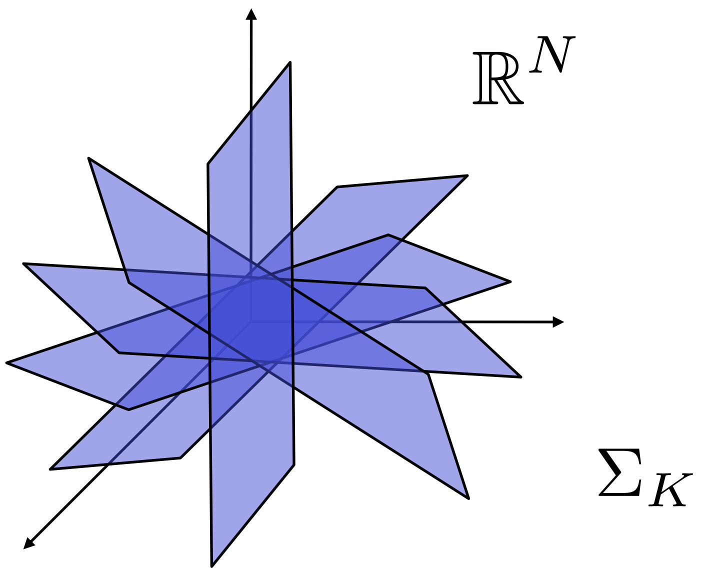
Sparse Approximation of Frequency-Sparse Signals

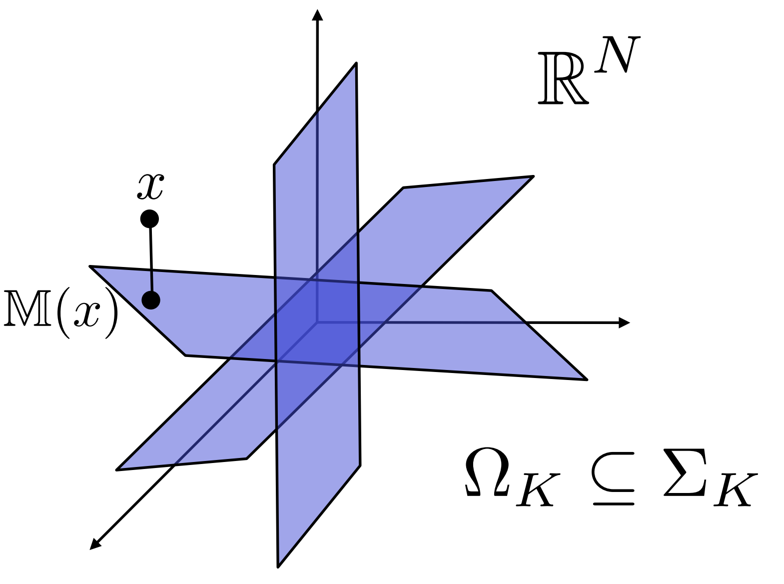


Signal is sum of 10 sinusoids at arbitrary frequencies

Structured Sparse Signals

- A K**-sparse** signal lives on the collection of K-dim subspaces aligned with coordinate axes
- A K**-structured sparse** signal lives on a particular (reduced) collection of K-dimensional canonical subspaces

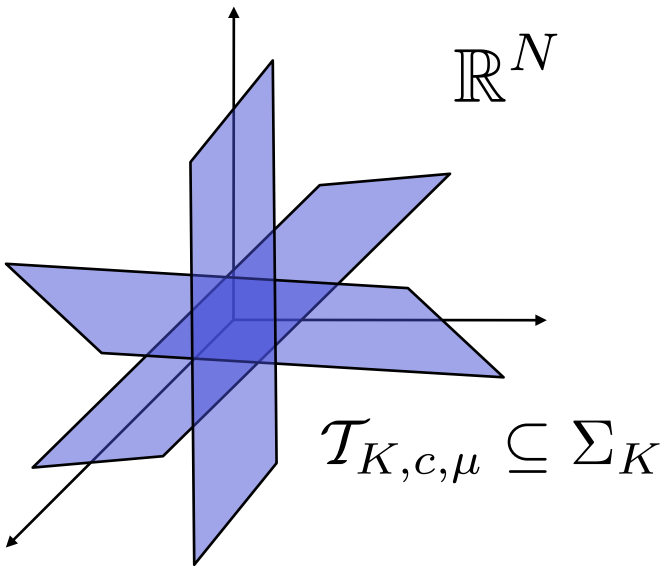




Novel ***structured sparse approximation algorithms*** find closest projection of arbitrary signals into union of subspaces

