The Null Space of the Delsarte-Goethals Frame

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1 Introduction

In compressed sensing (CS) [1–3], we wish to sense a signal $f \in \mathbb{R}^{\mathcal{C}}$ by taking its product with a matrix $\Phi \in \mathbb{R}^{N \times \mathcal{C}}$ to obtain a measurement vector $y \in \mathbb{R}^N$. We refer to the rows of Φ as the projection vectors, as measurements y correspond to projections, or inner products, of the signal f onto the rows of the matrix Φ. When $N \ll \mathcal{C}$, this acquisition scheme effectively compresses the signal f. Since in this case the signal recover problem is ill posed, one must exploit prior information on the signal such as sparsity or compressibility. For example, we say a signal is K-sparse if only K out of the $\mathcal{C}$ entries of f are nonzero.

CS relies on the use of sparse approximation algorithms, as well as specially tailored signal recovery algorithms based on sparsity, to recover the signal f from the measurements y and the CS matrix Φ. Most work in CS relies on random constructions on the matrix Φ; that is, the entries of the matrix are drawn independently from a suitable probability distribution such as Gaussian or Rademacher. Such matrices have been shown to provide enough information about a K-sparse signal f through the measurements y when $N = \mathcal{O}(K \log \mathcal{C})$.

1.1 The Delsarte-Goethals Frame

The Delsarte-Goethals Frame [4] (DGF) was proposed as a deterministic CS matrix construction that enables efficient recovery of almost all sparse signals without the use of randomness in the generation of the measurement matrix. The matrix uses the Delsarte-Goethals set of matrices $DG(m, r)$, with $m, r \in \mathbb{Z}$. This set is a vector space containing $2^{(r+1)m}$ binary symmetric matrices of size

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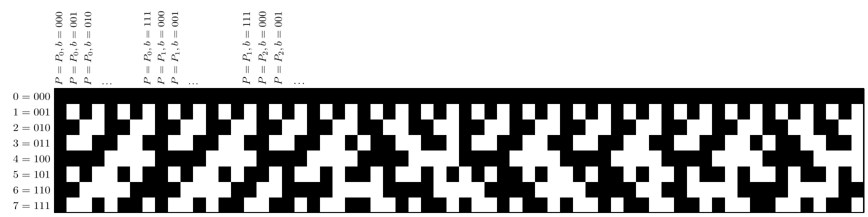

Figure 1: *Example Delsarte-Goethals frame φ with $\mathcal{C} = 64$ and $N = 8$. Each row is labeled by a binary vector $x \in \mathbb{F}_2^3$, and each column is labeled by a pair (P, b), with $P \in DG(m, r)$ and $b \in \mathbb{F}_2^3$.*

$m \times m$ with the property that the difference of any two distinct matrices has rank at least $m - 2r$.

The DGF φ is a CS matrix of size $N = 2^m$ and $\mathcal{C} = 2^m R$, with $R \in [1, 2^{m(r+1)}]$ an integer. We index the rows of φ with elements $x \in \mathbb{F}_2^m$, represented as binary vectors of length m. starting with 0, the zero vector. Similarly, we index the columns with the ordered pairs (P, b), where $P \in DG(m, r)$ and $b \in \mathbb{F}_2^m$. In this way, we label and define the entry of φ in row x and column (P, b) as follows:

$$\varphi_{P,b}(x) = i^{xPx^\top + 2bx^\top}.$$

Here $x^\top$ denotes the transpose of x. Note that all the arithmetic in the expressions $xPx^\top + 2bx^\top$ takes place in the ring of integers modulo 4, since the expression appears as an exponent for $i = \sqrt{-1}$. Given P, b, the vector $xPx^\top + 2bx^\top$ is a codeword in the Delsarte-Goethals code (defined over the ring of integers modulo 4). For a fixed matrix P, the 2^m columns $\varphi_{P,b}, b \in \mathbb{F}_2^m$ form an orthonormal basis Γ_P that can also be obtained by postmultiplying the Walsh-Hadamard basis by the unitary transform diag $\left[i^{xPx^\top}\right]$. Consequently, when P has zero diagonal, the resulting basis Γ_P has real-valued entries. We expand our discussion of real-valued DGFs in Section 3. Figure 1 shows an example DGF for $\mathcal{C} = 64$ and $N = 8$ ($m = 3$).

