MULTI-BRANCH BINARY MODULATION SEQUENCES FOR INTERFERER REJECTION

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ABSTRACT

When the techniques of random modulation are used in wideband communication receivers, one can design spectrally shaped sequences that mitigate interferers while preserving messages to reduce distortion caused by amplifier nonlinearity and noise. For sampling rates that are too high for standard modulation, one can instead rely on multi-branch architectures involving multiple modulators working at reduced sampling rates. In this paper, we propose an algorithm to design a set of binary sequences to be used in multi-branch modulation to mitigate a strong interferer while allowing for stable message recovery. The implementation consists of a quadratic program that is relaxed into a semidefinite program combined with a randomized projection. While interferer signals are often modeled as a subspace under the discrete Fourier transform, spectrum leakage occurs when the signal contains so-called off-grid frequencies. The Slepian basis provides a much better-suited representation for such bandlimited signals that mitigates spectrum leakage. We use both representations during the evaluation of our design algorithm, where numerical simulations show the advantages of our sequence designs versus the state of the art.

Index Terms— multi-branch modulation, sequence design, semidefinite programming, randomized projection, Slepian basis.

1. INTRODUCTION

Receivers for emerging wireless communication systems are expected to deal with a very wide spectrum and adaptively choose which parts of it to extract. A major issue for such receivers is to process spectra having very weak signals from a distant source mixed with strong signals from nearby interferers. Recently, random sequences for wideband signal modulation have been employed in the realization of communication system receivers [1–3]. In essence, the measurements from random modulation and low-pass filtering contain a baseband spectrum that is the linear combination of all frequency components of the input signal. As the size of the bandwidth of interest increases, the necessary modulator sampling rates can be reduced by relying on a multi-branch modulation architecture that employs multiple sequences.

In many cases, when the locations of one or more interferers is known, a modulation to mitigate the interferer is desirable to reduce the distortion due to the amplifier nonlinearity. Therefore, it is promising to replace commonly used pseudorandom sequences with spectrally shaped sequences that effectively implement a notch filter to suppress interferers. Furthermore, the sequences used in the different branches should provide a representation that is not only invertible but well conditioned in order to be amenable to the presence of noise in the received signal and in the operation of the hardware.

While the Fourier basis is commonly used to represent the interferer band as a subspace, it suffers from so-called spectrum leakage: the energy of a single-tone signal in the representation of the Fourier basis leaks to all basis elements whenever the frequency of the signal is not contained in the set of frequencies sampled by the Fourier basis, which are known as the on-grid frequencies. Alternatively, the Slepian basis captures most of the energy of a signal in just a few coefficients regardless of its frequency, as long as the frequency is within a sufficiently compact band of the spectrum [4–7].

In prior work, we proposed a algorithm to design a spectrally shaped binary modulation sequence that provides a passband and notch for the pre-determined message and interferer bands, respectively [8]. The sequence design problem was set as a *quadratically constrained quadratic program* (QCQP) that uses Fourier basis representations for the message and interferer bands. The optimization problem is relaxed into a *semi-definite program* (SDP) relaxation followed by a randomized projection.

In this paper, we extend this previous work towards an algorithm for the design of a set of binary sequences used in multi-branch modulation for signal acquisition. The sequences are capable of rejecting the interferer band while providing a stably invertible linear measurement operator, which will be amenable for use in real-world setting where the obtained observations are corrupted by noise and other nonidealities. We include numerical experiments that verify the good performance of our designs when a sufficient number of randomized projections far below the size of the exhaustive search space is used in the sequence design. We also compare the algorithm's performance when the Fourier and Slepian bases are used to represent interferer signals, focusing in cases where the interferer frequencies are on-grid and off-grid.

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2. BACKGROUND

We define an N-dimensional complex exponential vector as $\mathcal{F}(f) = \frac{1}{\sqrt{N}}\left[1, e^{j2\pi f}, \ldots, e^{j2\pi(N-1)f}\right]^T$, where $f \in \mathcal{M} = [0, 1]$ is the corresponding normalized frequency. The elements of the Fourier basis $\mathcal{F}_m = \mathcal{F}(f_m)$ ($m = 1, 2, \ldots, N$) sample the normalized frequency range Ω uniformly with the sampled frequencies $f_m = (m-1)/N \in \Omega$, which we also refer to as on-grid frequencies, while we refer to all other frequencies $f \in \mathcal{M}$ as off-grid frequencies.

