

Design of Spectrally Shaped Binary Sequences via Randomized Convex Relaxations

Dian Mo

Department of Electrical and Computer Engineering
University of Massachusetts
Amherst, MA 01003
mo@umass.edu

Marco F. Duarte

Department of Electrical and Computer Engineering
University of Massachusetts
Amherst, MA 01003
mduarte@ecs.umass.edu

Abstract—In wideband communication one often aims to detect a weak signal received together with a strong interferer. One can design a spectrally shaped sequence to be convolved with the received signal featuring a bandpass for the message and a notch for the interferer. Unfortunately, when the sequence must be quantized, commonly used constrained optimization schemes (e.g., semidefinite programs) are not amenable due to the spectrum distortion from quantization. We explore an alternative method that applies random projections on the solution of the semidefinite program. Our experimental evidence shows that our proposed method succeeds in finding suitable spectrally shaped sequences.

Index Terms—wideband communication, analog-to-digital conversion, sequence design, integer programming, semidefinite relaxation, rank-one approximation, random projection

I. INTRODUCTION

The bandwidth of wireless communications is increasing rapidly due to the intense demand of applications such as cognitive radio and ultrawideband [1, 2]. Dealing with such large bandwidth requires the receivers to process signals with a very wide spectrum, resulting in very fast sampling rates and pushing the receivers to their performance limits: the Nyquist sampling theorem requires the analog-to-digital converters (ADCs) to perform at extremely high sampling frequencies. Recently, compressive sensing techniques have introduced the random demodulator as an architecture that acquires large bandwidth signals using a low-rate ADC by leveraging a randomized convolution operation [3], which has also been described as a random modulation and/or pre-integration [4]. In essence, the random demodulator convolves the received signals with a pseudo-random sequence followed by low-rate sampling. The original signal is then recovered by exploiting the sparsity property of the underlying signal spectrum.

Another aspect that is commonly observed in wideband communications is that the received signals are commonly composed of weak signals of interest from distant sources (e.g., transmitters away from the receivers) in the presence of strong signals from nearby sources (e.g., interferers closer to the receivers). The former and latter components of the received signals are colloquially called the message and interferer. It is important to separate and eliminate the interferer before the message is processed. Thus, it would be desirable to use a spectrally shaped sequence allowing the message but

stopping the interferer, rather than the default pseudo-random sequence with a flat spectrum, during demodulation. A recent contribution on the design of spectrally shaped sequences has proposed a method using fast Fourier transform and element-wise operations [2]. However, while the sequence obtained by this method features excellent spectral shaping, it is also complex-valued and therefore is difficult to implement at the circuit level in certain practical applications such as random demodulation.

In this paper, we consider the problem of designing a spectrally shaped binary (± 1) sequence that improves the implementation feasibility of filtering during demodulation. In our examples in this paper, the spectral shape of the sequence is tailored for message passing and interferer cancellation; more specifically, we aim to find a binary sequence with sufficient large spectrum magnitudes for frequencies where message lies while keeping sufficient small magnitudes for the frequencies where the interferer may exist. To be more precise, assume that Ω_M and Ω_I are two disjoint subsets of normalized discrete-time frequencies in the range $[-\pi, \pi)$ that represent the frequency occupancies of the message and interferer, respectively. We then define our problem as the design of a binary sequence $s \in \{-1, 1\}^N$ for which $\|\mathcal{F}_M s\|_2^2$ is maximized while $\|\mathcal{F}_I s\|_2^2$ is bounded by some interferer power tolerance α , where \mathcal{F}_M and \mathcal{F}_I respectively denote subsets of the discrete Fourier transform basis elements corresponding to the frequencies in the message and interferer bands Ω_M and Ω_I , respectively. However, the binary constraint renders this optimization problem NP-hard.