1.2 The Haar Wavelet Basis

The Haar Wavelet Basis provides a set of piecewise smooth functions of different support sizes and magnitudes, thanks to its multiscale analysis properties. The first basis function has a constant value throughout its support; all other functions have only two distinct nonzero values, each covering half of the function's support. We now define notation for the Haar wavelet basis functions. Our description here is for 1-D wavelets for simplicity, but the concepts can easily be extended to 2-D wavelets.

A length-$\mathcal{C}$ Haar wavelet $\psi_{s,j}$ (where $\mathcal{C}$ is a power of 2) is labeled according to its scale $s = 0, \ldots, \log_2 \mathcal{C} - 1$, and its offset $j = 0, \ldots, 2^s - 1$. Its entries are denoted by $\psi_{s,j}(n), n = 0, \ldots, \mathcal{C}-1$. For the scale $s = 0$, we also define the Haar scaling function:

$$\phi(n) = \sqrt{1/\mathcal{C}},\ 0 \le n \le \mathcal{C}-1.$$

The mother Haar wavelet is defined as

$$\psi(n) = \begin{cases} \sqrt{1/\mathcal{C}} & 0 \le n < \mathcal{C}/2, \\ -\sqrt{1/\mathcal{C}} & \mathcal{C}/2 \le n < \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

The Haar wavelets at different scales and offsets are generated through dilation and translation:

$$\psi_{s,j}(n) = \sqrt{2^s}\psi(2^s n - \mathcal{C}j).$$

Specifically, for scales $0 \le s < \log_2 \mathcal{C}$ and offsets $0 \le j < 2^s$, the Haar wavelet is defined as follows:

$$\psi_{s,j}(n) = \begin{cases} \sqrt{2^s/\mathcal{C}} & 2^{-s}\mathcal{C}j \le n < 2^{-s}\mathcal{C}(j+1/2), \\ -\sqrt{2^s/\mathcal{C}} & 2^{-s}\mathcal{C}(j+1/2) \le n < 2^{-s}\mathcal{C}(j+1), \\ 0 & \text{otherwise.} \end{cases}$$

This structure for the support of the wavelets (i.e., the location of its nonzero values) is known as a dyadic structure: the wavelet's support is of size $2^{-s}\mathcal{C}$, and the offset is a multiple of its size. For simplicity, we denote by $\mathcal{D}_{s,j}$ the set of indices in the dyadic interval at scale s and offset j:

$$\mathcal{D}_{s,j} = \{2^{-s}\mathcal{C}j, 2^{-s}\mathcal{C}j + 1, \ldots, 2^{-s}\mathcal{C}(j+1) - 1\}.$$

For example, using this notation, we can write

$$\psi_{s,j}(n) = \sqrt{2^s/\mathcal{C}} \left(\chi_{\mathcal{D}_{s+1,2j}}(n) - \chi_{\mathcal{D}_{s+1,2j+1}}(n) \right),$$

where $\chi_{\mathcal{P}}(n)$ denotes the indicator function for the set $\mathcal{P}$ on n; thus we have that the support of $\psi_{s,j}$ is $\mathcal{D}_{s,j}$.

It is easy to check that the wavelets defined above have unit norm and are orthogonal to each other, making an orthonormal basis for $\mathbb{R}^{\mathcal{C}}$. We collect the functions $\psi_{s,j}$ as the columns of a Haar wavelet basis matrix Ψ, so that its columns are indexed by the ordered pairs $(s, j),\ 0 \le s < \log_2 \mathcal{C},\ 0 \le j < 2^s$ and its rows are indexed by $n,\ 0 \le n \le \mathcal{C}$.

