New Wavelet Coefficient Raster Scannings for Deterministic Compressive Imaging

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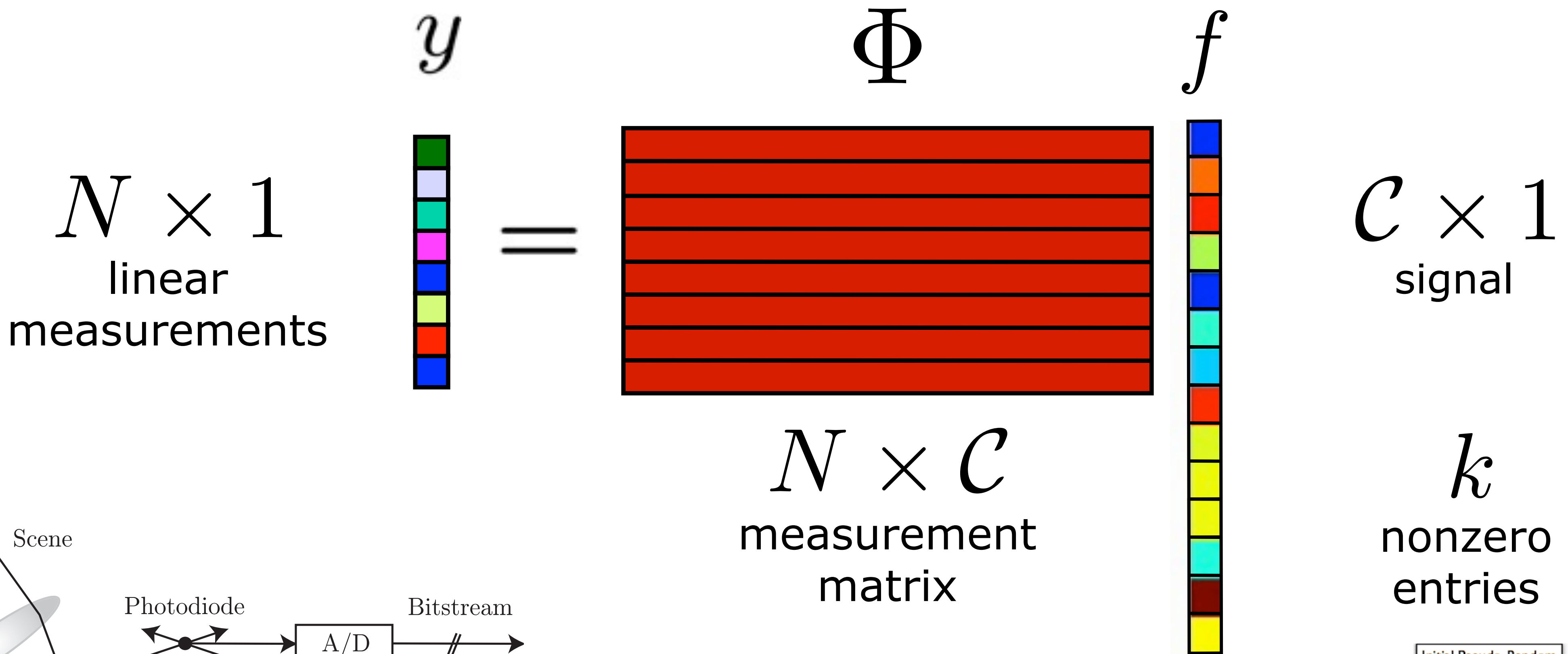
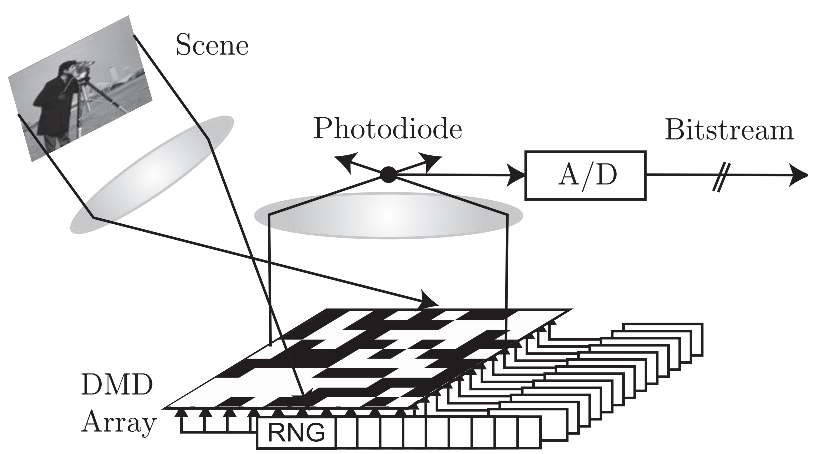
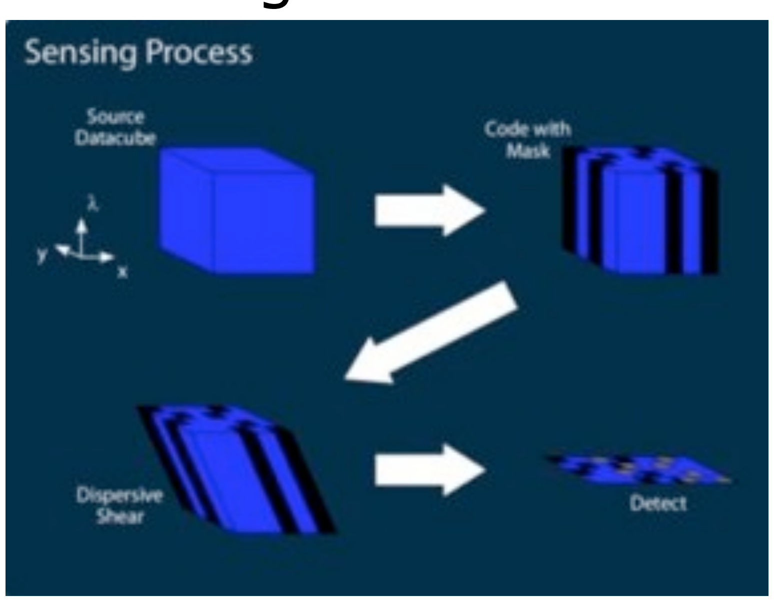
Joint work with Sina Jafarpour and Robert Calderbank