It is easy to show that the optimization problem defined above can be written as a convex semidefinite programming (SDP) problem with an additional (non-convex) rank-one constraint. Relaxing the SDP problem by dropping the rank constraint makes the problem convex and easily solvable but leaves behind the issue of extracting a feasible sequence from the resulting SDP matrix solution if it does not have unit rank. It is common to approximate the solution to the rank-constrained optimization by obtaining the best rank-one approximation to the SDP solution, which can be obtained from the eigenvector with the largest eigenvalue in a manner similar to principal component analysis. Nonetheless, while the resulting eigenvector does exhibit the desired spectral

shaping (as will be shown in the sequel, cf. Figure 2), this eigenvector does not meet the binary constraint of our original problem, and the necessary quantization required on this eigenvector severely affects the spectral shape of the sequence. It has also been shown recently that an alternative SDP relaxation can return a matrix solution with rank at most two [5]. However, the formulation requires sparsity for the matrices involved in the quadratic objective function and constraints, a property that is not present in our application.

As an alternative to the methods listed above, we leverage a random projection technique as an alternative approach to obtain quantized solutions to the spectrally shaped sequence design problem, as originally proposed by Goemans and Williamson [6] for the max-cut binary optimization problem. A significant amount of literature extends this method to solve the problem approximately under a variety of settings [7–9]. In a nutshell, a randomization strategy returns an approximate solution that maximizes or minimizes the expected value of the objective function while the constraints are satisfied in expectation as well. A single (best) approximation will be selected from repeated draws of the randomization process with respect to a suitable criterion for optimality (e.g., by evaluating the values of the objective function and constraints).

In this paper, we present an algorithm for the design of spectrally shaped binary sequences that combines the use of a SDP relaxation and randomized projections that provide multiple feasible binary sequences. We provide a suitable criterion to choose a particular instance from the randomized projections so that the resulting sequence best matches the specific constraints on the message band and interferer band for applications in wideband communications. We show through our experimental results that, for small-scale versions of the problem, our randomized method returns the same binary sequence as the exhaustive search over all possible candidates. While exhaustive search becomes prohibitive when the sequence length is moderately large, we also show examples of sequences found with our approach that provide suitable spectral shaping for practical applications.

II. BACKGROUND

A. Spectrally Shaped Binary Sequence Design

A spectrally shaped sequence for modulation purposes provides a passband and a notch for pre-determined message and interferer bands. A power-based approach for designing a length- N binary sequence can be written as

$$\begin{aligned} \hat{s} = \arg \max_{s \in \{-1,1\}^N} & \quad \|\mathcal{F}_M s\|_2^2 \\ \text{s.t.} & \quad \|\mathcal{F}_I s\|_2^2 \leq \alpha, \end{aligned} \quad (1)$$

for some interferer power tolerance $\alpha > 0$. Here \mathcal{F}_M and \mathcal{F}_I collect all discrete Fourier transform basis elements corresponding to the message band Ω_M and the interferer band Ω_I , respectively. It is commonly known that such an integer optimization problem is NP hard; furthermore, an exhaustive search is too inefficient to be used except in cases where the sequence length is very small.

It is equivalent to formulate the sequence design (1) as a quadratic optimization problem. By defining a rank-one matrix $S = ss^T$, the objective function in (1) has a linear representation with respect to S as follows:

$$\begin{aligned} \|\mathcal{F}_M s\|_2^2 &= s^T \mathcal{F}_M^H \mathcal{F}_M s = \text{trace}(\mathcal{F}_M^H \mathcal{F}_M s s^T) \\ &= \text{trace}(\mathcal{F}_M^H \mathcal{F}_M S), \end{aligned} \quad (2)$$

where $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian (conjugate transpose) operations, respectively. Similarly, the constraint can be written as $\|\mathcal{F}_I s\|_2^2 = \text{trace}(\mathcal{F}_I^H \mathcal{F}_I S)$. On the other hand, any matrix S can be factorized as ss^T if S is positive semidefinite and $\text{rank}(S) = 1$. Therefore, we can reformulate the sequence design (1) as the following quadratic optimization problem:

$$\begin{aligned} \hat{S} = \arg \max_{S \in \mathbb{R}^N} & \quad \text{trace}(\mathcal{F}_M^H \mathcal{F}_M S) \\ \text{s.t.} & \quad \text{trace}(\mathcal{F}_I^H \mathcal{F}_I S) \leq \alpha, \\ & \quad S_{ii} = 1, \quad i = 1, 2, \dots, N, \\ & \quad S \succeq 0, \\ & \quad \text{rank}(S) = 1. \end{aligned} \quad (3)$$

Here, $S \succeq 0$ denotes that S is positive semidefinite. The constraints on the diagonal elements $S_{ii} = 1$ guarantee the binary nature of the sequence s .