1.3 Contributions

The DGF stands out because of its deterministic nature and the fast signal recovery algorithms enabled by its construction [4]. However, the DGF must be applied directly on a sparse or compressible vector. That is, the signal f measured using the CS matrix φ must be sparse or compressible in the canonical domain. While most natural images do not have this property, transforms used for image compression are suitable to obtain sparsity or compressibility,

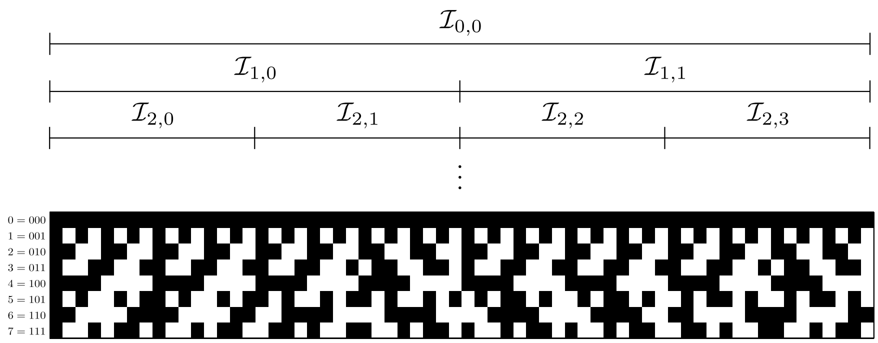

Figure 2: *Example dyadic column intervals for a DGF of size* $\mathcal{C} = 64$*,* $N = 8$ *(*$m = 3$*).*

with examples including the 2-D discrete cosine transform and the 2-D discrete wavelet transform.

In this report, we show the performance of DGF for signal recovery is sensitive to the presence of nonzero clusters in the sparse vector to be recovered. Such a property appears, for example, in piecewise smooth signals that are sparse in a wavelet domain. Specifically, we show that there exist many pairs of Haar wavelets of sufficiently coarse scales whose linear combination lies in the null space of the DGF. In the case of canonically sparse signals, this property means that the DGF has sub-optimal recovery performance for sparse signals whose support is clustered within the coefficient vector, as there is a nonnegligible component of the signal that projects to the null space of the DGF. The sensitivity can be bypassed by reordering the elements of the vector being sensed or, equivalently, permuting the columns of the DGF before sensing [5].

2 Dyadic Column Sums of the DGF

We now consider the sum of columns of the DGF that correspond to the support of a dyadic wavelet. For each scale-offset pair (s, j), we let the set $\mathcal{I}_{s,j} \subseteq DG(m,r) \times \mathbb{F}_2^m$ denote the pairs (P, b) for the columns at the positions contained in the dyadic interval $\mathcal{D}_{s,j}$; we call $\mathcal{I}_{s,j}$ a *dyadic column interval.* Some examples are shown in Figure 2. Thus, one can write $f_{s,j} = \varphi\psi_{s,j}$, which yields

$$f_{s,j}(x) = \sum_{(P,b)\in\mathcal{I}_{s+1,2j}} \varphi_{P,b}(x) - \sum_{(P,b)\in\mathcal{I}_{s+1,2j}} \varphi_{P,b}(x). \qquad (1)$$

We begin by stating some properties of these subsets $\mathcal{I}_{s,j}$.

Proposition 1. *If* $s \geq \log_2(\mathcal{C}/N) = \log_2 R$*, then* $\mathcal{I}_{s,j} = \{P_{s,j}\} \times \mathcal{F}_{s,j}$*, where* $P_{s,j}$ *is a fixed matrix in* $DG(m,r) \setminus DG(m,0)$*, and* $\mathcal{F}_{s,j} = a_{s,j} \oplus \mathbb{F}_2^{\log_2 \mathcal{C} - s}$ *, with* $a_{s,j} \in \mathbb{F}_2^{s - \log_2 R}$.