Compressive Imaging Architectures

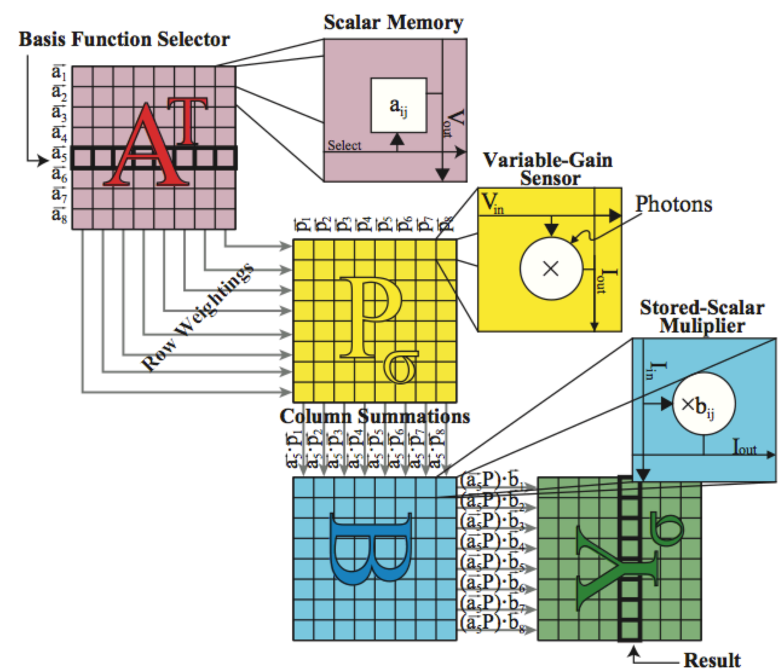




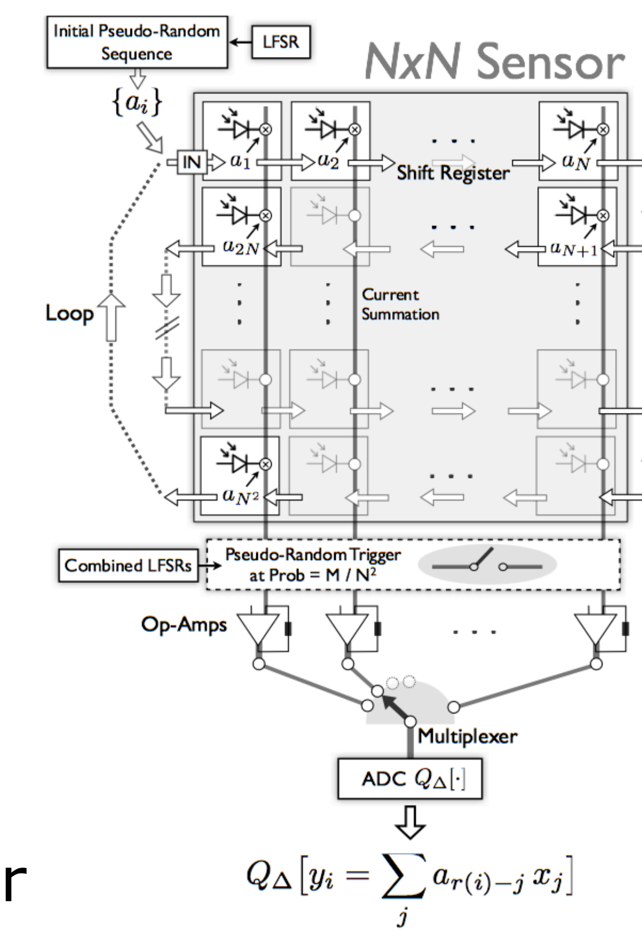
Rice Single Pixel Camera



Duke CASSI

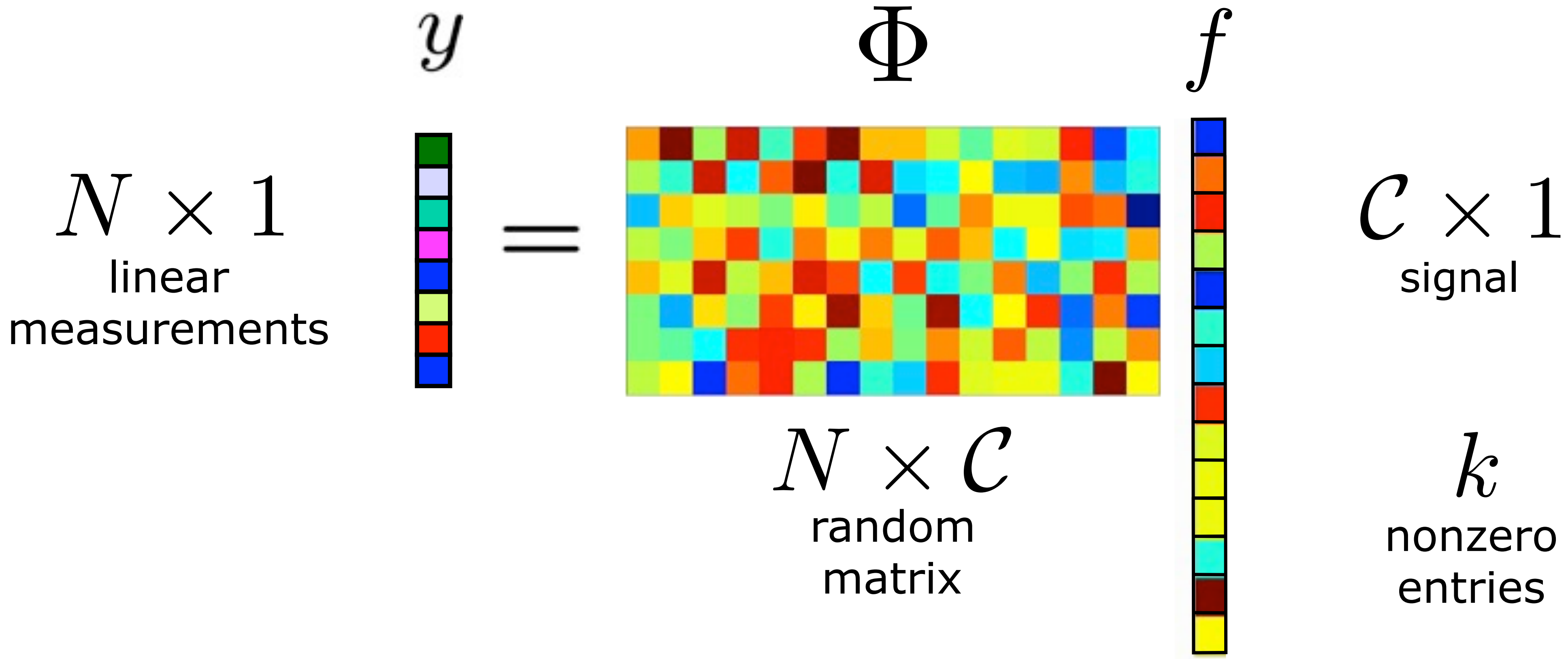


Georgia Tech Analog Imager



EPFL Convol. Imager

Compressive Imaging via Random Matrices



- Random matrices with i.i.d. subgaussian entries
- Can recover all k-sparse signals if $N = \mathcal{O}(k \log(\mathcal{C}/k))$
- Signal can be sparse/compressible in arbitrary basis
- ***High complexity*** for signal recovery -
  costly storage/matrix product for random matrices

Deterministic CS matrices

- ***Delsarte-Goethals Frame***: $\varphi_{P,b}(x) = i^{xPx^\top + 2bx^\top}$

$$\varphi = [H \;\; D_2 H \;\; D_3 H \;\; \ldots \;\; D_R H] \qquad D_j = \mathrm{diag}\left(\{i^{xP_j x^T}\}_{x \in \mathbb{F}_2^m}\right)$$

[Calderbank, Howard, Jafarpour 2009]

  - $N = 2^m$ rows indexed by $x \in \mathbb{F}_2^m$
  - $\mathcal{C} = 2^m R$ columns, $R \in \{1, \ldots, 2^{m(r+1)}\}$, indexed by (P, b), $P \in DG(m, r)$, $b \in \mathbb{F}_2^m$

Deterministic CS matrices

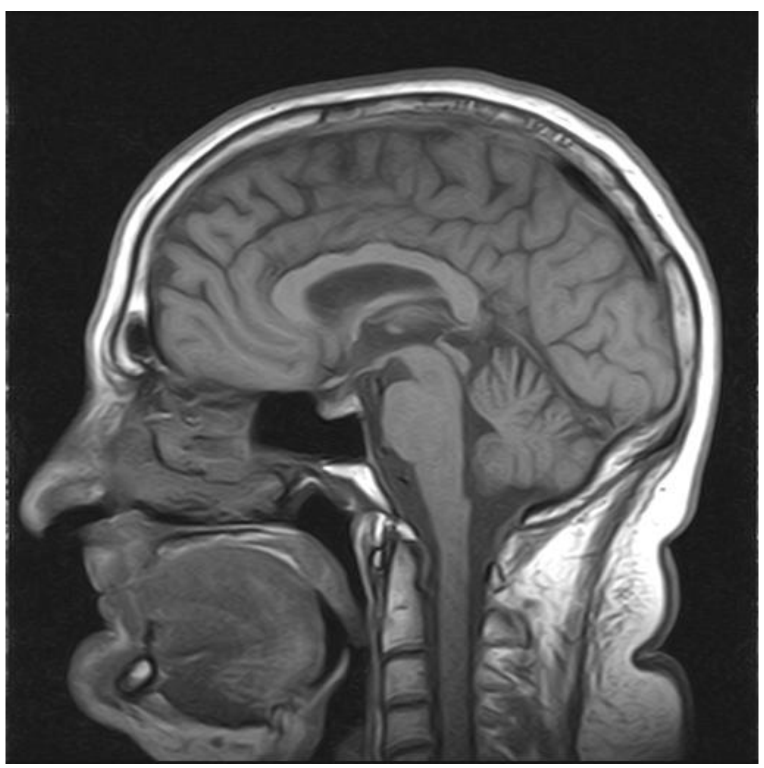
- DGF structure allows for ***efficient*** matrix multiplication
- Since DGF has small coherence and spectral norm, can ***recover most*** sufficiently sparse signals via ℓ_1-norm minimization [Tropp 2008][Calderbank, Howard, Jafarpour 2009]

- No characterization of ***failure modes*** (sparse vectors in null space of DGF φ)

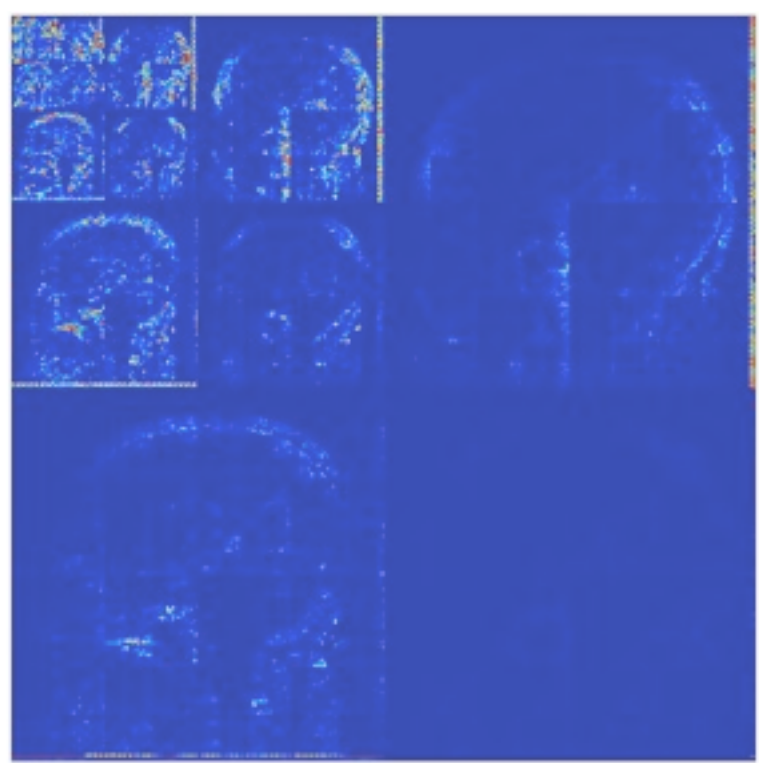
Prior Work: DG Frame Imaging

- Apply DGF on image's ***wavelet coefficients***:

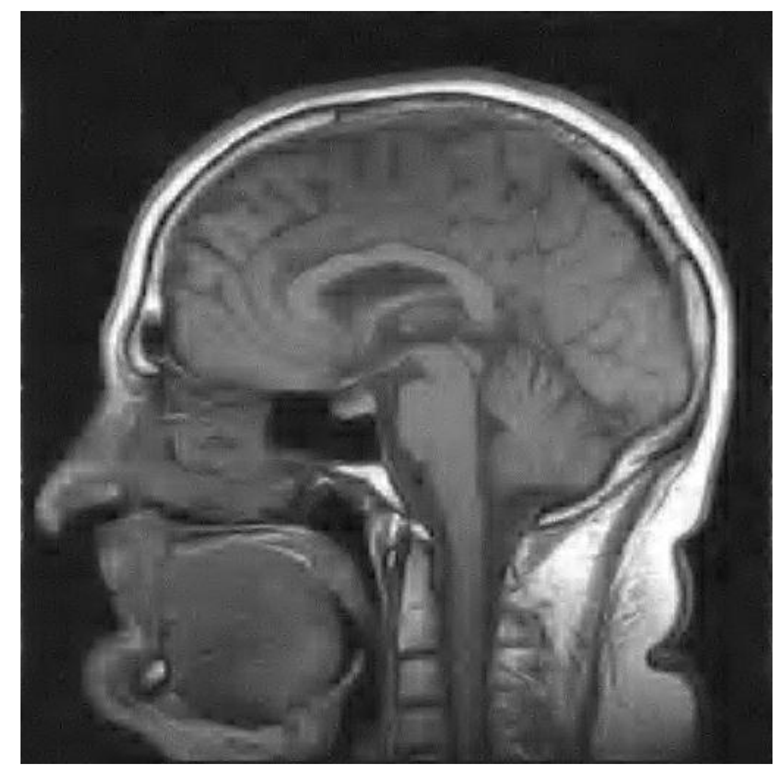
$$y = \varphi\theta = \varphi\Psi^T f$$



Original



Wavelet Coefficients



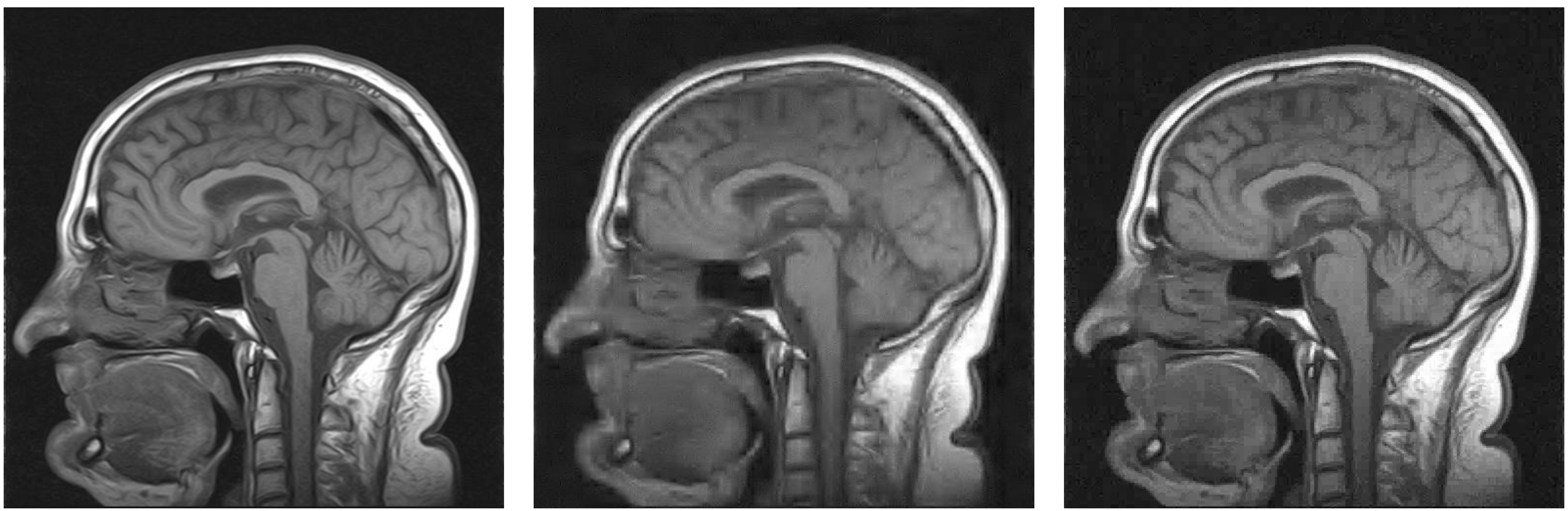
Recovery via BP

$\mathcal{C} = 512 \times 512 = 2^{18}$ $\quad N = 65536 = 2^{16}$ $\quad k = 0.07\mathcal{C} \approx 18000$

[Ni, Datta, Mahanti, Roudenko, Cochran 2010]

Prior Work: DG Frame Imaging

- ***Two-Stage Approach*** (TSA) for DGF Image Recovery:
  - Estimate $\widehat{\theta}_1 = H^{-1}y$
  - Calculate residual $r = y - H\widehat{\theta}_1$
  - Estimate $[\theta_2 \;\; \ldots \;\; \theta_R]$ (remainder of θ) from r using standard CS recovery algorithms