[Baraniuk, Cevher, Duarte, Hegde 2010]

Structured Frequency-Sparse Signals

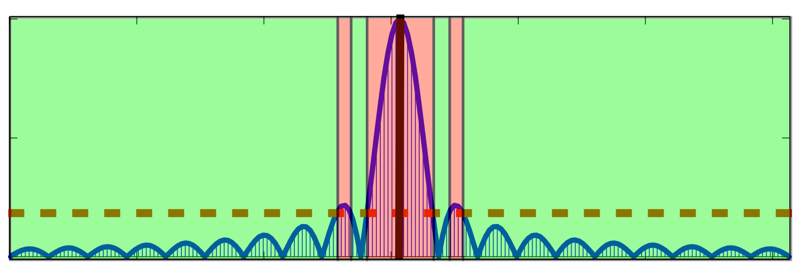


- A ***K*-structured frequency-sparse** signal x consists of K sinusoids that are mutually incoherent:

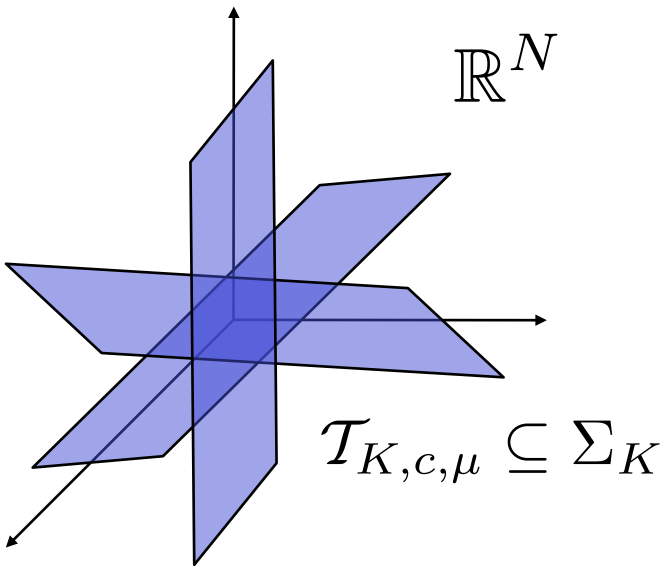
$$x = \sum_{k=1}^{K} a_k e(f_k) \in \mathcal{T}_{K,c,\mu} \text{ if}$$

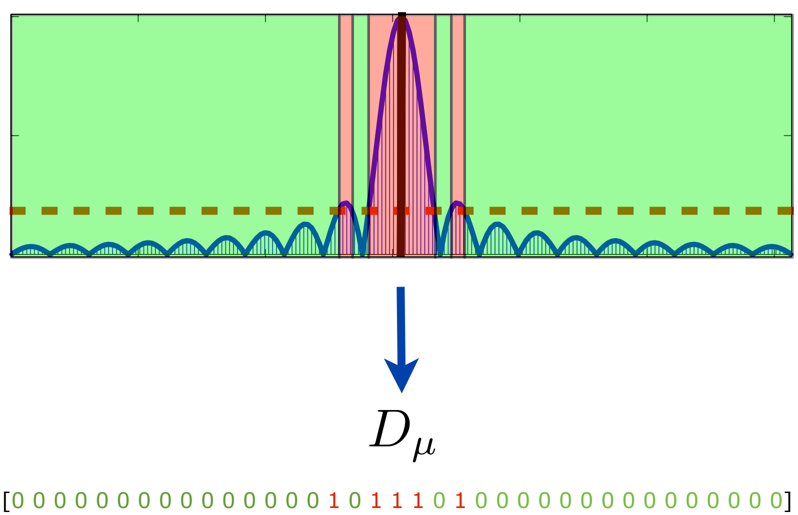
$$cf_K \in \mathbb{Z}, \; |\langle e(f_k), e(f_{k'}) \rangle| \leq \mu \; \forall \; k \neq k'$$

- If x is K-structured frequency-sparse, then there exists a K-sparse vector θ such that $x = \Psi(c)\theta$ and the nonzeros in θ are spaced apart from each other.