In words, $\mathcal{F}_{s,j}$ is a subset of $\mathbb{F}_2^m$ whose elements share the $s - \log_2 R$ most significant bits; fluctuations on later bits span the subset. Thus, for $s \geq \log_2(\mathcal{C}/N) = \log_2 R$, the subset $\mathcal{I}_{s,j}$ is defined by the matrix $P_{s,j}$ and the "header" $a_{s,j}$ containing the fixed most significant bits of b over $\mathcal{I}_{s,j}$.

Proposition 2. *If $s \leq \log_2(\mathcal{C}/N) = \log_2 R$, then $\mathcal{I}_{s,j} = \mathcal{P}_{s,j} \times \mathbb{F}_2^m$, where $\mathcal{P}_{s,j}$ is a subset of $DG(m,r) \setminus DG(m,0)$ containing $2^{-s}\mathcal{C}$ matrices.*

Proof sketch. Since the offset and size of the interval $\mathcal{I}_{s,j}$ is a multiple of $N = 2^m$, the selected columns include the set of bases Γ_P for a subset of matrices P used in the construction of the DGF (as defined in Section 1.1). $\square$

Armed with this properties, we consider the behavior of "dyadic sums" of columns of φ, defined as

$$S_{s,j}(x) := \sum_{(P,b)\in\mathcal{I}_{s,j}} \varphi_{P,b}(x),$$

which simplifies the calculation in (1) to

$$y_{s,j}(x) = \sqrt{2^{s+1}/\mathcal{C}}(S_{s+1,2j}(x) - S_{s+1,2j+1}(x)).$$

Lemma 1. *For $s \leq \log_2(\mathcal{C}/N)$, we have*

$$S_{s,j}(x) = \begin{cases} |\mathcal{I}_{s,j}| & x = 0, \\ 0 & x \neq 0. \end{cases}$$

Proof. We write

$$S_{s,j}(x) = \sum_{(P,b)\in\mathcal{I}_{s,j}} \varphi_{P,b}(x) = \sum_{P\in\mathcal{P}_{s,j}} \sum_{b\in\mathbb{F}_2^m} i^{xPx^\top + 2bx^\top} = \sum_{P\in\mathcal{P}_{s,j}} i^{xPx^\top} \sum_{b\in\mathbb{F}_2^m} i^{2bx^\top}.$$

For $x \neq 0$ the second sum is equal to zero, rendering $S_{s,j}(x) = 0$. When $x = 0$, the second sum is equal to N. Additionally, since $x = 0$, the first sum is equal to $|\mathcal{P}_{s,j}|$, and so we have

$$S_{s,j}(x) = |\mathcal{P}_{s,j}|N = |\mathcal{I}_{s,j}|,$$

proving the lemma. $\square$

For the next lemma, we denote by $\mathcal{B}_{l_1:l_2}(x)$ the subset of bits $\{l_1, \ldots, l_2\}$ from the binary vector $x \in \mathbb{F}_2^m$. For example, $\mathcal{B}_{1:l}(x)$ contains the l most significant bits of the binary vector x. Similarly, $\mathcal{B}_{l+1:m}(x)$ contains the remaining bits of x.

Lemma 2. *For $s \geq \log_2(\mathcal{C}/N)$, we have that $S_{s,j}(x)$ can only take one of these values: $\{0, 2^{-s}\mathcal{C}, -2^{-s}\mathcal{C}\}$.*