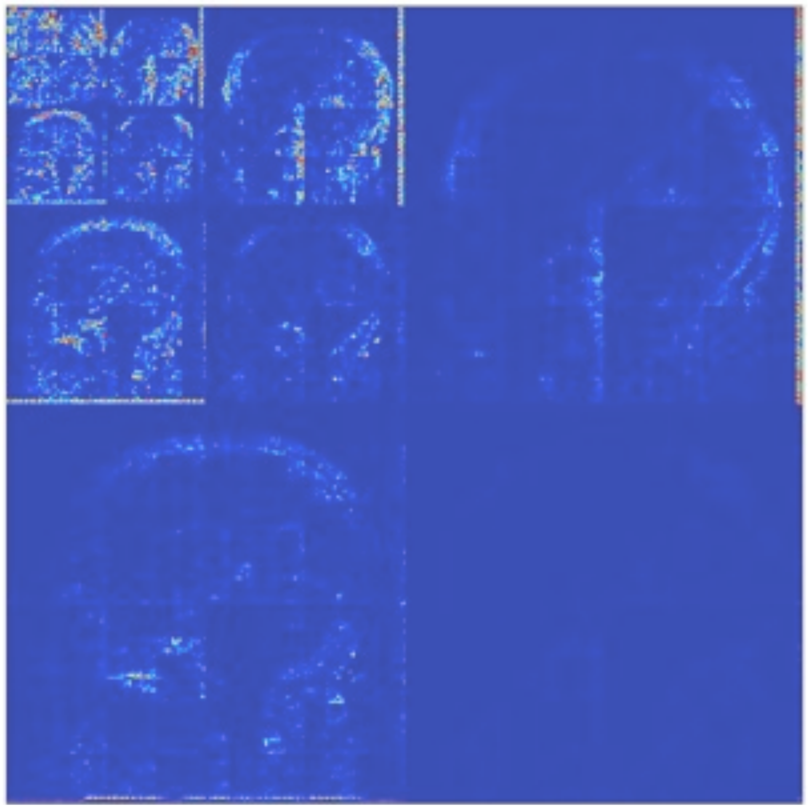


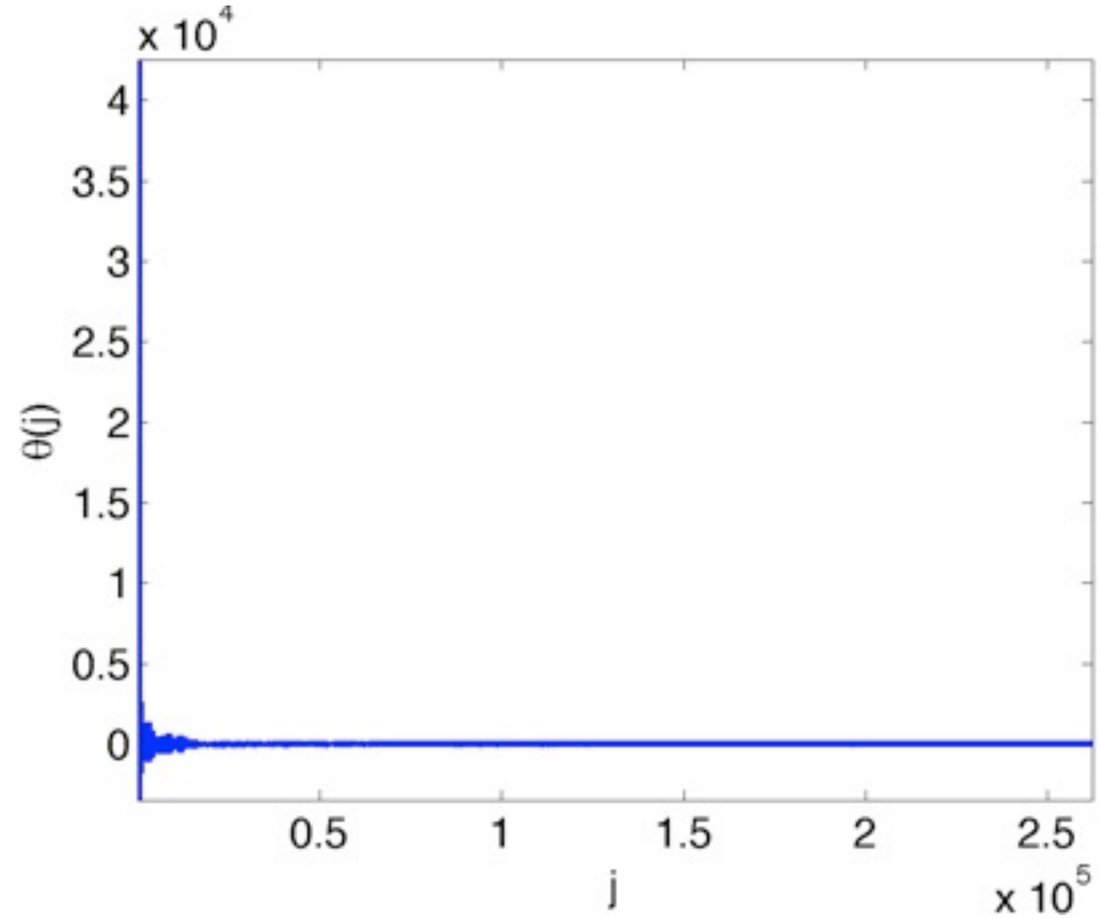
Original　　Recovery via BP　　Recovery via TSA/BP

$$\mathcal{C} = 512 \times 512 = 2^{18} \quad N = 65536 = 2^{16} \quad k = 0.07\mathcal{C} \approx 18000$$

[Ni, Datta, Mahanti, Roudenko, Cochran 2010]

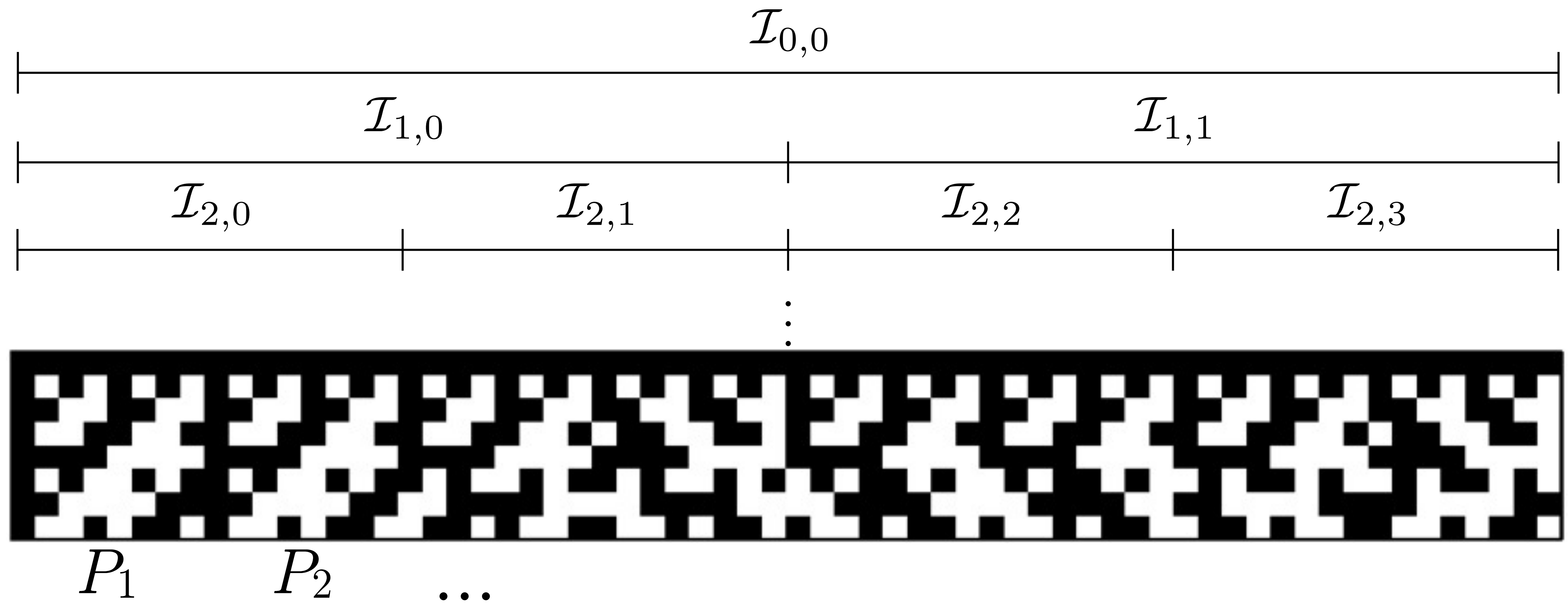
Clustering in Coefficient Vectors





- 2D wavelet coefficients are raster scanned into vectors ***from coarsest to finest scales***
- More large coefficients present at coarsest scales, ***clustered*** at beginning of rasterized vector
- Can clusters be ***to blame*** for loss in performance?
- Study if clustered vectors appear in ***null space*** of φ

Dyadic Column Partitionings



- Begin by considering groupings of columns of φ
- ***Dyadic partitionings:*** $I_{s,j} \subseteq DG(m,r) \times \mathbb{F}_2^m$
  2^s sets of $2^{m-s}R$ columns, $s \in [0, \log_2 \mathcal{C}), j \in [0, \ldots, 2^s)$
- Dyadic partitionings can be written as

$$I_{s,j} = \begin{cases} \{P_l\}_{l \in \mathcal{P}_{s,j}} \times \mathbb{F}_2^m & s \leq \log_2(R), \\ \{P_{\lfloor jR/2^s \rfloor}\} \times \left(a_{s,j} \oplus \mathbb{F}_2^{\log_2 \mathcal{C} - s}\right) & s > \log_2(R) \end{cases}$$

