Regression Performance of Group Lasso for Arbitrary Design Matrices

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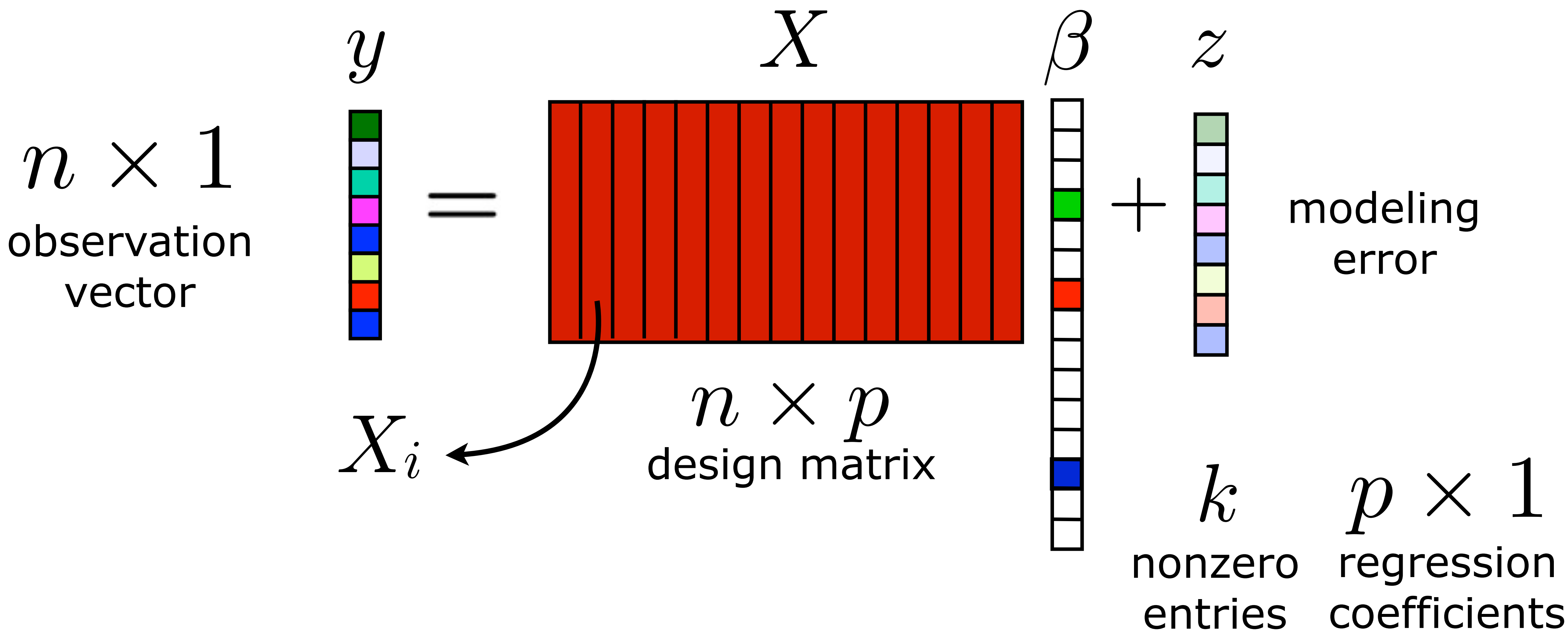
Linear Regression and the Lasso