Proof. We write

$$
\begin{aligned}
S_{s,j}(x) &= \sum_{(P,b)\in\mathcal{I}_{s,j}} \varphi_{P,b}(x) = \sum_{b\in\mathcal{F}_{s,j}} i^{xP_{s,j}x^\top + 2bx^\top} \\
&= i^{xP_{s,j}x^\top} \sum_{b\in\mathbb{F}_2^{\log_2 \mathcal{C}-s}} i^{2a_{s,j}\mathcal{B}_{1:s-\log_2 R}(x)+2b\mathcal{B}_{s-\log_2 R+1:m}(x)^\top} \\
&= i^{xP_{s,j}x^\top + 2a_{s,j}\mathcal{B}_{1:s-\log_2 R}(x)} \sum_{b\in\mathbb{F}_2^{\log_2 \mathcal{C}-s}} i^{2b\mathcal{B}_{s-\log_2 R+1:m}(x)^\top}. \qquad (2)
\end{aligned}
$$

For $\mathcal{B}_{s-\log_2 R+1:m}(x) \neq 0$ the sum is equal to zero, rendering $S_{s,j}(x) = 0$. When $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$, the sum is equal to $2^{-s}\mathcal{C}$. Finally, note that the exponent on the first term is even, proving the lemma. □

Now comes our final result. We consider the linear combination of two Haar wavelets projected by the DGF at the same scale $\eta_{s,j_1,j_2,\pm}(x) = y_{s,j_1} \pm y_{s,j_2}$, and look to determine which combinations of wavelets belong in the nullspace of φ.

Theorem 1. *For $1 \leq s \leq \log_2(\mathcal{C}/N)$ and $0 \leq j_1, j_2 < 2^s$, $j_1 \neq j_2$, we have $\eta_{s,j_1,j_2,+} = 0$ if*

- *$s < \log_2(\mathcal{C}/N)$ (for all j_1, j_2);*
- *$\log_2(\mathcal{C}/N) \leq s \leq \log_2(\mathcal{C}/N) + 2r$, $a_{s,j_1} = a_{s,j_2}$ (that is, the two dyadic wavelet intervals are at the same position within the domain of the corresponding DG matrices $P_{s,j_1} \neq P_{s,j_2}$) and $x(P_{s,j_1} - P_{s,j_2})x^\top = 0 \pmod 4$ for all $x \in \mathbb{F}_2^m$ such that $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$; or*
- *$s \geq \log_2(\mathcal{C}/N)$ and $x(P_{s,j_1} - P_{s,j_2})x^\top = 2(a_{s,j_2} - a_{s,j_1})\mathcal{B}_{1:s-\log_2 R}(x)^\top \pmod 4$ for all $x \in \mathbb{F}_2^m$ such that $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$.*

The conditions also hold for $\eta_{s,j_1,j_2,-} = 0$ by adding a term of 2 in the equalities (mod 4).

Note that it is no clear whether the third condition can hold, and the number of scales for which the second condition can hold is dependent on the set $DG(m,r)$ used in the DGF. For the Kerdock set $DG(m,0)$, $s = \log_2(\mathcal{C}/N)$.

Proof. We begin by writing

$$
\eta_{s,j_1,j_2,\pm}(x) = \sqrt{2^{s+1}/\mathcal{C}}[(S_{s+1,2j_1} - S_{s+1,2j_1+1}) \pm (S_{s+1,2j_2} - S_{s+1,2j_2+1})]. \quad (3)
$$

Lemmas 1 and 2 show that the value of a dyadic sum of a given scale can take only two or five distinct values, respectively, depending on the scale s used. When $s < \log_2(\mathcal{C}/N)$, Lemma 1 provides $\eta_{s,j_1,j_2,\pm} = 0$ for all j_1, j_2; this gives the first condition in the theorem. When $s \geq \log_2(\mathcal{C}/N)$, the sums from (2) involved in (3) vanish for all x with $\mathcal{B}_{s-\log_2 R+2:m}(x) \neq 0$, and so we have

$$
\eta_{s,j_1,j_2,\pm}(x) = 0 \text{ if } \mathcal{B}_{s-\log_2 R+2:m}(x) \neq 0.
$$