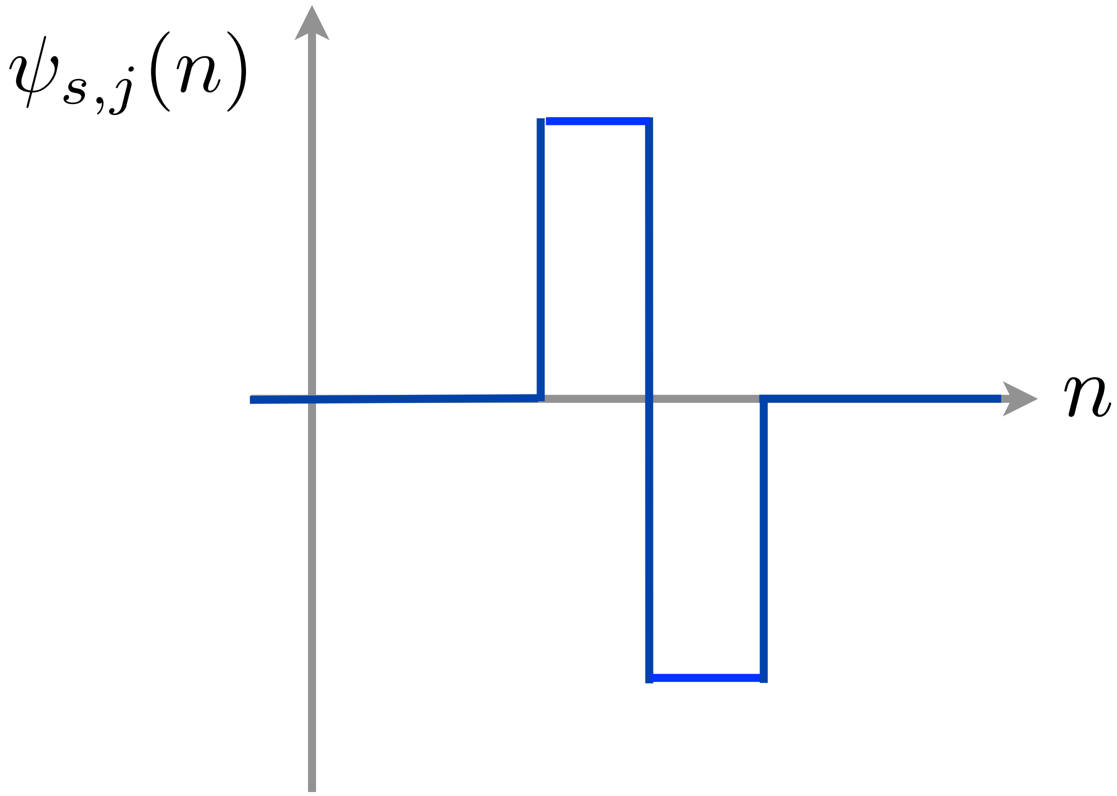
Behavior of Dyadic Column Sums

$$S_{s,j}(x) = \sum_{(P,b)\in\mathcal{I}_{s,j}} \varphi_{P,b}(x)$$

- If $S_{s,j}(x) = 0$ for all $x \in \mathbb{F}_2^m$, then columns in $\mathcal{I}_{s,j}$ are ***linearly dependent***.
- Properties from ***group theory*** allow us to prove:
  - If $s \leq \log_2 R$ then $S_{s,j}(x) = \begin{cases} |\mathcal{I}_{s,j}| & x = 0, \\ 0 & x \neq 0 \end{cases}$
  - If $s > \log_2 R$ then $S_{s,j}(x) \in \{0, \pm|\mathcal{I}_{s,j}|\}$
- ***Many sums vanish***, others can cancel each other - canceling adjacent dyadic interval sums?

The Amazing Vanishing Haar Wavelets

$$\psi_{s,j}(x) = \begin{cases} \sqrt{2^s/\mathcal{C}} & 2^{-s}\mathcal{C}j \leq n < 2^{-s}\mathcal{C}(j+1/2) \\ -\sqrt{2^s/\mathcal{C}} & 2^{-s}\mathcal{C}(j+1/2) \leq n < 2^{-s}\mathcal{C}(j+1) \\ 0 & \text{otherwise} \end{cases}$$



- Product $\eta_{s,j} = \varphi\psi_{s,j}$ can be expressed as sum of dyadic intervals:

  $\eta_{s,j}(x) = S_{s+1,2j}(x) - S_{s+1,2j+1}(x)$

- With results on dyadic sums, we have $\eta_{s,j} = 0$ if $s < \log_2 R$
- How about wavelets at finer scales?

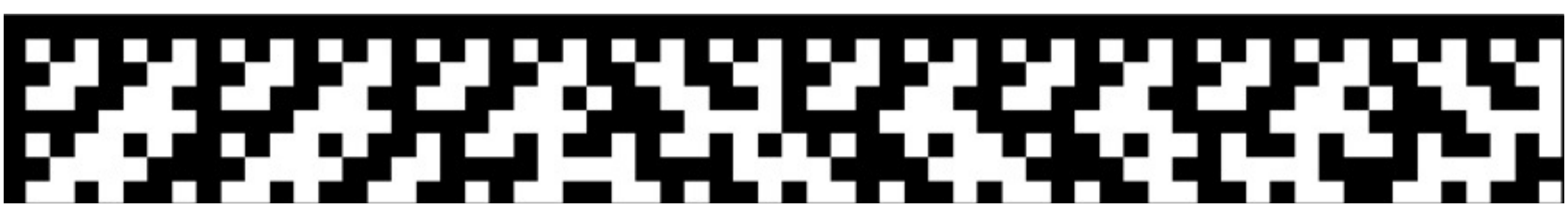
Main Result

Theorem:

Denote the projected difference of Haar wavelets by $\eta_{s,j_1,j_2} = \varphi(\psi_{s,j_1} - \psi_{s,j_2})$. Then $\eta_{s,j_1,j_2} = 0$ if

- $s < \log_2 R$, for all j_1, j_2;
- $\log_2 R \leq s \leq \log_2 R + 2r$, $a_{s,j_1} = a_{s,j_2}$, and $x(P_{s,j_1} - P_{s,j_2})x^\top = 0 \bmod 4$ for all x with $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$
- $\log_2 R \leq s \leq \log_2 R + 2r$ and $x(P_{s,j_1} - P_{s,j_2})x^\top = 2(a_{s,j_2} - a_{s,j_1})\mathcal{B}_{1:s-\log_2 R}(x)^\top \bmod 4$ for all x with $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$

where $\mathcal{B}_{p:q}(x)$ denotes bits p to q of x.



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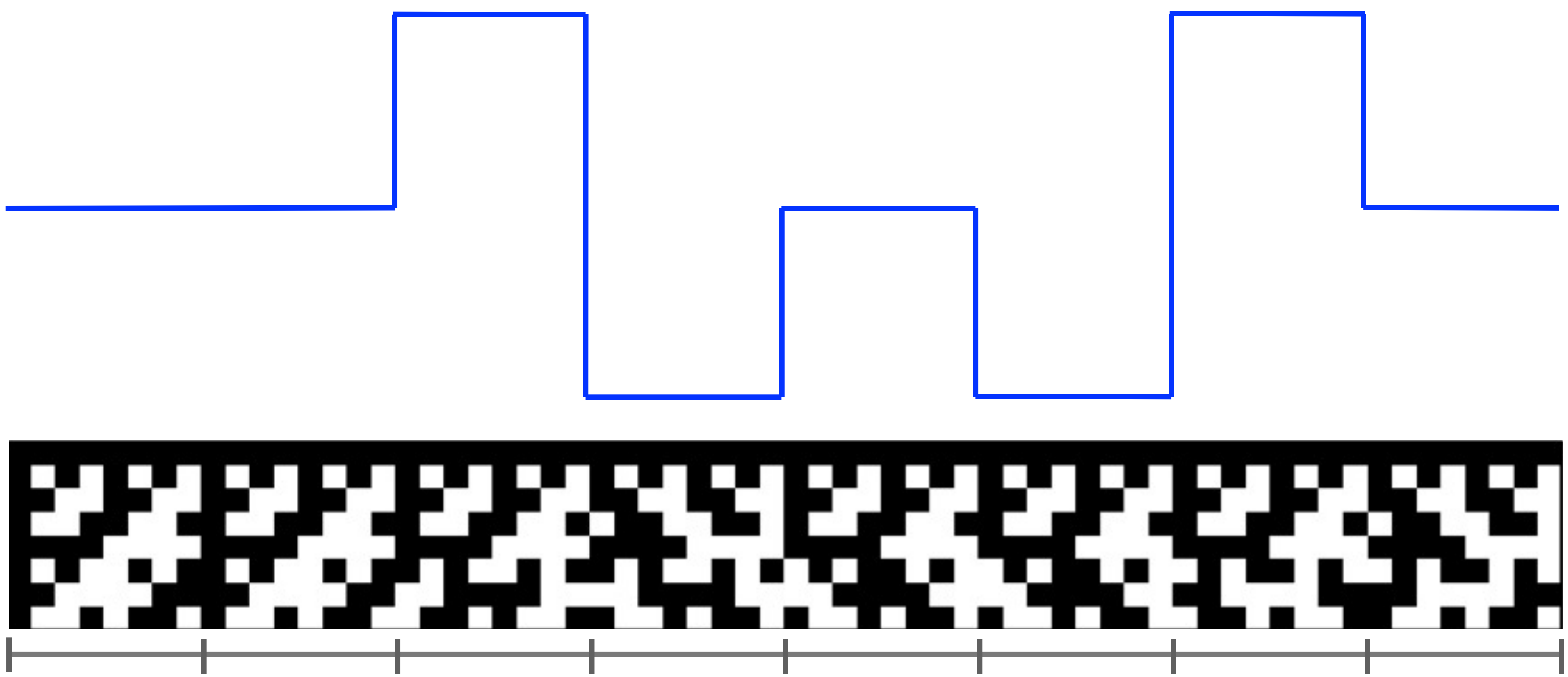