Spectrally Shaped Binary Sequence Design: In prior work [8], we introduced an algorithm to design a spectrally shaped binary sequence based on an SDP relaxation and randomized projection [9–16]. A spectrally shaped sequence is designed to provide a passband and notch for fixed message and interferer bands, denoted by $\mathcal{P} \subseteq \{f_1, \ldots, f_N\}$ and $\mathcal{S} \subseteq \{f_1, \ldots, f_N\}$ respectively. We denote by $\mathcal{F}_\mathcal{P} = \{\mathcal{F}_i : f_i \in \mathcal{P}\}$ and $\mathcal{F}_\mathcal{S} = \{\mathcal{F}_i : f_i \in \mathcal{S}\}$ as the collections of all discrete Fourier transform basis elements corresponding to the message and interferer bands, respectively. Our least-squares approach for designing a binary sequence of length N can be written as the QCQP

$$\begin{aligned} \hat{\mathbf{s}} = \arg\max_{\mathbf{t}\in\mathbb{R}^N} \quad & \|\mathcal{F}_\mathcal{P}\mathbf{t}\|_2^2 \\ \text{s.t.} \quad & \|\mathcal{F}_\mathcal{S}\mathbf{t}\|_2^2 \le \alpha, \\ & {t_n}^2 = 1, \quad n = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

for some interferer tolerance $\alpha > 0$, where t_n denotes the n^{th} entry of $\mathbf{t}$. Such an integer optimization is NP-hard [16].

An SDP relaxation for the QCQP (1) can be obtained by noting that $\|\mathcal{F}_\mathcal{P}\mathbf{s}\|_2^2 = \text{tr}\left(\mathcal{F}_\mathcal{P}^H\mathcal{F}_\mathcal{P}\mathbf{s}\mathbf{s}^T\right)$ and $\|\mathcal{F}_\mathcal{S}\mathbf{s}\|_2^2 = \text{tr}\left(\mathcal{F}_\mathcal{S}^H\mathcal{F}_\mathcal{S}\mathbf{s}\mathbf{s}^T\right)$, where $\text{tr}(\cdot)$ denotes the trace of a matrix. Lifting $\mathbf{s}$ to $\mathbf{S} = \mathbf{s}\mathbf{s}^T$ and dropping the rank constrain $\text{rank}(\mathbf{S}) = 1$ provide us with the optimization

$$\begin{aligned} \widehat{\mathbf{S}} = \arg\max_{\mathbf{T}\in\mathbb{S}^N} \quad & \text{tr}\left(\mathcal{F}_\mathcal{P}^H\mathcal{F}_\mathcal{P}\mathbf{T}\right) \\ \text{s.t.} \quad & \text{tr}\left(\mathcal{F}_\mathcal{S}^H\mathcal{F}_\mathcal{S}\mathbf{T}\right) \le \alpha, \\ & T_{n,n} = 1, \quad n = 1, 2, \ldots, N, \end{aligned} \tag{2}$$

where $T_{n,n}$ denotes the n^{th} diagonal entry of $\mathbf{T}$. After solving the SDP relaxation, the next important step is to extract a feasible solution $\tilde{\mathbf{s}}$ to (1) from the optimal solution $\widehat{\mathbf{S}}$ resulting from (2) when the rank constrain is not met.

Randomized projection is an efficient way to obtain feasible solutions. Denote the eigendecomposition of $\widehat{\mathbf{S}}$ as $\widehat{\mathbf{S}} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T$, where the columns of $\mathbf{U} \in \mathbb{R}^{N\times r}$ are the eigenvectors and the diagonal entries of $\mathbf{\Sigma} \in \mathbb{R}^{r\times r}$ are the eigenvalues, and let $\mathbf{v} \in \mathbb{R}^r$ be a standard Gaussian random vector i.e., $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. Then $\mathbf{w} = \mathbf{U}\mathbf{\Sigma}^{1/2}\mathbf{v}$, where $\mathbf{\Sigma}^{1/2}$ is the entry-wise square root of $\mathbf{\Sigma}$, maximizes the expected value of the objective function in (1) and satisfies the corresponding constraints in expectation. A candidate binary sequence $\tilde{\mathbf{s}}$ is then obtained by quantizing the approximation vector $\mathbf{w}$ as $\tilde{\mathbf{s}} = \text{sign}(\mathbf{w})$. Such randomized projection and binary quantization are repeated multiple times to provide a set of candidate sequences and outputs the best sequences that maximizes the score function (e.g., the object function in (1)) is selected.