If $\mathcal{B}_{s-\log_2 R+2:m}(x) = 0$, all terms inside the sum in (2) are equal to one and each of the sums involved in (3) are equal to $2^{-s-1}\mathcal{C}$. With this information, and plugging (2) in (3), we obtain

$$\begin{aligned}
\eta_{s,j_1,j_2,\pm}(x) &= \sqrt{2^{-s-1}\mathcal{C}}\left[i^{xP_{s,j_1}x^\top+2a_{s,j_1}\mathcal{B}_{1:s-\log_2 R}(x)^\top}\left((-1)^{0\cdot\mathcal{B}_{s-\log_2 R+1}(x)}-(-1)^{1\cdot\mathcal{B}_{s-\log_2 R+1}(x)}\right)\right.\\
&\qquad \left.\pm i^{xP_{s,j_2}x^\top+2a_{s,j_2}\mathcal{B}_{1:s-\log_2 R}(x)^\top}\left((-1)^{0\cdot\mathcal{B}_{s-\log_2 R+1}(x)}-(-1)^{1\cdot\mathcal{B}_{s-\log_2 R+1}(x)}\right)\right]\\
&= \begin{cases} \sqrt{2^{-s+1}\mathcal{C}}\left[i^{xP_{s,j_1}x^\top+2a_{s,j_1}\mathcal{B}_{1:s-\log_2 R}(x)^\top}\right. & \\ \qquad\qquad \left.\pm i^{xP_{s,j_2}x^\top+2a_{s,j_2}\mathcal{B}_{1:s-\log_2 R}(x)^\top}\right] & \text{if } \mathcal{B}_{s-\log_2 R+1}(x)=0,\\ 0 & \text{if } \mathcal{B}_{s-\log_2 R+1}(x)=1. \end{cases}
\end{aligned}$$

We then have that $\eta_{s,j_1,j_2,+}(x) = 0$ for all $x \in \mathbb{F}_2^m$ if for all $x \in \mathbb{F}_2^m$ such that $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$,

$$xP_{s,j_1}x^\top + 2a_{s,j_1}\mathcal{B}_{1:s-\log_2 R}(x)^\top = xP_{s,j_2}x^\top + 2a_{s,j_2}\mathcal{B}_{1:s-\log_2 R}(x)^\top \pmod 4$$

and $\eta_{s,j_1,j_2,-}(x) = 0$ if

$$xP_{s,j_1}x^\top + 2a_{s,j_1}\mathcal{B}_{1:s-\log_2 R}(x)^\top = xP_{s,j_2}x^\top + 2a_{s,j_2}\mathcal{B}_{1:s-\log_2 R}(x)^\top + 2 \pmod 4,$$

which is the third condition given in the theorem. We focus on $\eta_{s,j_1,j_2,+}(x)$ and study a couple of special cases.

- If $P_{s,j_1} = P_{s,j_2}$, then we need $2a_{s,j_1}\bar{x}^\top = 2a_{s,j_2}\bar{x}^\top$ for all $\bar{x} \in \mathbb{F}_2^{s-\log_2 R}$. This is only possible if $a_{s,j_1} = a_{s,j_2}$, which is a contradiction with $j_1 \neq j_2$ and $P_{s,j_1} = P_{s,j_2}$.
- If $P_{s,j_1} \neq P_{s,j_2}$ and $a_{s,j_1} = a_{s,j_2} = a_0$, then we obtain

  $$\eta_{s,j_1,j_2,\pm}(x) = \sqrt{2^{-s+1}\mathcal{C}}\; i^{2a_0\mathcal{B}_{1:s-\log_2 R}(x)^\top}\left[i^{xP_{s,j_1}x^\top} \pm i^{xP_{s,j_2}x^\top}\right]. \qquad (4)$$

  This implies that $(P_{s,j_1} + P_{s,j_2})$ has rank at most $m - s + \log_2(\mathcal{C}/N)$. Since the DGF class of matrices forces this rank to be at least $m - 2r$, we may have $\eta_{s,j_1,j_2,\pm} = 0$ (i.e., sums of wavelets at scale s are in the null space) only if $s \leq \log_2(\mathcal{C}/N) - 2r$. Therefore this can only occur if $\log_2(\mathcal{C}/N) \leq s \leq \log_2(\mathcal{C}/N) + 2r$.