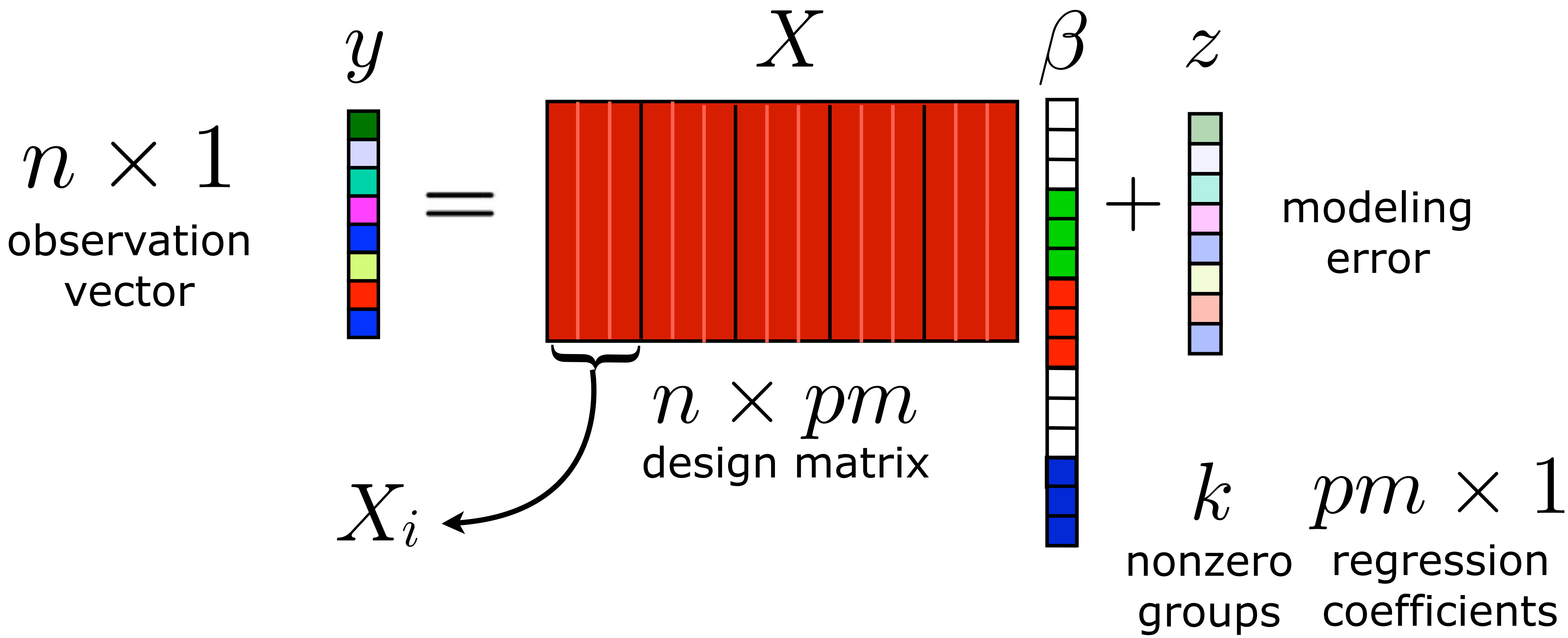


- One regression variable per unit-norm column of X
- Modeling error assumed to be i.i.d. $z \sim \mathcal{N}(0, \sigma^2 I)$
- Lasso: obtain sparse regression coefficient vector as

$$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \|y - X\beta\|_2 + 2\lambda\sigma\|\beta\|_1$$

[Tibshirani, 1996]

Group Linear Regression



- Correlated variables grouped in m column submatrices
- Group Lasso: use mixed norm on coefficient vector

$$\widehat{\beta} = \arg \min_{\beta \in \mathbb{R}^{pm}} \|y - X\beta\|_2 + 2\lambda\sigma\sqrt{m}\|\beta\|_{2,1}$$

where $\|\beta\|_{2,1} = \sum_{i=1}^{p} \|\beta_i\|_2$ [Yuan and Lin, 2006]

Existing Performance Guarantees for Group Lasso

- ***Asymptotic*** convergence for linear regression, coefficient estimation over ***random design matrix***
  [Bach 2008][Meier, van de Geer, Bühlmann 2008]
- ***Asymptotic*** convergence for linear regression, coefficient estimation, model selection over random noise
  [Liu and Zhang 2009][Nardi and Rinaldo 2010]

- Non-asymptotic convergence of linear regression, coefficient estimation over random noise via ***combinatorially complex matrix conditions***
  [Chesneau and Hebiri][Huang and Zhang 2010]
- **Today**: non-asymptotic convergence for linear regression over random noise via ***simple matrix conditions***

Recent Analytical Tools for Lasso

- ***Probabilistic model*** on regression coefficient vector:
  - support $I \subseteq \{1, \ldots, p\}$ of β_I selected uniformly at random
  - signs of k nonzero entries β are i.i.d. and equally likely ± 1

- ***Simple metrics*** on design matrix
  - spectral norm $\|X\|_2$
  - worst-case coherence $\mu(X) = \max_{1 \leq i \neq j \leq p} |\langle X_i, X_j \rangle|$

[Candès, Plan 2010]

Recent Analytical Tools for Lasso

Theorem: Assume that

- $k \le \frac{C_0 p}{\|X\|_2^2 \log p}$ and
- $\mu(X) \le C_1 / \log p$.

If $\lambda = \sqrt{2 \log p}$ and $z \sim \mathcal{N}(0, \sigma^2 I)$, then the output of lasso obeys we can guarantee that

$$\|X\beta - X\widehat{\beta}\|_2^2 \le C k \sigma^2 \log p$$

with probability at least $1 - \mathcal{O}(p^{-1})$.

[Candès, Plan 2010]

New Analytical Tools for Group Lasso

- ***New definition*** of sign patterns for grouped regression coefficient vector $\beta = [\beta_1 \;\; \beta_2 \;\; \ldots \;\; \beta_p]$

$$\overline{\text{sign}}(\beta_i) = \beta_i / \|\beta_i\|_2$$

- ***Probabilistic model*** on regression coefficient vector that accounts for correlations within groups:
  - active groups $I \subseteq \{1, \ldots, p\}$ of β selected uniformly at random
  - group signs of nonzero groups of β are statistically independent:

$$\mathbb{P}(\cup_{i \in I} \overline{\text{sign}}(\beta_i) \in \mathcal{A}_i) = \prod_{i \in I} \mathbb{P}(\overline{\text{sign}}(\beta_i) \in \mathcal{A}_i)$$

  - nonzero regression coefficients have zero median:

$$\mathbb{E}(\text{sign}(\beta)) = \mathbf{0}$$

Simple Metrics on Design Matrix

- Spectral norm $\|X\|_2$
- Worst-case coherence:

$$\mu(X) = \max_{1 \leq i \neq i' \leq p, 1 \leq j \neq j' \leq m} |\langle X_{i,j}, X_{i',j'} \rangle|$$

- Worst-case ***block coherence:***

$$\mu_B(X) = \max\left\{ \max_{1 \leq i \neq i' \leq p} \|X_i^T X_{i'}\|_2, \max_{1 \leq i \leq p} \|X_i^T X_i - I\|_2 \right\}$$

See also [Eldar, Rauhut 2010]

Near-Optimal Group Linear Regression

Theorem: Assume that

- $k \le \frac{C_0 p}{\|X\|_2^2 \log(pm)}$,
- $\mu(X) \le 1/m$, and
- $\mu_B(X) \le C_1 / \log(pm)$.