Slepian Basis: It is well-known that any bandlimited signal must be infinite in the time domain and no signal with finite length in the time domain can be bandlimited. In [4, 5], Slepian provided a remarkable representation for bandlimited, approximately finite-length discrete-time signals using *discrete prolate spheroidal sequences* (DPSSs). Given a length N and a half-bandwidth $W \in (0, 0.5)$, the DPSSs are a collections of N discrete infinite-length signals that are strictly bandlimited to the frequency range $[-W, W]$ but highly concentrated in their first N entries. DPSSs are defined as the eigenvectors of a procedure that suppresses all entries of an infinite-length signal except the first N entries and then filters out all components of the signal outside the frequency range $[-W, W]$. The Slepian basis $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ are the first N entries of those DPSSs and forms an orthonormal basis for a subspace approximation to the set of signals bandlimited to $[-W, W]$. Recently, the fast Slepian transform was proposed for efficient computation of approximated projections onto the leading Slepian basis elements [6, 7], making the Slepian basis a competitive alternative to the Fourier basis.

The first $2NW$ elements of the Slepian basis are usually sufficient to express the N-length samples of any signal bandlimited to the frequency range $[-W, W]$ [6, 7]. By modulating the baseband Slepian basis with an element of the Fourier basis, one can obtain a subspace approximation of signals restricted to any frequency subset of $[0, 1]$. For example, the modulated Slepian basis $\{\mathcal{F}_m \circ \mathcal{G}_1, \ldots, \mathcal{F}_m \circ \mathcal{G}_N\}$ can be used to compactly represent signals bandlimited to the range $[f_m - W, f_m + W]$, where $\circ$ denotes an entry-wise product.

The most significant difference between the Fourier and Slepian representations appears for signals containing off-grid frequencies. For example, when $W = 1/N$, a complex exponential $\mathbf{x} = \mathcal{F}(f)$ for $f \in [f_{m-1}, f_{m+1}]$ can be approximated by a linear combination of three elements of the Fourier basis $\mathbf{A} = [\mathcal{F}_{m-1}, \mathcal{F}_m, \mathcal{F}_{m+1}]$ or three elements of the Slepian basis $\mathbf{A} = [\mathcal{F}_m \circ \mathcal{G}_1, \mathcal{F}_m \circ \mathcal{G}_2, \mathcal{F}_m \circ \mathcal{G}_3]$ with coefficients $\hat{\mathbf{c}} = \mathbf{A}^H\mathbf{x}$, where $\mathbf{A}^H$ denotes the conjugate transpose of $\mathbf{A}$. Figure 1 shows the energy of the coefficients for $\mathbf{x}$ under both bases as a function of f, with $m = 4$. Although the Fourier basis compacts the signal energy to a single coefficient when the frequency is on-grid (i.e., $f \in \{f_{m-1}, f_m, f_{m+1}\}$), some energy is leaked to other coefficients when the frequency is off-grid. In contrast, the top three coefficients of the signal in the Slepian basis capture almost all energy of a signal at all values of the frequency within the band of interest. Nonetheless, the Fourier basis has better rejection than the Slepian basis for signals with on-grid frequencies outside the bandwidth of interest, which also affects its suitability to model signals restricted to a bandwidth

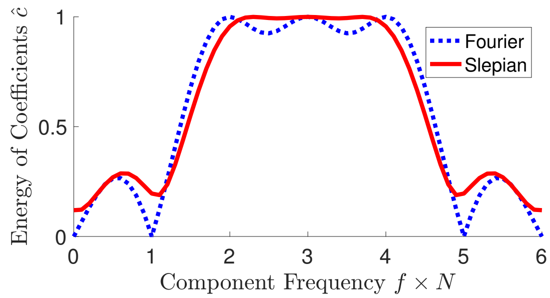

Fig. 1. *Energy of coefficients $\hat{\mathbf{c}}$ for a complex exponential $\mathcal{F}(f)$ projected onto subsets of the Fourier and Slepian bases as a function of the component frequency f.*

within our design approach.