Although (3) is as difficult to be solved as (1), the only non-convex constraint included in (3) is the rank constraint $\text{rank}(S) = 1$, and the objective function and all other constraints are convex with respect to S . Thus, previous research work has focused on addressing this single non-convex constraint.

B. Semidefinite Relaxation

To approximately solve (3), one may choose to simply drop the rank constraint to obtain the following relaxation:

$$\begin{aligned} \hat{S} = \arg \max_{S \in \mathbb{R}^N} & \quad \text{trace}(\mathcal{F}_M^H \mathcal{F}_M S) \\ \text{s.t.} & \quad \text{trace}(\mathcal{F}_I^H \mathcal{F}_I S) \leq \alpha, \\ & \quad S_{ii} = 1, \quad i = 1, 2, \dots, N, \\ & \quad S \succeq 0. \end{aligned} \quad (4)$$

The resulting convex problem is the SDP relaxation of (1) and can be efficiently solved by many modern optimization toolboxes, e.g., SeDuMi or SDPT3.

Through this relaxation, the difficulty of binary sequence design has been pushed to the extraction of a binary sequence \hat{s} that is an optimal feasible solution to (1) from the matrix solution \hat{S} resulting from the SDP (4). If \hat{S} is of rank one, then one can always get the sequence by factorizing $\hat{S} = \hat{s}\hat{s}^T$ and \hat{s} will be the feasible and optimal solution for problem (3). Otherwise, if the rank of \hat{S} is larger than one, alternative approaches are needed to extract a sequence \hat{s} from the solution \hat{S} , while observing optimality and feasibility. Nonetheless, we note that in both cases the extracted sequence will not be a feasible solution for (1) unless it happens to be binary, a coincidence that is unlikely to occur in practice.

A common approach to approximate the rank-constrained solution to (3) leverages the principal eigenvector of \hat{S} (i.e., the eigenvector with the largest eigenvalue) to construct the approximation. Specifically, when $\text{rank}(\hat{S}) = r$, then \hat{S} has r eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ and eigenvectors $u_1, u_2, \dots, u_r \in \mathbb{R}^N$ such that its eigendecomposition is given by $\hat{S} = \sum_{k=1}^r \lambda_k u_k u_k^T = U \Lambda U^T$, where $U = [u_1, u_2, \dots, u_r]$ and Λ is the diagonal matrix such that $\text{diag}(\Lambda) = [\lambda_1, \lambda_2, \dots, \lambda_r]$. Since $\lambda_1 u_1 u_1^T$ is the best rank one approximation of \hat{S} in the Frobenius norm sense, $\tilde{s} = \sqrt{\lambda_1} u_1$ would provide the corresponding approximation to the optimal sequence, provided that it is feasible to (1). Otherwise, a feasible solution \hat{s} can be obtained by projecting \tilde{s} into the feasible solution space. For example, in our binary case, binary quantization by $\hat{s} = \text{sign}(\tilde{s})$, where $\text{sign}(\cdot)$ returns the signs of all entries, will return a feasible solution.

Although the eigendecomposition above is a simple way of obtaining a rank-one approximation from the SDP solution \hat{S} , it is not suitable in our problem. As shown in Figure 1, while the eigenvector with largest eigenvalue provides a very good match to the desired bandpass filter for the message (which also filters out the interferer), the binary quantized eigenvector presents a spectrum that is much less suitable - in particular, the strength of the spectrum for the message and interferer are comparable to one another. A more sophisticated scheme to obtain a binary sequence from the relaxation (4) is needed to provide better quantized sequences.

