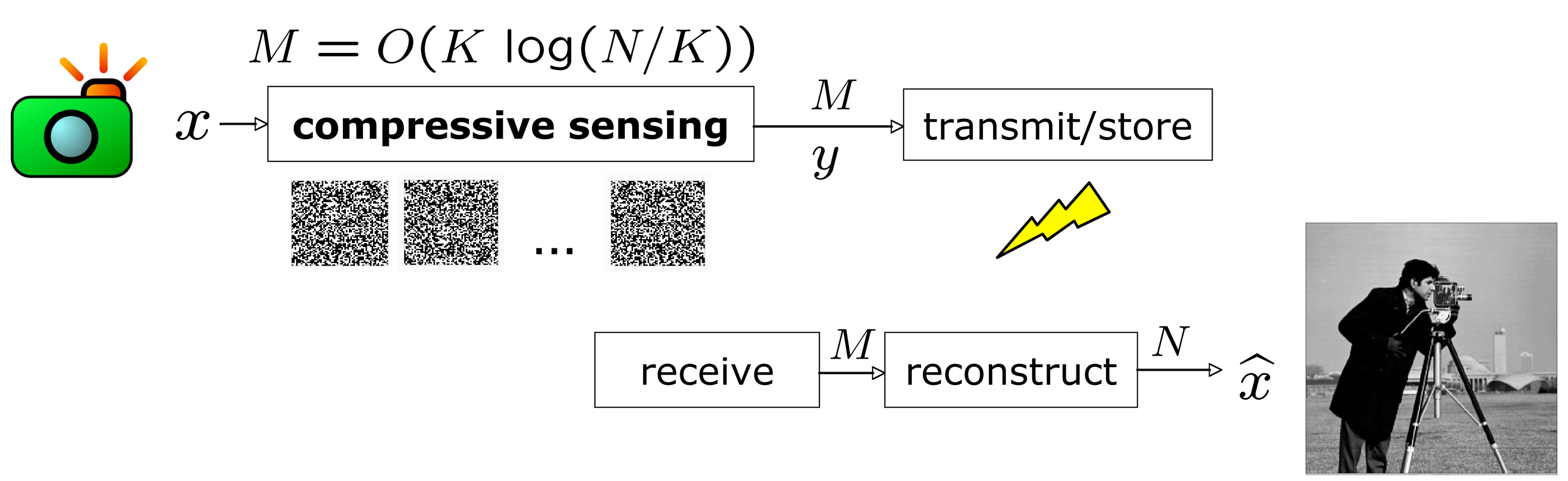
Performance Limits for Jointly Sparse Signals via Graphical Models

Marco F. Duarte, Shriram Sarvotham, Dror Baron, Michael B. Wakin, Richard G. Baraniuk

dsp.rice.edu/cs

Compressive Sensing (CS)

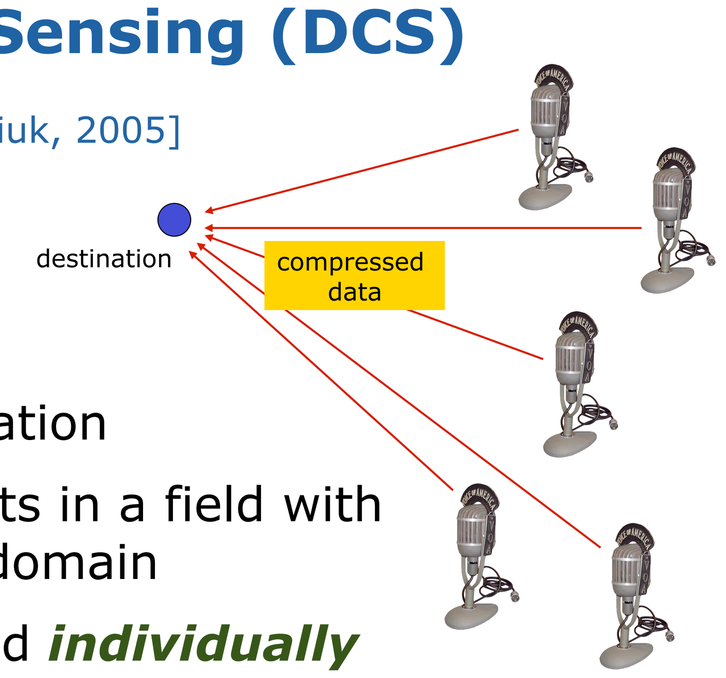
[Candès/Romberg/Tao, Donoho, 2005]



Sparse signal x, matrix Φ, measurements $y = \Phi x$

Distributed Compressive Sensing (DCS)

[Baron, Wakin, Duarte, Sarvotham, Baraniuk, 2005]



- Types of structure in signal ensembles
  - Intrasignal and Intersignal correlation
- Example: temperature measurements in a field with correlations in the time and spatial domain
- Compressive measurements obtained ***individually*** by each sensor - reduced communication requirement

Joint Sparsity Model (JSM)