3. SEQUENCE DESIGN

We seek a set of binary sequences that are used to modulate the received signals in a multi-branch modulation architecture. To block the interferer, the binary sequences should work as a band-stop filter to provide a notch at the interferer band. Furthermore, in order to obtain a stable reconstruction of the message (i.e., a well-conditioned measurement operator), the sequences should be as close to mutually orthogonal as possible after projecting onto the message subspace.

We denote by $\mathbf{A}$ and $\mathbf{B}$ the basis matrices for the interferer and message band subspaces, respectively. The elements of both $\mathbf{A}$ and $\mathbf{B}$ can come from either Fourier basis $\mathcal{F}$ or the Slepian basis $\mathcal{G}$. We denote the number of branches by C and the sequence oversampling factor by $R \geq 1$, so that the sequences are of length RN. Although we emphasize that such oversampling is necessary to achieve good performance, we leave its analysis for future work.

The goal for the sequence design of each branch is to find a binary sequence $\mathbf{s}$ such that the magnitude of its projection onto the interferer space is minimized while meeting a target level of approximate orthogonality against previously obtained sequences. Thus, an approach for sequence design for the k^{th} branch ($k = 1, 2, \ldots, C$) can be written as the QCQP

$$\begin{aligned} \hat{\mathbf{s}}_k = \arg\min_{\mathbf{t}\in\mathbb{R}^{RN}} \quad & \left\|\mathbf{A}^H\mathbf{t}\right\|_2^2 \\ \text{s.t.} \quad & \left|\hat{\mathbf{s}}_i^T\mathbf{B}\mathbf{B}^H\mathbf{t}\right|^2 \leq \alpha RN, \quad i = 1, \ldots, k-1, \\ & t_n^2 = 1, \quad n = 1, \ldots, RN, \end{aligned} \tag{3}$$

where $\hat{\mathbf{s}}_i$ is the designed sequence for the i^{th} branch and α is the orthogonality tolerance. Following the framework prescribed in Section 2, the SDP relaxation for the QCQP (3) can be obtained by noting that $\|\mathbf{A}^H\mathbf{t}\|_2^2 = \text{tr}\left(\mathbf{A}\mathbf{A}^H\mathbf{t}\mathbf{t}^T\right)$ and $|\hat{\mathbf{s}}_i^T\mathbf{B}\mathbf{B}^H\mathbf{t}|^2 = \text{tr}\left(\mathbf{B}\mathbf{B}^H\hat{\mathbf{s}}_i\hat{\mathbf{s}}_i^T\mathbf{B}\mathbf{B}^H\mathbf{t}\mathbf{t}^T\right)$. By lifting $\mathbf{t}$ to

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Algorithm 1 *Multi-Branch Sequence Design*

Input: interferer band basis $\mathbf{A}$, message band basis $\mathbf{B}$, coherence tolerance α, oversampled length RN, number of branches C, number of randomized projections L

Output: spectrally shaped sequences $\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \ldots, \hat{\mathbf{s}}_C$

for $k = 1, 2, \ldots, C$ **do**
    obtain optimal solution $\widehat{\mathbf{S}}_k$ to SDP relaxation (4)
    compute SVD for $\widehat{\mathbf{S}}_k = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$
    for $\ell = 1, 2, \ldots, L$ **do**
        generate random vector $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
        obtain approximation by projecting $\mathbf{w}_\ell = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{v}$
        obtain candidate by quantization $\tilde{\mathbf{s}}_\ell = \text{sign}\left(\mathbf{w}_\ell\right)$
    end for
    select best binary sequence

$$\hat{\mathbf{s}}_k = \arg\min_{\tilde{\mathbf{s}}_\ell : 1\leq\ell\leq L} \left\{ \left\|\mathbf{A}^H\tilde{\mathbf{s}}_\ell\right\|_2^2 : \left|\hat{\mathbf{s}}_i^T\mathbf{B}\mathbf{B}^H\tilde{\mathbf{s}}_\ell\right| \leq \alpha RN \right\}$$

end for

$\mathbf{T} = \mathbf{t}\mathbf{t}^T$, we obtain the SDP relaxation

$$\begin{aligned} \widehat{\mathbf{S}}_k = \arg\min_{\mathbf{T}\in\mathbb{S}^{RN}} \quad & \text{tr}\left(\mathbf{A}\mathbf{A}^H\mathbf{T}\right) \\ \text{s.t.} \quad & \text{tr}\left(\mathbf{B}\mathbf{B}^H\hat{\mathbf{s}}_i\hat{\mathbf{s}}_i^T\mathbf{B}\mathbf{B}^H\mathbf{T}\right) \leq \alpha RN, \\ & i = 1, \ldots, k-1, \\ & T_{n,n} = 1, \quad n = 1, 2, \ldots, RN. \end{aligned} \tag{4}$$