III. RANDOMIZED MATRIX APPROXIMATION

As an alternative to the eigendecomposition above, randomization also provides a path to obtain approximate solutions for (1) starting from the matrix solution \hat{S} of (4) [6]. Assume that $v \in \mathbb{R}^r$ is a random vector whose entries are drawn independently and identically according to the standard Gaussian distribution, i.e., $v \sim \mathcal{N}(0, I)$, where I is the identity matrix. Let $\tilde{s} = U \Lambda^{1/2} v$, where $\Lambda^{1/2}$ denotes the element-wise square root of Λ . We then obtain

$$\tilde{s} \sim \mathcal{N}\left(0, (U \Lambda^{1/2})(U \Lambda^{1/2})^T\right) = \mathcal{N}(0, U \Lambda U^T) = \mathcal{N}(0, \hat{S});$$

this implies that $\hat{S} = \text{E}(\tilde{s} \tilde{s}^T)$, where $\text{E}(\cdot)$ denotes element-wise expectation. When \hat{S} is the optimal solution for (4), \tilde{s} also maximizes $\text{E}(\|\mathcal{F}_M \tilde{s}\|_2^2)$ due to the linearity of the trace:

$$\begin{aligned} \text{E}(\|\mathcal{F}_M \tilde{s}\|_2^2) &= \text{E}(\text{trace}(\mathcal{F}_M^H \mathcal{F}_M \tilde{s} \tilde{s}^T)) \\ &= \text{trace}(\mathcal{F}_M^H \mathcal{F}_M \text{E}(\tilde{s} \tilde{s}^T)) \\ &= \text{trace}(\mathcal{F}_M^H \mathcal{F}_M \hat{S}). \end{aligned} \quad (5)$$

Similarly, $\text{E}(\|\mathcal{F}_I \tilde{s}\|_2^2) = \text{trace}(\mathcal{F}_I^H \mathcal{F}_I \hat{S}) \leq \alpha$. Thus, the random variable \tilde{s} maximizes the expectation of the objective function in (1) and satisfies the corresponding constraint in expectation. Therefore, the random projection \tilde{s} can be interpreted as a statistical approximation to the solution of (4). We can further apply binary quantization to this random vector to obtain a binary random variable $\hat{s} = \text{sign}(\tilde{s})$.

Algorithm 1 Randomized Binary Sequence Design

Input: message band Ω_M , interferer band Ω_I , random search size R , selection score function $g(\cdot)$

Output: binary sequence \hat{s}

- 1: generate Fourier transform bases \mathcal{F}_M and \mathcal{F}_I for message and interferer bands
 - 2: formulate the relaxed SDP problem according to (4)
 - 3: solve optimal solution \hat{S}
 - 4: decompose $\hat{S} = U \Lambda U^T$
 - 5: **for** $\ell = 1, 2, \dots, R$ **do**
 - 6: generate random vector $v \sim \mathcal{N}(0, I)$
 - 7: obtain approximation by projecting $\tilde{s}_\ell = U \Lambda^{1/2} v$
 - 8: quantize $\hat{s}_\ell = \tilde{s}_\ell$
 - 9: **end for**
 - 10: select best binary sequence $\hat{s} = \arg \max_{1 \leq \ell \leq R} g(\hat{s}_\ell)$
-

We propose this combination of SDP and randomized projection for the design of spectrally shaped binary sequences, as shown in Algorithm 1. We begin by formulating the relaxed SDP problem (4) according to the pre-determined message and interferer bands. We note that in practice the message and interferer bands do not need to be restricted to a single continuous interval. After solving (4), we obtain the eigendecomposition of the solution matrix \hat{S} and use it to project multiple draws of the random vector v that achieve the desired spectral shaping in expectation; the number of draws R is a parameter of the algorithm. Subsequently, we apply binary quantization on the multiple random projections $\{\tilde{s}_\ell\}_{\ell=1}^R$ to obtain multiple approximated feasible solutions $\{\hat{s}_\ell\}_{\ell=1}^R$ to the original problem (1). Finally, a selection score function $g(\cdot)$ that measures the quality of the approximation is used to select the best-performing option among the randomized approximations.