Structured Sparse Approximation





Algorithm 1: $\mathbb{T}(\theta, K, c, \mu)$

Integer Program

Inputs:

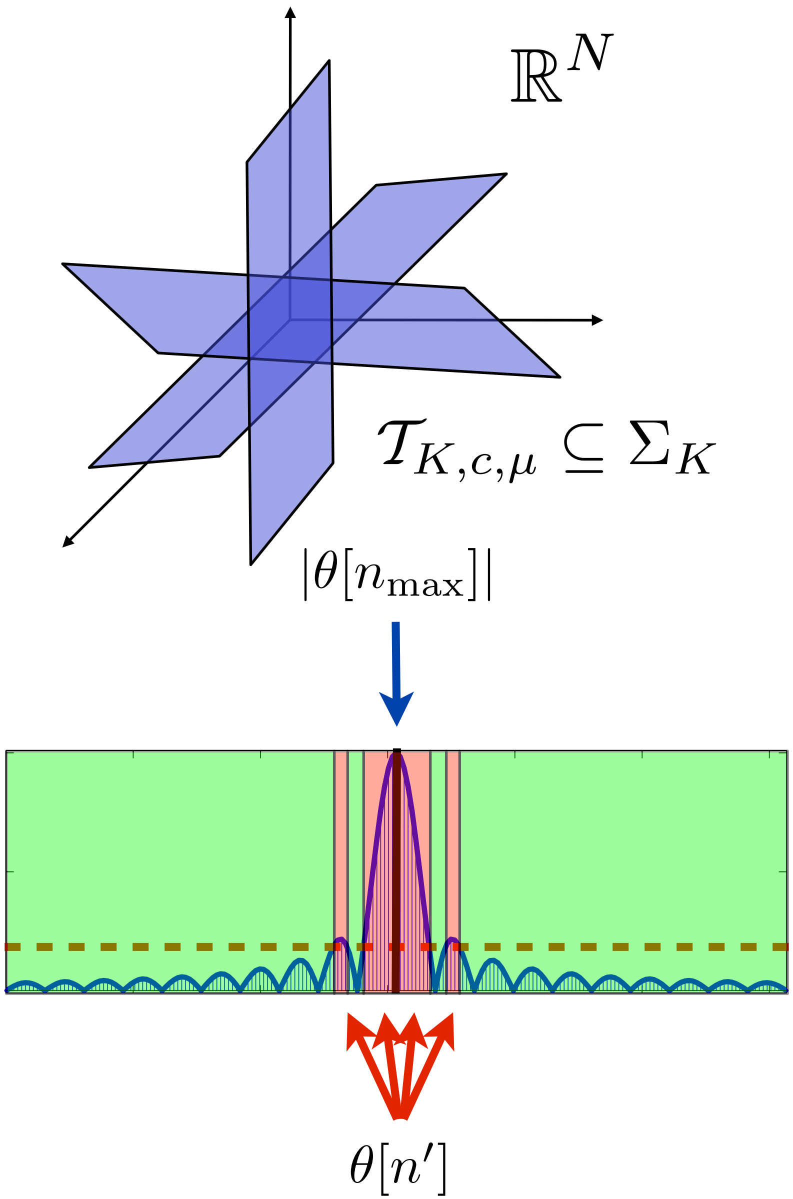
- Coefficient vector θ
- Target sparsity K
- Redundancy factor c
- Maximum coherence μ

Output:

- Sparse coefficient vector $\widehat{\theta}$

- Initialize:
  $w_\theta[i] = \theta[i]^2, \; i = 0, \ldots, cN-1$
- Solve support:
  $$s = \arg \max_{s \in \{0,1\}^{cN}} w_\theta^T s$$
  $$\text{s.t. } D_\mu s \leq \mathbf{1}, s^T \mathbf{1} \leq K$$
- Mask coefficients:
  $\widehat{\theta}[i] \leftarrow \theta[i] s[i], i = 0, \ldots, cN-1$