If $\lambda = \sqrt{2\log(pm)}$ and $z \sim \mathcal{N}(0, \sigma^2 I)$, then the output of group lasso obeys

$$\|X\beta - X\widehat{\beta}\|_2^2 \le Cmk\sigma^2 \log(pm)$$

with probability at least $1 - \mathcal{O}((pm)^{-1})$

Note that for $m=1$, group lasso is same as lasso, and we obtain the result of [Candès, Plan 2010]

Near-Group-Isometries in Expectation

Lemma [Duarte, Bajwa, Calderbank 2010]

Define i.i.d. Bernoulli random variables $\delta_1, \ldots, \delta_p$ with parameter $\delta = k/p$ and form a block submatrix $X_{I'} = [X_i \ : \ \delta_i = 1]$. Then for $q = 2\log(pm)$, we have the bound

$$[\mathbb{E}\|X_{I'}^* X_{I'} - I\|_2^q]^{1/q} \leq \ 20\mu_B(X)\log(pm) + \delta\|X\|_2^2 + 9\sqrt{\delta\log(pm)(1+(m-1)\mu(X))\|X\|_2}$$

- Generalizes results on conditioning of ***random subdictionaries*** [Tropp 2008] to grouped submatrices
- Theorem's conditions on $\mu_B(X), \ \mu(X), \ \|X\|_2$ provide $O(1)$ bounds on $[\mathbb{E}\|X_{I'}^* X_{I'} - I\|_2^q]^{1/q}$

Proof Sketch

(Well) conditioning in expectation, combined with distribution on z, imply that with the given probability these three properties hold simultaneously:

- ***invertibility***: $X_I^* X_I$ is invertible and $\left\|(X_I^* X_I)^{-1}\right\|_2 \leq 2$
- ***orthogonality***: $\|X^* z\|_{2,\infty} \leq \sqrt{2m}\lambda$
- ***complementary size***:

$$2\lambda\sqrt{m}\|X_{I^C}^* X_I (X_I^* X_I)^{-1} \overline{\text{sign}}(\beta_I)\|_{2,\infty} + \|X_{I^C}^* X_I (X_I^* X_I)^{-1} X_I^* z\|_{2,\infty} \leq (2-\sqrt{2})\lambda\sqrt{m}$$

where $\|\beta\|_{2,\infty} = \max_{1\leq i\leq p} \|\beta_i\|_2$

(similar to [Candès, Plan 2010])

Group Lasso and Multiple Measurement Vectors

- m sparse ***correlated*** vectors $B = [\beta_1 \ \beta_2 \ \ldots \ \beta_m] \in \mathbb{R}^{p \times m}$
- Vectors β_i share ***common support***
- ***Single*** design matrix X
- Observation model $Y = XB + Z$, error $Z \in \mathbb{R}^{n \times m}$
- MMV problem can alternatively be expressed as group sparse linear regression:

$$y' = \mathrm{vect}\left(Y^T\right) \qquad \beta' = \mathrm{vect}\left(B^T\right) \qquad z' = \mathrm{vect}\left(Z^T\right)$$

$$X' = X^T \otimes I$$

$$y' = X'\beta' + z'$$

Group Lasso and Multiple Measurement Vectors

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Guarantees available for many MMV recovery algorithms:

- rely on ***random*** design matrix,
- focus on ***asymptotic*** behavior,
- focus on ***coefficient estimation*** or ***model selection***,
- rely on ***combinatorially complex matrix metrics***, or
- apply only for ***noiseless*** model

[Tropp 2006][Tropp, Gilbert, Strauss 2006][Gribonval, Rauhut, Schnass, Vandergheynst 2008]
[Obozinski, Wainwright, Jordan 2009][Lee, Bresler 2010][Kim, Lee, Ye 2010]
[Davies, Eldar 2010][Eldar, Kuppinger, Bölcksei 2010][Eldar, Rauhut 2010]

Summary and Future Work

- Recent tools for average case analysis of linear regression via lasso can be ***extended*** to group lasso
- New probability model that ***captures correlations*** present within each group of predictor variables
- Can apply this probability model to ***standard*** lasso: group lasso ***relaxes*** lasso's requirement from $\mu_B(X) \leq C_1/m \log(pm)$ to $\mu_B(X) \leq C_1/\log(pm)$
- ***Extending*** guarantees to model selection, coefficient estimation for grouped variables

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