After obtaining $\widehat{\mathbf{S}}_k$, we use the same randomized projection and binary quantization procedure described in Section 2 to extract a binary sequence $\hat{\mathbf{s}}_k$ for the k^{th} branch, and we proceed iteratively for subsequent branches as shown in Algorithm 1. It is important to generate a sufficiently large number of candidate sequences to meet the constrains and return a suitable sequence for each branch. Nonetheless, Section 4 shows that the size of the proposed randomized search is far smaller than that of the exhaustive search.

For a signal $\mathbf{x}$, each obtained sequence $\hat{\mathbf{s}}_k$ is used to obtained a measurement $y_k = \hat{\mathbf{s}}_k^T\mathbf{x}$. When $N = C$, we can collect the measurements from all branches $\mathbf{y} = \mathbf{S}^T x$ in order to recover $\mathbf{x}$, where $\mathbf{S} = [\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \ldots, \hat{\mathbf{s}}_C]$.

4. NUMERICAL EXPERIMENTS

We conduct several experiments to test the performance of the proposed algorithm for the design of binary sequences for multi-branch modulation using both the Fourier and Slepian bases and compare to the baselines described earlier. In the following experiments, we set the sequence length $N = 15$, the oversampling rate $R = 15$ for all sequence designs tested, and the branch number $C = 15$. The half bandwidth of the interferer band is $W = 1/RN$ such that the interferer band covers the frequency range $\mathcal{S} = [f_{m-1}, f_{m+1}] \subset \mathcal{M}$, so that the on-grid frequency $f_m = (m-1)/RN$ ($m = 2, 3, \ldots, N-1$) is the center frequency of $\mathcal{S}$. The message band covers the rest of spectrum, i.e., $\mathcal{P} = (f_1, f_{m-1}) \cup (f_{m+1}, f_N)$.

For the purpose of simplicity, we use the elements of the Fourier basis corresponding to the on-grid frequencies in the message band as the message basis (i.e., $\mathbf{B} = \{\mathcal{F}_i : f_i \in \mathcal{P}\}$). However, we use two kinds of bases for the interferer band: the first contains only the elements of the Fourier basis corresponding to the on-grid frequencies in the interferer band (i.e., $\mathbf{A} = [\mathcal{F}_{m-1}, \mathcal{F}_m, \mathcal{F}_{m+1}]$); the second is the top three elements of the Slepian basis for bandwidth $W = 1/N$ modulated by the element of the Fourier basis $\mathcal{F}_m$ (i.e., $\mathbf{A} = [\mathcal{F}_m \circ \mathcal{G}_1, \mathcal{F}_m \circ \mathcal{G}_2, \mathcal{F}_m \circ \mathcal{G}_3]$). We refer to the sequences obtained from the Fourier and Slepian basis as Fourier and Slepian sequences, respectively. We compare the performance of those two sequences with pseudorandom binary sequences (PRBS), and a single-branch sequence obtained from the approach in [8]; the equivalent measurement matrix $\mathbf{S}$ is the Toeplitz matrix obtained from this sequence.

We use two metrics to judge the performance of the sequence designs. We first focus on the band gain. We define the gain for the measurement operator $\mathbf{S}$ at frequency f as the power of a modulated single tone at f where the original tone has unit power $R(f)_{dB} = 10\log_{10}(\|\mathbf{S}^T\mathcal{F}(f)\|_2^2/RN)$, so that we have $R(f) \leq 0$ dB. We use the minimum gain among the message band $\mathcal{P}$ to represent the message gain. We desire for modulation to provide large message gain while providing small interferer gain.