Single model for the signal ensemble describes signal correlations

Example: Sparse common and innovations

J signals $x_1, \ldots, x_J$ of length N

$$x_j = z_C + z_j, \; j = 1, \ldots, J$$

common $\|z_C\|_0 = K_C$

innovation $\|z_j\|_0 = K_j$

M_j individual measurements $y_j = \Phi_j x_j$

Single Matrix/Vector Representation

Measurements conveniently represented as $Y = \Phi X$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_J \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_J \end{bmatrix} \text{ and } \Phi = \begin{bmatrix} \Phi_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \Phi_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \Phi_J \end{bmatrix}$$

$JN \times 1$; $\sum_{j=1}^{J} M_j \times 1$

Location Matrices and Value Vectors

Consider single signal case, $x \in \mathbb{R}^N, \|x\|_0 = K$

Location matrix P: identity submatrix for sparsity pattern

Value vector $\theta \in \mathbb{R}^K$: values for nonzero entries of x

Unique sparsest representation $x = P\theta$

For Joint Sparsity Models:

- Apply location matrix and value vector model to signal ensemble to obtain $X = P\Theta$
- JSM defined as a ***group*** $\mathcal{P}$ of allowed location matrices P
- ***Joint sparsity*** (dimensionality) of signal ensemble:

$$D = \min \dim(\Theta) \text{ such that } X = P\Theta, P \in \mathcal{P}$$

- Minimal location matrices:

$$\mathcal{P}_M(X) = \{P \in \mathcal{P} \text{ s.t. } P \in \mathbb{R}^{JN \times D}, \exists \Theta \text{ s.t. } X = P\Theta\}$$

Example: Sparse common and innovations

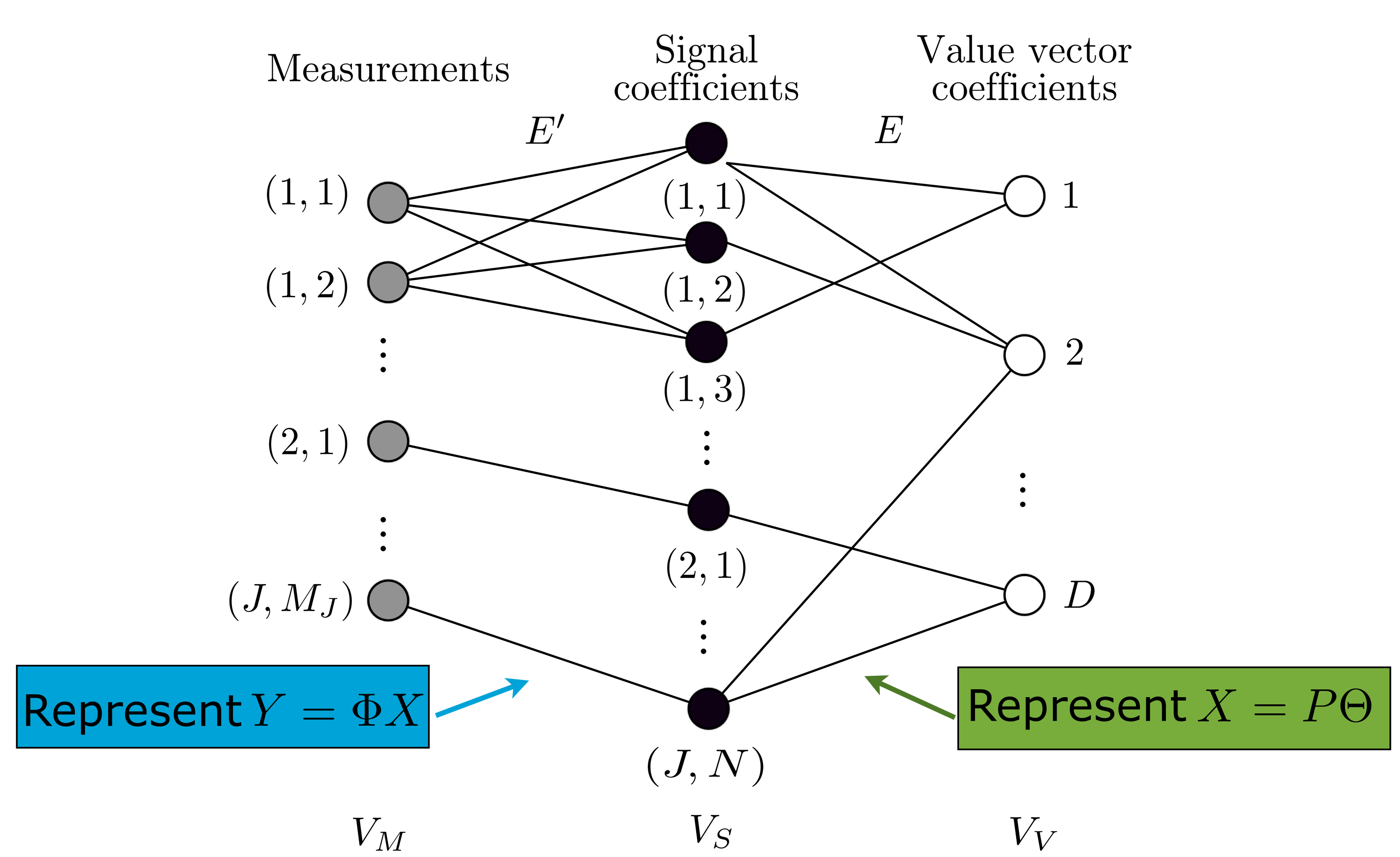
$$z_C = P_C \theta_C \text{ and } z_j = P_j \Theta_j$$