Structured Sparse Approximation



Algorithm 2: $\mathbb{T}_h(\theta, K, c, \mu)$

Inhibition Heuristic

Inputs:

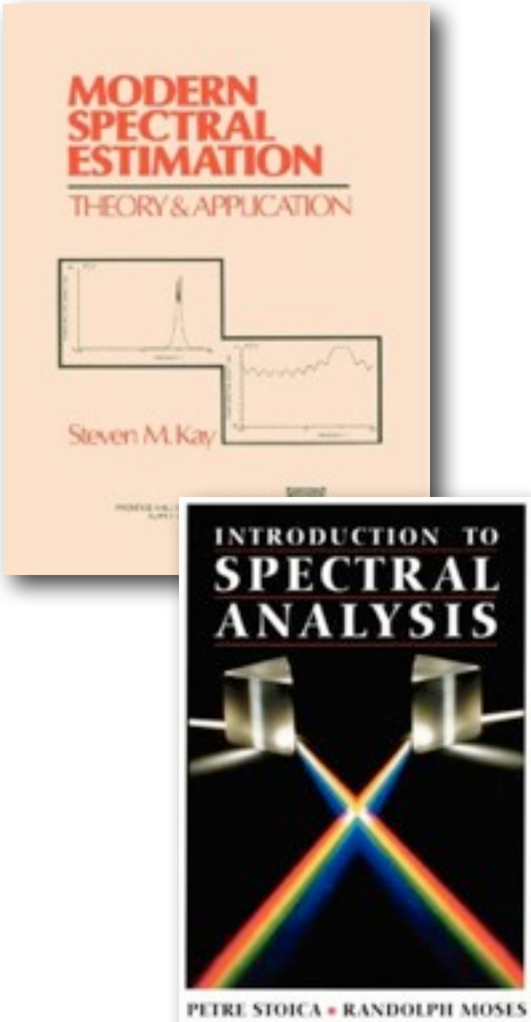
- Coefficient vector θ
- Target sparsity K
- Redundancy factor c
- Maximum coherence μ

Output:

- Sparse coefficient vector $\widehat{\theta}$

- Initialize: $\widehat{\theta}[d] = 0, \ d = 0, \ldots, cN - 1$
- While θ is nonzero and $\|\widehat{\theta}\|_0 \leq K$,
  - Find max abs entry $|\theta[n_{\max}]|$ of θ
  - Copy entry $\widehat{\theta}[n_{\max}] = \theta[n_{\max}]$
  - Inhibit "coherent" entries $\theta[n'] = 0$
  - Repeat