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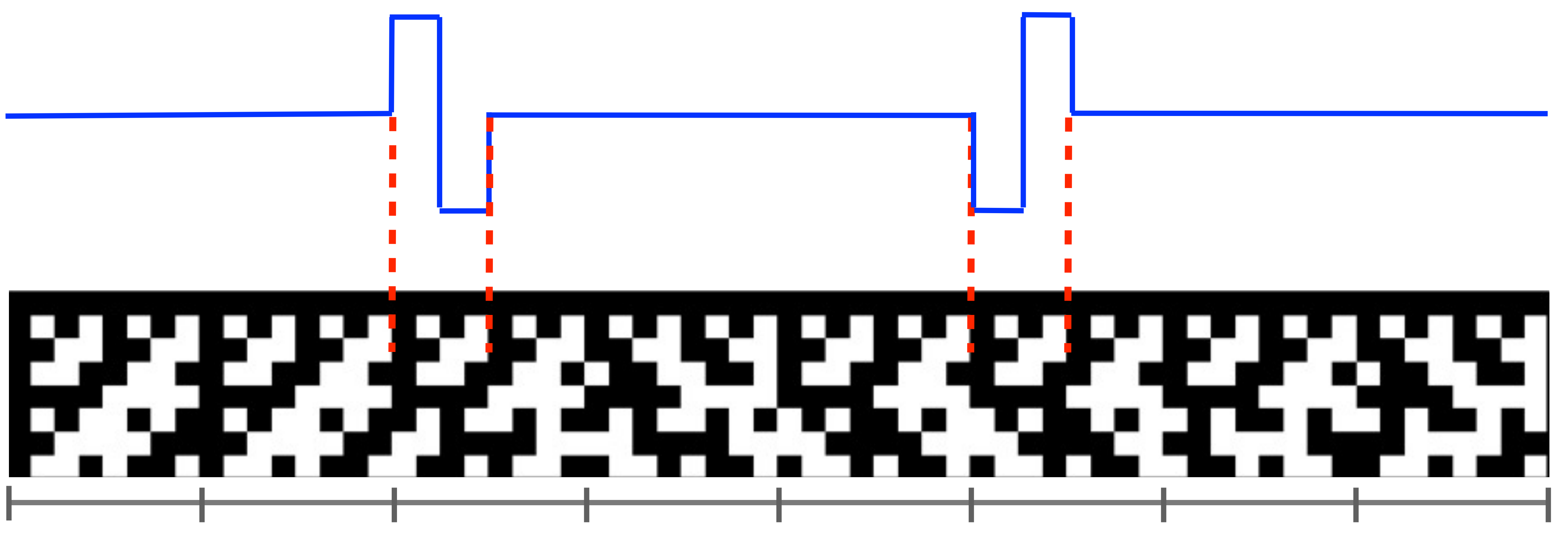

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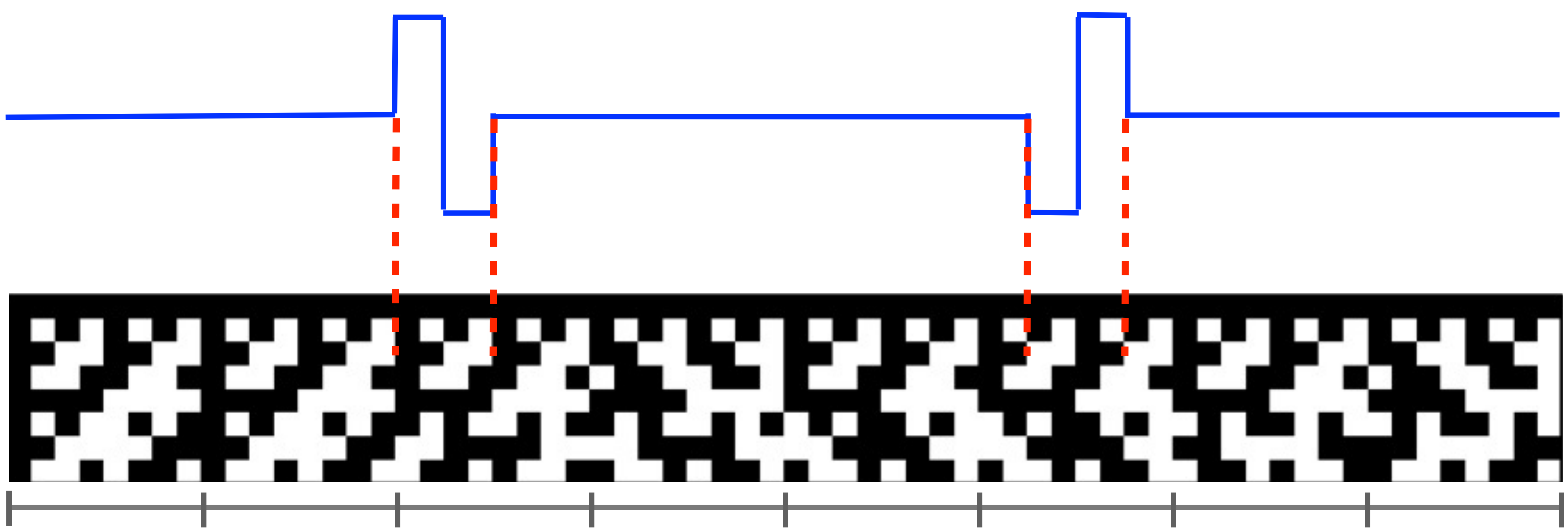
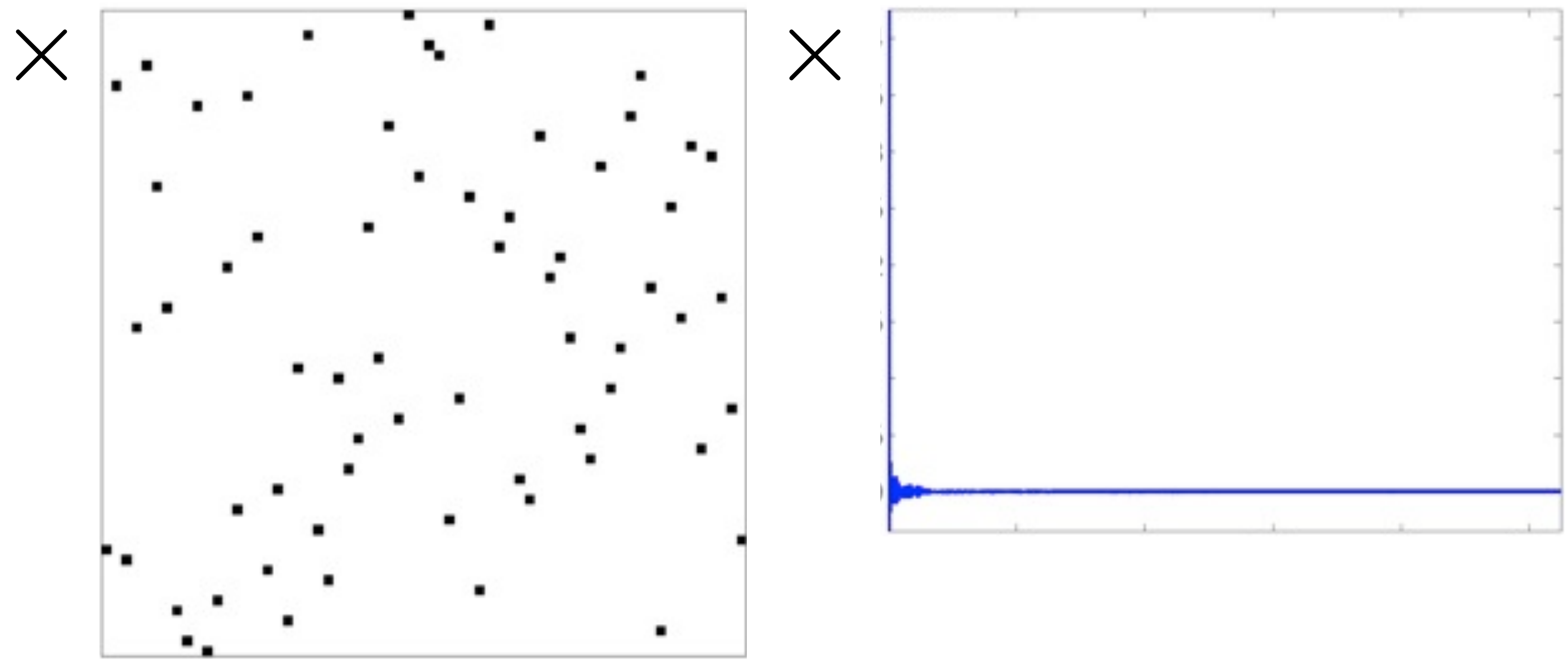
Denote the projected difference of Haar wavelets by $\eta_{s,j_1,j_2} = \varphi(\psi_{s,j_1} - \psi_{s,j_2})$. Then $\eta_{s,j_1,j_2} = 0$ if