It is intuitive to use the objective function in (1) as the selection score function, i.e., $g(\tilde{s}_\ell) = \|\mathcal{F}_M \tilde{s}_\ell\|_2^2$. However, there is no guarantee that maximizing the message spectrum power will provide attenuation for the interferer with respect to the message; an example exhibiting such lack of attenuation is shown in Figure 2. To ensure the presence of such attenuation, we can use a selection score function as the ratio between the minimum magnitude of the spectrum in the message band and the maximum magnitude of the spectrum in the interferer band; this ratio is formally written as

$$g(\tilde{s}_\ell) = \frac{\min |\mathcal{F}_M \tilde{s}_\ell|}{\max |\mathcal{F}_I \tilde{s}_\ell|}, \quad (6)$$

where the absolute value is taken in an entry-wise fashion and the minimum and maximum are evaluated over the entries of the corresponding vectors. Maximizing this ratio has the potential to select desirable sequences to meet the spectral shaping requirements, as our numerical experiments in the next section will show.

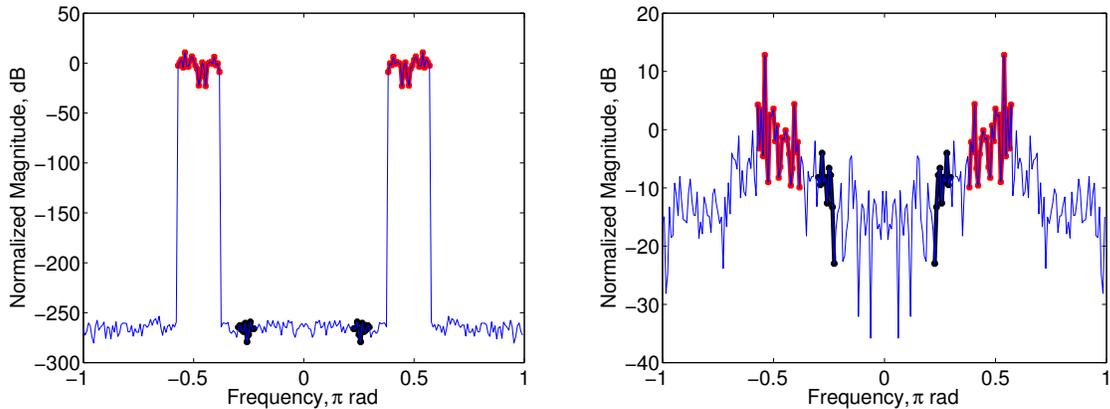


Fig. 1. Normalized spectra for the original eigenvector (left) and the eigenvector after binary quantization (right). Black and red markers denote the interferer and message bands. Spectra are normalized with respect to the mean magnitude of the message band.

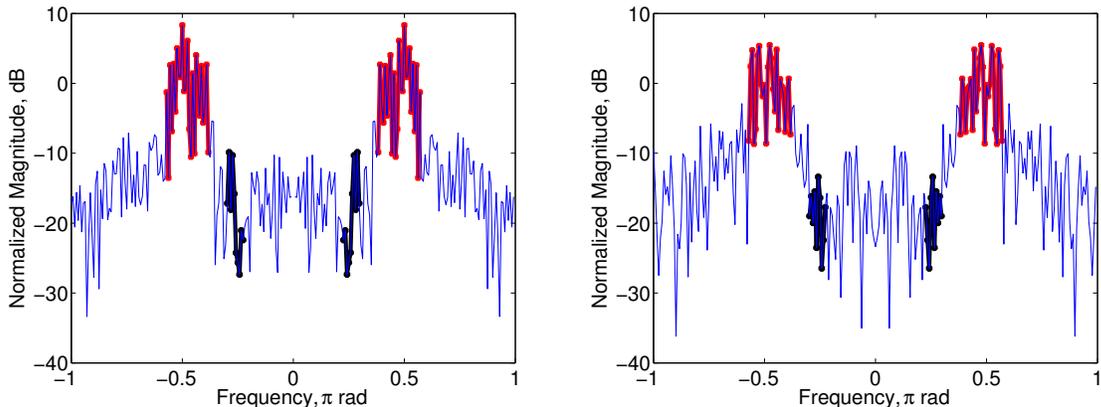


Fig. 2. Normalized spectra of example binary sequence that, among the $R = 10000$ random projection draws, maximizes the message spectrum power (left) and the message-to-interferer spectrum magnitude ratio (6) (right). Black and red markers denote the interferer and message bands. Spectra are normalized with respect to the mean magnitude of the message band.