Structured Sparse Approximation



Algorithm 3: $\mathbb{T}_l(x, K)$

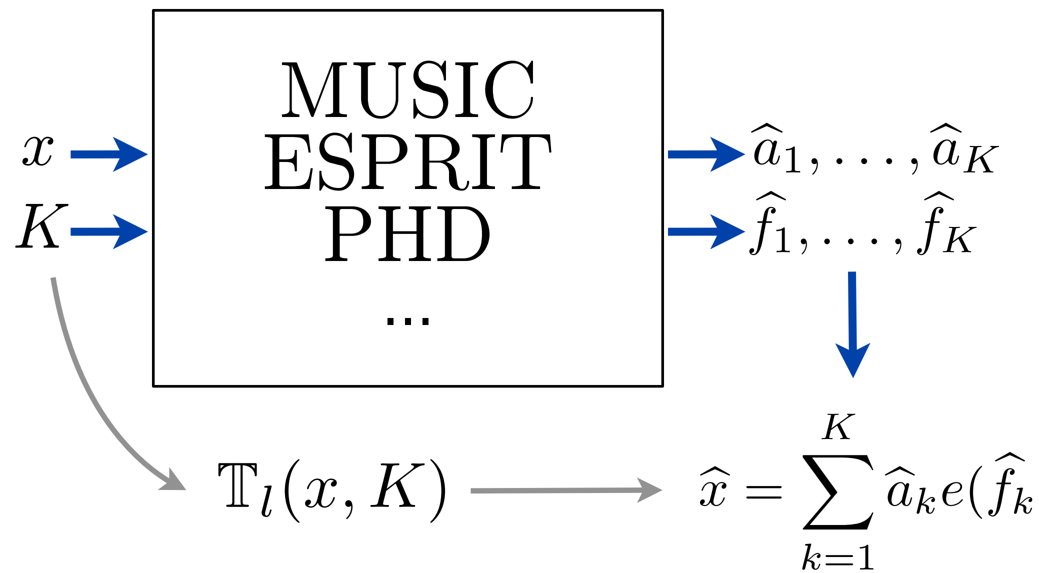
Line Spectral Estimation

Inputs:

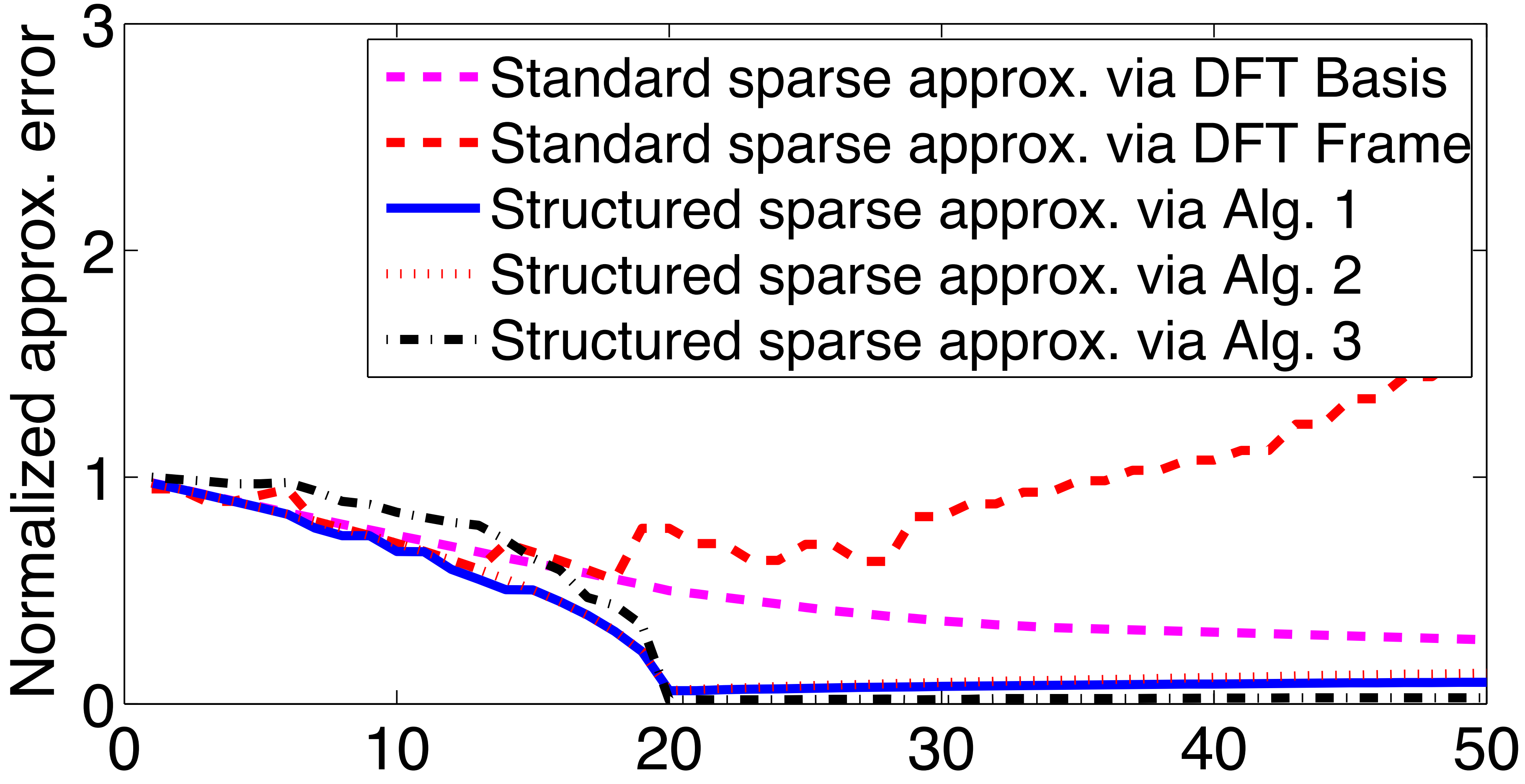
- Signal vector x
- Target sparsity K

Output:

- Parameter estimates $\widehat{a}_1, \ldots, \widehat{a}_K$
  $\widehat{f}_1, \ldots, \widehat{f}_K$
- Signal estimate $\widehat{x}$



Sparse Approximation of Frequency-Sparse Signals



Signal is sum of 10 sinusoids at arbitrary frequencies

Standard Sparse Signal Recovery

Iterative Hard Thresholding

Inputs:

- Measurement vector y
- Measurement matrix Φ
- Sparsity K

Output:

- Signal estimate $\widehat{\theta}$

- Initialize: $\widehat{\theta}_0 = 0,\ r = y,\ i = 0$
- While halting criterion false,
  - $i \leftarrow i + 1$
  - $b \leftarrow \widehat{\theta}_{i-1} + \Phi^T r$ *(estimate signal)*
  - $\widehat{\theta}_i \leftarrow \mathcal{T}(b, K)$ *(obtain best sparse approx.)*
  - $r = y - \Phi\widehat{\theta}_i$ *(calculate residual)*
- Return estimate $\widehat{\theta}$

Structured Sparse Signal Recovery

Spectral Iterative Hard Thresholding (SIHT)

Inputs:

- Measurement vector y
- Measurement matrix Φ
- DFT Frame $\Psi(c)$
- Structured sparse approx. algorithm $\mathbb{T}(\theta, K, c, \mu)$

Output:

- Signal estimate $\widehat{x}$

- Initialize: $\widehat{\theta}_0 = 0,\ r = y,\ i = 0$
- While halting criterion false,
  - $i \leftarrow i + 1$
  - $b \leftarrow \mathbb{T}(\widehat{\theta}_{i-1} + \Psi^T \Phi^T r, cN, c, \mu)$ *(estimate signal)*
  - $\widehat{\theta}_i \leftarrow \mathbb{T}(b, K, c, \mu)$ *(obtain best* ***structured*** *sparse approx.)*
  - $r = y - \Phi \widehat{\theta}_i$ *(calculate residual)*
- Return estimate $\widehat{x} = \Psi(c)\widehat{\theta}_i$

Algorithms 1, 2 & 3

Model-Based SCS Recovery via SIHT

Theorem:
Assume we obtain noisy CS measurements of a signal ensemble $y = \Phi\Psi(c)\theta + n$. If $\Phi\Psi(c)$ has the model-based RIP with $\delta < 0.1$, then the output of the SIHT algorithm obeys

$$\underbrace{\|\theta - \widehat{\theta}\|_2}_{\text{CS recovery error}} \leq C_1 \underbrace{\|\theta - \mathbb{T}(\theta, K, c, \mu)\|_2}_{} + \frac{C_2}{\sqrt{K}} \underbrace{\|\theta - \mathbb{T}(\theta, K, c, \mu)\|_1}_{} + C_3 \underbrace{\|n\|_2}_{\text{noise}}$$