As described in Section 3, it is necessary to have a large enough set of candidate sequences to return sequences with satisfied performance. Figure 2 shows the average gain over 10 sequences obtained from Algorithm 1 for each of the possible choice of on-grid interferer using a varying number of randomized projections L. Values of $R(f)$ below 60 dB are thresholded in the figure for clarity. Both Fourier and Slepian sequences have decreasing interferer gain as the number of randomized projection increases and outperform their baseline alternatives, while message gain is essentially invariant.

We also measure the gain as a function of the interferer frequency offset, i.e., the absolute difference between the interferer frequency and the center frequency of the interferer band $d = |f - f_m|RN \in [0, 1]$. We use Algorithm 1 with $L = 100000$ random projections to generate both Fourier sequences and Slepian sequence for the interferer band centered at all possible on-grid frequencies. Figure 2 shows the average gain over all choices for the center frequency f_m as a function of the offset d. While the gain from PRBS is independent of the offset, the single, Fourier, and Slepian sequences have decreasing performance for off-grid interferers. Nonetheless, the proposed sequences outperform the alternatives for all $d > 0$, with Slepian sequences providing better rejection than Fourier sequences over most values of d.

Next, we focus on the condition number (CN) of $\mathbf{B}^H\mathbf{S}$, which is the ratio between the largest singular value and the smallest singular value of $\mathbf{B}^H\mathbf{S}$, and can be used to measure the stability of the measurement operator. Figure 3 shows the average CN over 100 sequences obtained from Algorithm 1

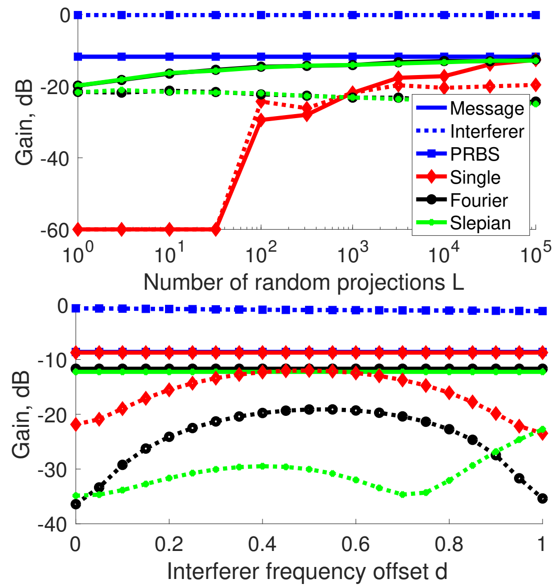

Fig. 2. *Average gain levels of modulation sequences for message and interferer components as a function of (top) the number of random projections L and (bottom) the interferer frequency offset d.*

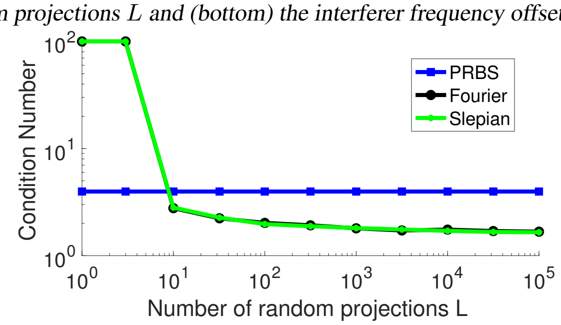

Fig. 3. *Average CN for multi-branch and PRBS measurement operators as a function of the number of random projections L.*

using a varying number of randomized projections L. The CNs of the Fourier and Slepian sequence matrices are slightly smaller than that of the PRBS matrix, while those for single sequence matrices are no less than 100 and thus are not shown in the figure. Note that the range of values for L used here is far less than that the size 2^{RN} of the exhaustive search.

5. CONCLUSIONS

In this paper, we proposed an algorithm to design binary sequence sets for multi-branch modulation that provide a notch for a fixed interferer band. The sequence design is implemented as an SDP combined with randomized projection. Our design considered two options for the basis (Fourier and Slepian) that represent an approximate subspace for signals contains in the interferer band. We have numerically shown that the performance of the designed sequences improves as the number of randomized projections in the relaxation increases, both in terms of rejection and stable invertibility. Our numerical evidence also shows that sequences based on a Slepian basis representation are more robust to interferers at off-grid frequencies, but are outperformed by the Fourier basis representation for frequencies on or near the grid.

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