$$P = \begin{bmatrix} P_C & P_1 & \mathbf{0} & \ldots & \mathbf{0} \\ P_C & \mathbf{0} & P_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_C & \mathbf{0} & \mathbf{0} & \ldots & P_J \end{bmatrix} \text{ and } \Theta = \begin{bmatrix} \theta_C \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_J \end{bmatrix}$$

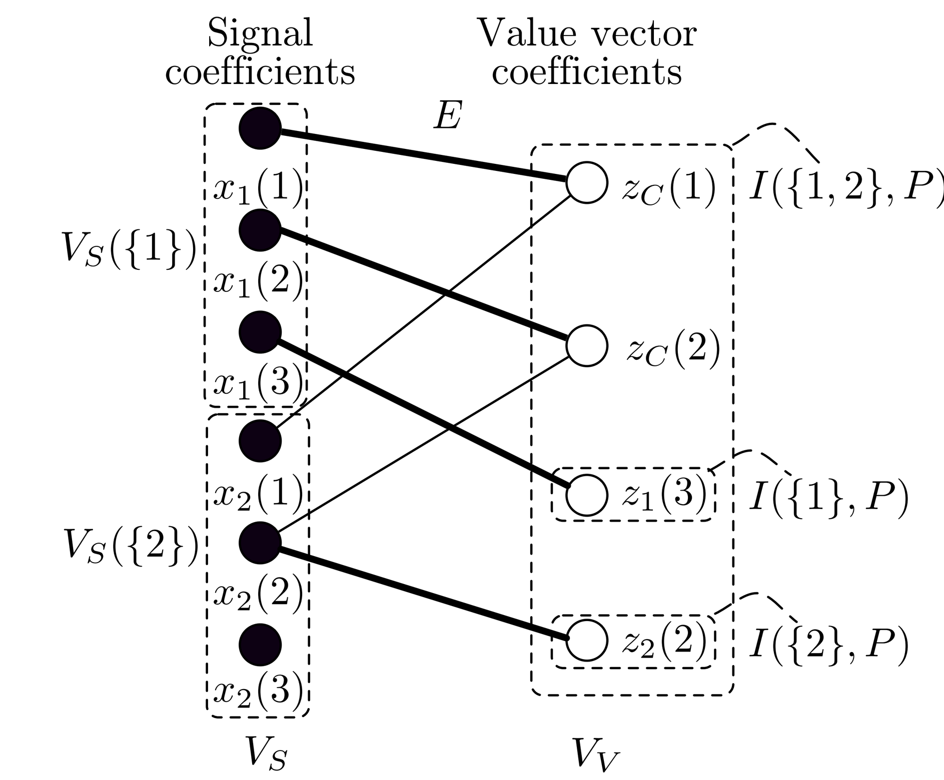
$$D = K_C + \sum_{j=1}^{J} K_j$$

Bipartite Graph Formulation

Represent relationships between measurements (V_M), value vector coefficients (V_V), and signal coefficients (V_S)



Quantifying dependencies



- Γ: subset of signal indices $\{1, \ldots, J\}$
- $V_S(\Gamma)$: set of signal vertices for all signals in Γ
- Measurements from Γ must recover two groups of value vector coefficients:
  - $I(\Gamma, P)$: set of value vector coefficients that are linked only to $V_S(\Gamma)$
  - $K_{C,\Gamma}(P)$: Number of common component coefficients that overlap with innovations for all signals outside Γ

Result

Given $P \in \mathcal{P}_M(X)$, Gaussian random measurements:

- ***Converse measurement region:***

$$\sum_{j \in \Gamma} M_j < |I(\Gamma, P)| + K_{C,\Gamma}(P)$$

- ***Achievable measurement region***:

$$\sum_{j \in \Gamma} M_j \geq |I(\Gamma, P)| + K_{C,\Gamma}(P) + |\Gamma|$$

Example: Sparse common and innovations

$$\sum_{j \in \Gamma} M_j \geq \sum_{j \in \Gamma} K_j + K_{C,\Gamma}(P) + |\Gamma|$$

Discussion

- Theorem applies to:
  - joint compressive sensing (***same*** measurement sum bound)
  - single signal compressive sensing (***matches*** bounds for ℓ_0 minimization)
- Similar conditions for location matrices with ***linearly independent columns***
- Dimensionality has volumetric notion similar to ***entropy***