CS recovery error

signal K-term

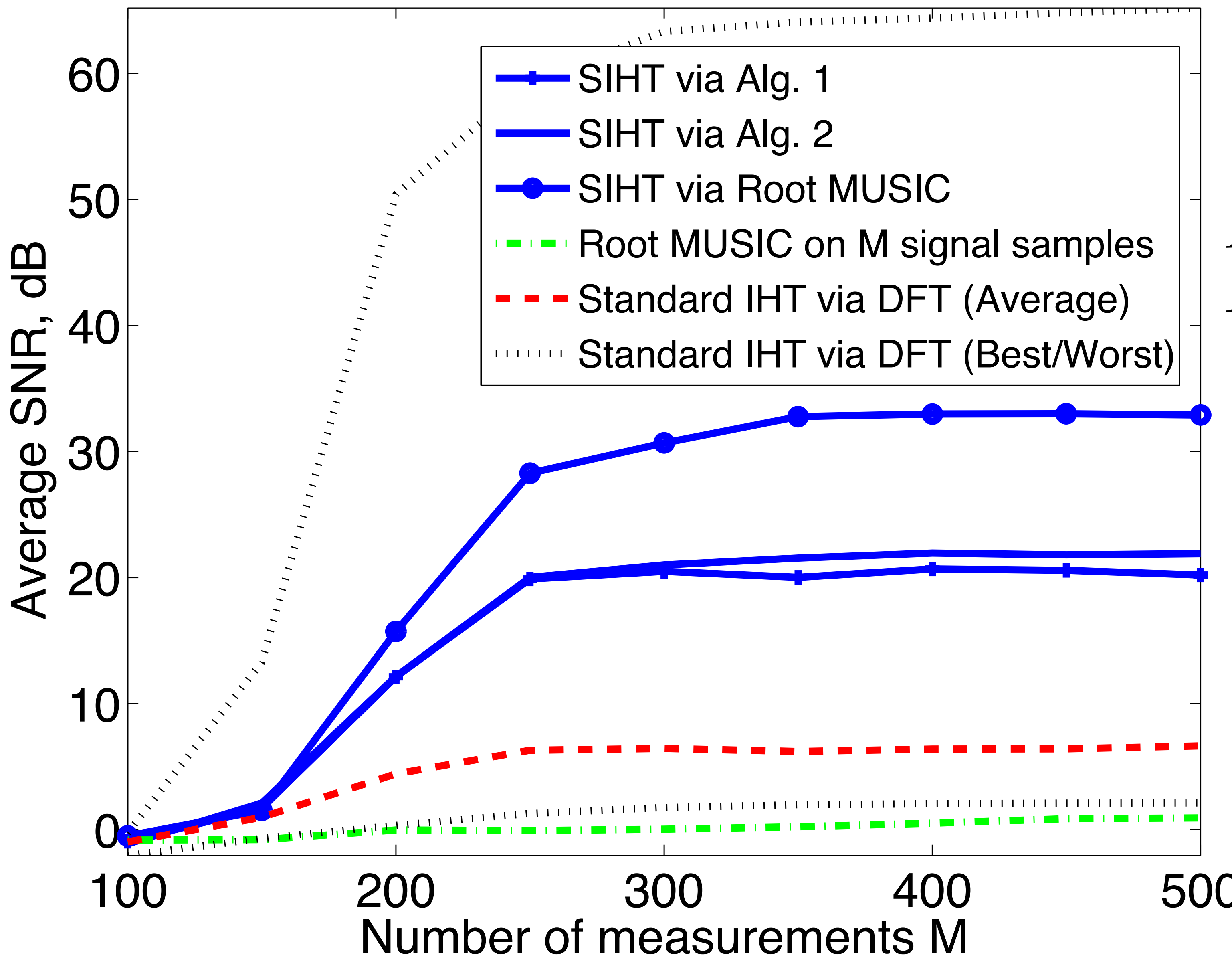
structured sparse approximation error

noise

In words, *instance optimality* based on *structured sparse approximation*

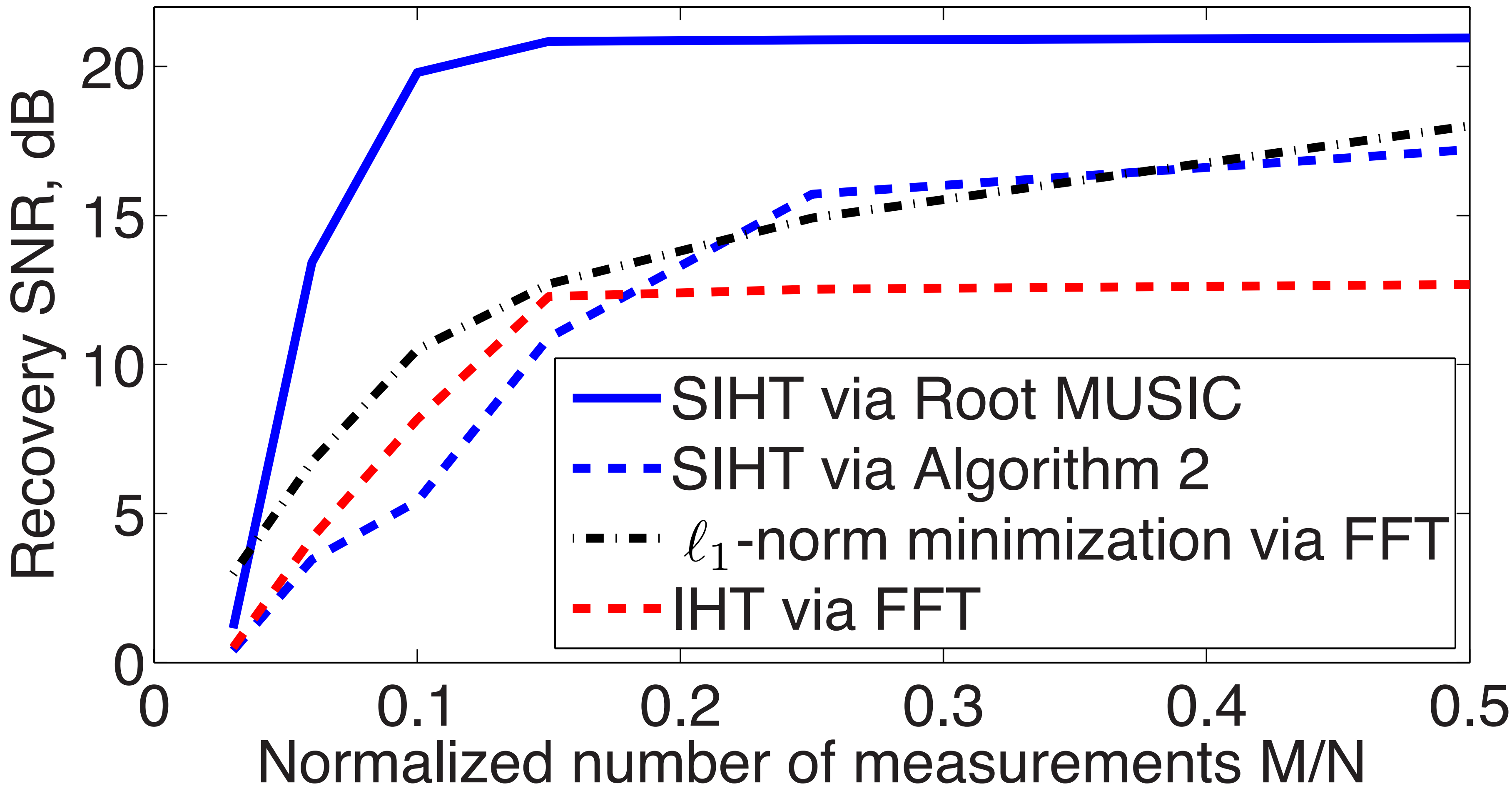
[Baraniuk, Cevher, Duarte, Hegde 2010]

Structured CS: Performance



$N = 1024$
$K = 20$

Real-World Frequency-Sparse Signals



Amplitude Modulated Signal, $N = 32768$

[Tropp, Laska, Duarte, Romberg, Baraniuk 2010]

Conclusions

- Spectral CS provides significant improvements on frequency-sparse signal recovery
  - address ***coherent dictionaries*** via structured sparsity
  - ***simple-to-implement*** modifications to recovery algs
  - can leverage decades of work on ***spectral estimation***
  - robust to model mismatch, presence of noise
- Current and future work:
  - leveraging more sophisticated algorithms
  - extension: CS recovery via ***parametrizable dictionaries***
  - localization, bearing estimation, radar imaging, ...
  - characterization of parameter estimation from compressive measurements
  - dictionary elements as samples from manifold models

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