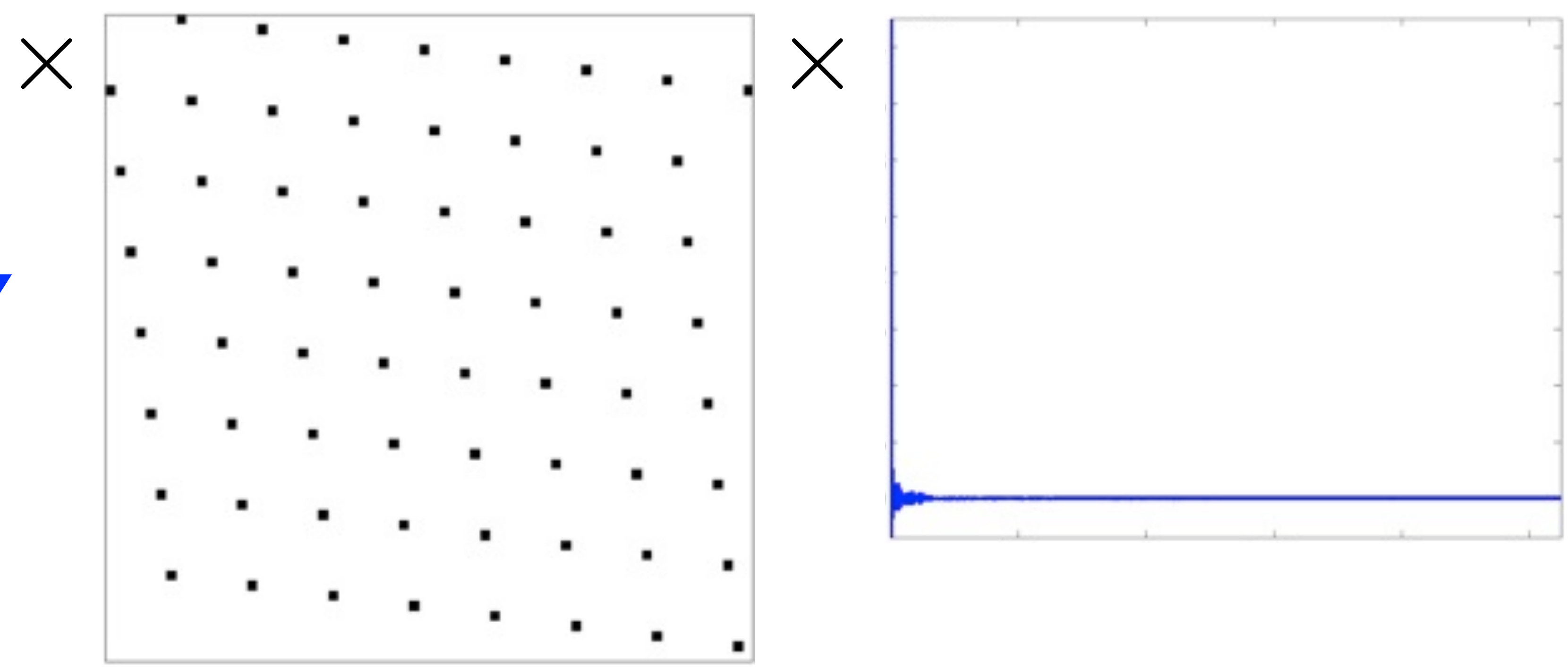
- $\log_2 R \leq s \leq \log_2 R + 2r$, $a_{s,j_1} = a_{s,j_2}$, and $x(P_{s,j_1} - P_{s,j_2})x^\top = 2(a_{s,j_2} - a_{s,j_1})\mathcal{B}_{1:s-\log_2 R}(x)^\top \bmod 4$ for all x with $\mathcal{B}_{s-\log_2 R+1:m}(x) = 0$



How to Vanish Vanishing Clusters

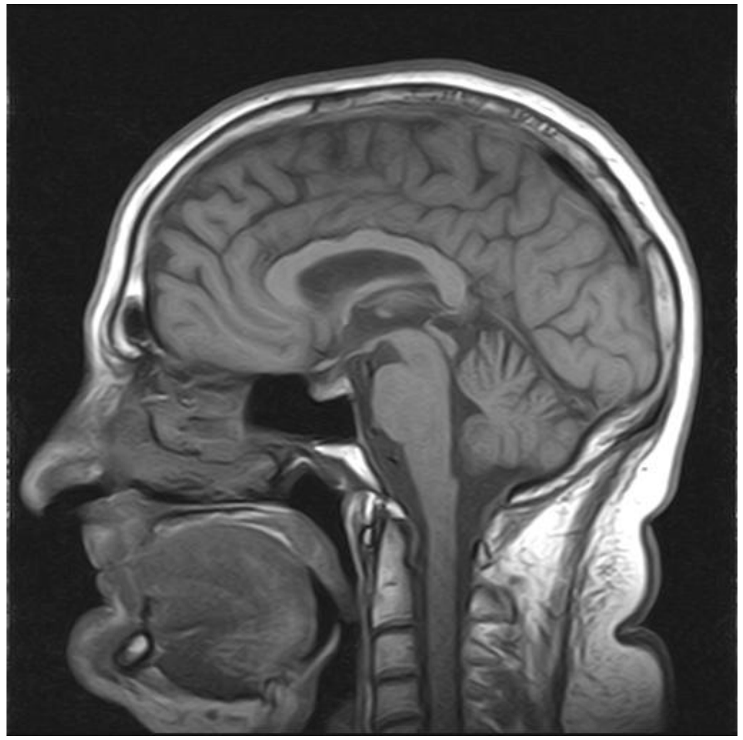


- ***Randomly*** permute entries of vector

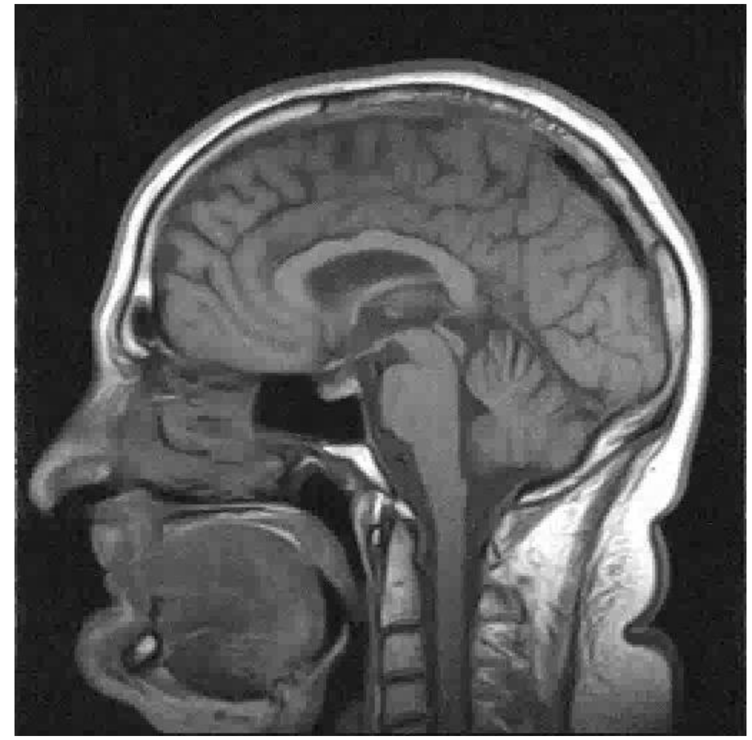


- ***Deterministically*** permute entries of vector
- Equivalent to permuting columns of matrix.