IV. NUMERICAL EXPERIMENTS

To show the performance of our binary spectrally shaped sequence design procedure (Algorithm 1), we conduct several experiments under varying sequence lengths N and number of random draws R . The message and interferer bands corresponding to consecutive intervals of widths $|\Omega_M|$ and $|\Omega_I|$, respectively, are separated from each other by Δ discrete-time frequency samples, and are chosen uniformly at random from all feasible choices in the N discrete-time frequency samples in the frequency range $[-\pi, \pi)$. We also set the power tolerance for the interferer band to $\alpha = 0.1|\Omega_I|$.

In the first experiment, we test the performance of our proposed algorithm on the design of spectrally shaped sequences with small length. As a baseline, we use the same selection score function (6) to select a binary sequence in an exhaustive search over all 2^N available options. Such an exhaustive search is only feasible for sufficiently small values of N ; in this case, we set $N = 16$. To compare the performance of our proposed approach against the baseline, we define the approximation accuracy for Algorithm 1 as the ratio

$$A = \frac{g(\hat{s})}{\min_{s \in \{-1, 1\}^N} g(s)}.$$

Figure 3 shows the mean and median approximation accuracy over 1000 trials (with each trial corresponding to a different draw of message and interferer bands) as a function of the size of the random search R . Here we set $|\Omega_M| = |\Omega_I| = \Delta = 2$ and the random search size R varies in the range of $[50, 500]$. It is clear from the median accuracy that our proposed randomized method has the potential to return the same sequences with the largest spectrum ratio (6) as the exhaustive search. This fact is particularly remarkable given that its search size is less than 1% of the size of exhaustive search.

In the second experiment, we test the performance of the proposed algorithm on the design of larger-scale sequences. Figure 4 presents the average spectrum ratio (measured in dB) as a function of both the width of the interferer band $|\Omega_I|$ and the spacing Δ between message and interferer bands; we fix $N = 256$ and $|\Omega_M| = 25$, $|\Omega_I|$ and Δ both vary between 1 to 10, and we run 100 random trials for each pair of values. The performance of our proposed algorithm decreases as the interferer width increases; this behavior is intuitive because a large interferer bandwidth will restrict the space of sequences that are feasible. More surprisingly, we have that the band spacing has little effect on the performance of the design of binary sequence using the random approach.

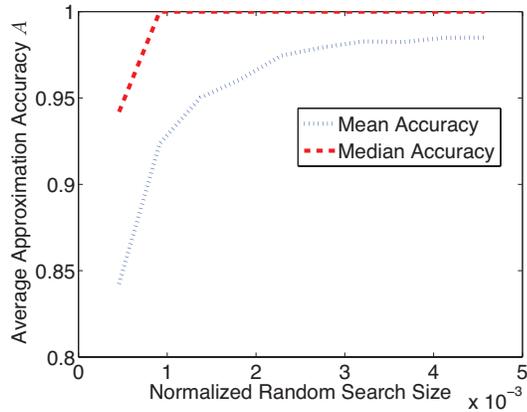


Fig. 3. Mean and median approximation accuracy A for the proposed randomized method as a function of normalized random search size with respect to the exhaustive search size, i.e., $R/2^N$.

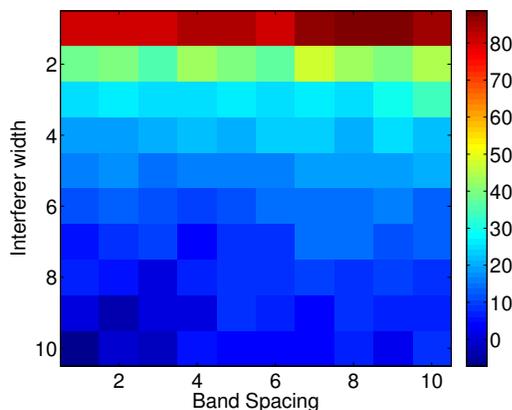


Fig. 4. Average spectrum ratio (6) in dB as a function of interferer width $|\Omega_I|$ and band spacing Δ for the proposed randomized method. The average is taken over 100 randomized trials for each pair of values.