This gives the second condition in the theorem. □

3 Real-valued versions of the DGF

In certain cases, we desire a CS matrix with real-valued entries; there are two possible approaches to adapt the DGF to a real-valued CS matrix. We now show that for each one of these options, the argument in the proof of Theorem 1 can be easily adapted, and therefore the same vectors are in the null space of the real-valued versions of the DGF.

l	$g(l)$
1	$[1\ 1]$
i	$[1\ -1]$
-1	$[-1\ -1]$
$-i$	$[-1\ 1]$

Table 1: Gray map table.

First, one can restrict the matrices $P \in DG(m,r)$ to the subset of the DG set of matrices with zero-valued diagonal entries. With such a restriction, the term $xPx^\top = 2\sum_{0\le i<j<2^m} x_i x_j P_{i,j}$ is an even number, rendering the entries of φ real-valued. In this case, Theorem 1 can be applied without change, since the result is not dependent on the particular choice of matrices P.

Alternatively, one can create a CS matrix having twice as many rows as the DGF by applying the Gray map $g : \{1, -1, i, -i\} \to \{-1, 1\}^2$ to the entries of φ, given in Table 1. The Gray map has the property that the norm of the difference between any two powers of i is equal to the norm of the difference of their Gray map image vectors. The new Gray-mapped CS matrix, which we denote by φ^G, has 2^{m+1} rows and $2^{m(r+1)}$ columns indexed by $(h,x) \in \mathbb{F}_2 \times \mathbb{F}_2^m$ and $(P,b) \in DG(m,r) \times \mathbb{F}_2^m$, respectively. By defining the equivalence $P = P' + d_P$, where d_P is the extracted diagonal of P and P' is the remainder of P, the entries of φ^G can be written as follows:

$$\varphi^G_{P,b}(h,x) = i^{xP'x^\top + 2bx^\top + 2h\epsilon_1(d_P x^\top) + \epsilon_2(d_P x^\top)},$$

where

$$\epsilon_1(l) = l \bmod 2 \text{ and } \epsilon_2(l) = \begin{cases} 0 & \text{if } l = 0,1 \bmod 4, \\ 2 & \text{if } l = 2,3 \bmod 4, \end{cases}$$

i.e., ϵ_1 and ϵ_2 extract the two bits of the binary representation of $l \bmod 4$. Since the dependence of $\varphi^G_{P,b}(h,x)$ on b is the same as that of $\varphi_{P,b}(x)$, it is easy to see that the proof of Theorem 1 is extendable to dyadic sums of the Gray-mapped DGF.

References

[1] D. L. Donoho, “Compressed sensing,” *IEEE Trans. Info. Theory*, vol. 52, no. 4, pp. 1289–1306, September 2006.

[2] E. J. Candès, “Compressive sampling,” in *Proc. International Congress of Mathematicians*, Madrid, Spain, 2006, vol. 3, pp. 1433–1452.

[3] R. G. Baraniuk, “Compressive sensing,” *IEEE Signal Processing Mag.*, vol. 24, no. 4, pp. 118–120, 124, July 2007.

[4] R. Calderbank, S. Howard, and S. Jafarpour, "Construction of a large class of deterministic sensing matrices that satisfy a statistical isometry property," *IEEE J. Selected Topics in Signal Processing*, vol. 4, no. 2, pp. 358–374, Apr. 2010.

[5] M. F. Duarte, S. Jafarpour, and R. Calderbank, "New wavelet coefficient raster scannings for compressive imaging," in *Int. Conf. Sampling Theory and Applications (SAMPTA)*, Singapore, May 2011.