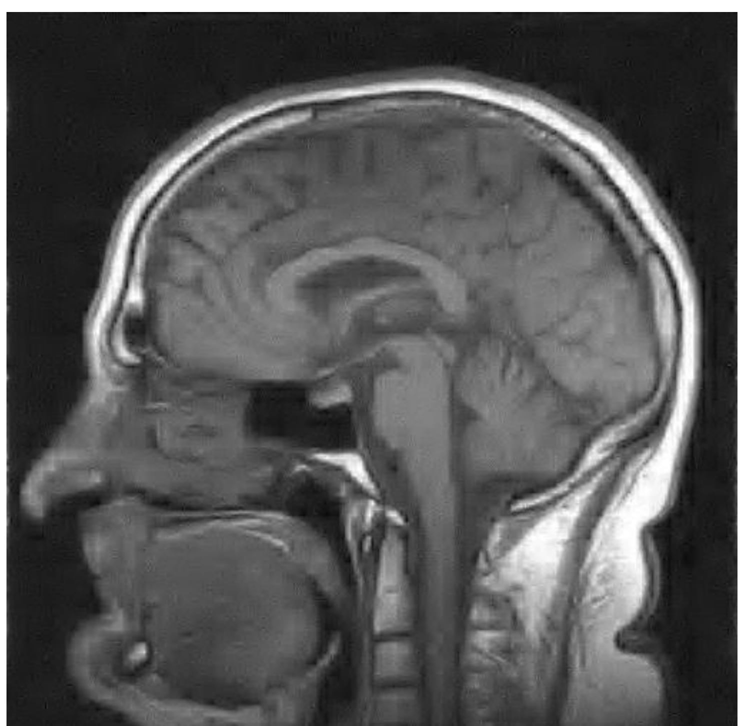
Experimental Results



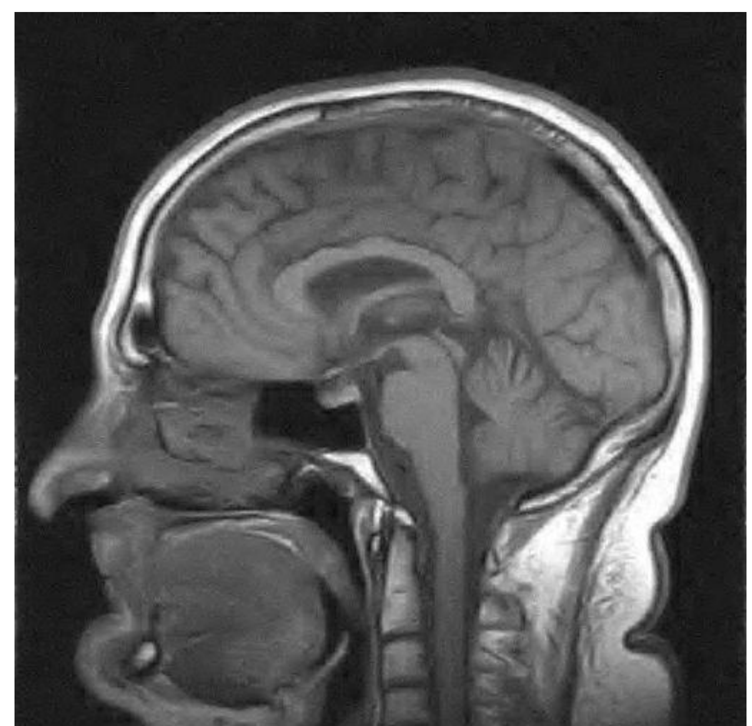
Original



TSA - 21.74dB

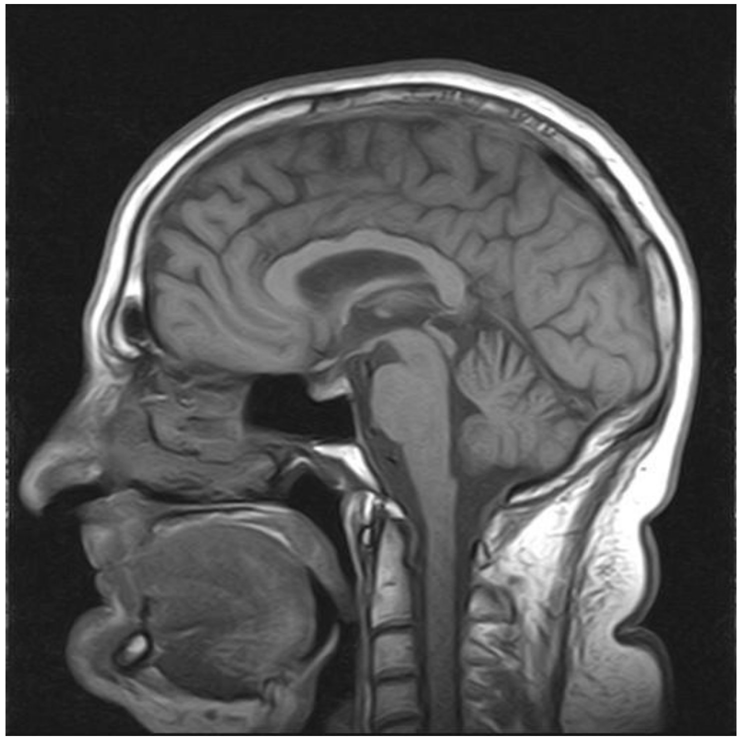


BP

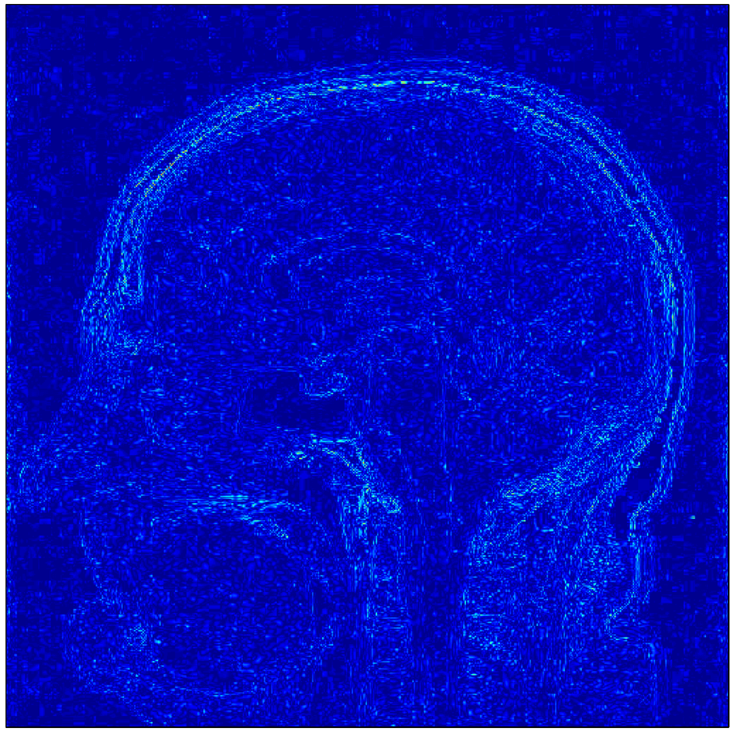


BP w/RP - 23.6dB

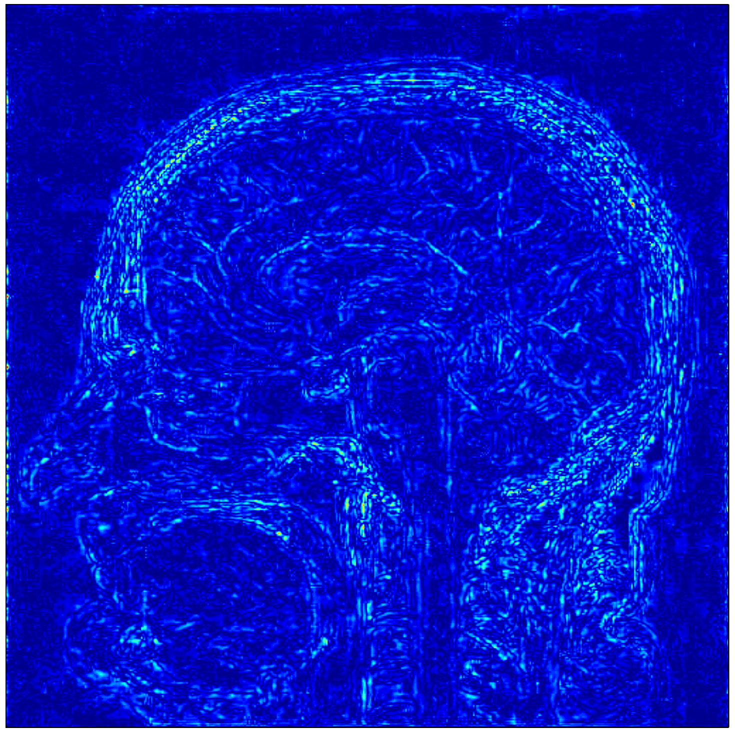
Experimental Results



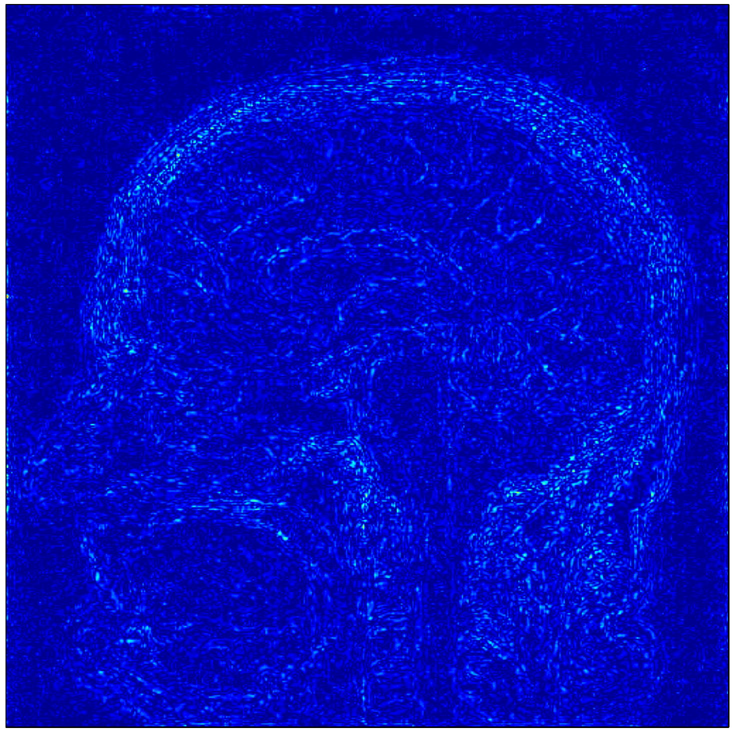
Original



TSA - 21.74dB



BP



BP w/RR - 23.6dB

Experimental Results

Algorithm	SNR (dB)	Time (s)
TSA	21.74	1008
BP + Random Raster	23.60	820
IHT + Random Raster	22.25	804
BP + Deterministic Raster	23.52	822
IHT + Deterministic Raster	20.98	813

Algorithm	$\sigma = 0.01$	$\sigma = 0.05$	$\sigma = 0.1$
TSA	21.64	19.51	16.37
BP + Random Raster	23.41	20.71	18.18
IHT + Random Raster	22.24	20.82	17.20

Summary and Future Work

- Established theory that explains ***shortcomings*** of Delsarte-Goethals frame for ***clustered*** sparse and compressible signals
- Designed ***new raster scannings*** for 2D wavelet vectors for natural images
- ***Standard recovery algorithms*** can be used once clusters are dissipated
- In progress: ***full characterization*** of null space of Delsarte-Goethals frame
- It is possible to show that the null space of DGF contains $2\sqrt{N}$-sparse vectors that are clustered

http://www.cs.duke.edu/~mduarte