V. CONCLUSION

In this paper, we propose an algorithm to design a spectrally shaped binary sequence. Our specific spectral shaping provides a passband and a notch for a pair of pre-determined message and interferer bands, respectively. We first pose the sequence design problem as a SDP problem (which is a common convex relaxation) and combine it with a random projection based on the solution to the SDP to obtain an approximation to the optimal sequence in a statistical sense. The statistical nature of the approximation process implies that the quality of the approximation can be increased by sampling the underlying random process an increasing number of times and selecting the draw that maximizes a particular suitability metric, which, in our case, is the ratio between the spectra of the message and interferer bands. Our experiments show that for small sequence lengths the randomized method is able to obtain the same optimal sequences as the exhaustive search at a fraction of the search cost, which shows promise for the use of our randomized method in spectrally shaped binary sequence design featuring larger length. Additionally, we find from the experiments that for longer sequences the interferer width plays a more significant factor in the quality of the obtained

binary sequences than the band spacing.

Many questions remain open both on the analysis and possible refinements of our algorithm. For example, the binary constraint on the sequence places a significant limitation on the design space for the desired sequence. More flexible quantization schemes that allow for multiple levels in the values of the sequence may improve the performance of our randomized method. Furthermore, one could consider changes to the objective function and the constraints (e.g., switching the two) and to the selection score function in order to make them more relevant to other types of applications. Possible examples include considering the dynamic range of the message and interferer spectra or the allocation of transmission power to different parts of the spectrum.

ACKNOWLEDGEMENT

We thank Dennis Goeckel, Robert Jackson, and Tamara Sobers for many helpful comments on our work. This paper is based in part upon work supported by the National Science Foundation under grant ECCS-1201835.

REFERENCES

- [1] C. Nunn and L. R. Moyer, "Spectrally - compliant waveforms for wideband radar," *IEEE Aerospace Electron. Syst. Mag.*, vol. 27, pp. 11–15, Aug. 2012.
- [2] W. Rowe, P. Stoica, and J. Li, "Spectrally constrained waveform design," *IEEE Signal Proc. Mag.*, vol. 31, no. 3, pp. 157–162, May 2014.
- [3] J. A. Tropp, J. N. Laska, M. F. Duarte, J. K. Romberg, and R. G. Baraniuk, "Beyond nyquist: Efficient sampling of sparse bandlimited signals," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 520–544, Jan 2010.
- [4] J. N. Laska, S. Kirolos, M. F. Duarte, T. S. Ragheb, R. G. Baraniuk, and Y. Massoud, "Theory and implementation of an analog-to-information converter using random demodulation," in *Proc. IEEE Int. Symp. Circuits Syst. (ISCAS)*, May 2007, pp. 1959–1962.
- [5] R. Madani, G. Fazelnia, and J. Lavaei, "Rank-2 matrix solution for semidefinite relaxation of arbitrary polynomial optimization problems," 2014, preprint.
- [6] M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," *J. ACM*, vol. 42, no. 6, pp. 1115–1145, Nov. 1995.
- [7] Y. Ye, "Approximating quadratic programming with bound and quadratic constraints," *Math. Program.*, vol. 84, no. 2, pp. 219–226, 1999.
- [8] S. Zhang, "Quadratic maximization and semidefinite relaxation," *Math. Program.*, vol. 87, no. 3, pp. 453–465, May 2000.
- [9] Z.-Q. Luo, N. D. Sidiropoulos, P. Eseng, and S. Zhang, "Approximation bounds for quadratic optimization with homogeneous quadratic constraints," *SIAM J. Optim.*, vol. 18, no. 1, pp. 1–28, Feb. 2007.