Compressive Direction-of-Arrival Estimation Off the Grid

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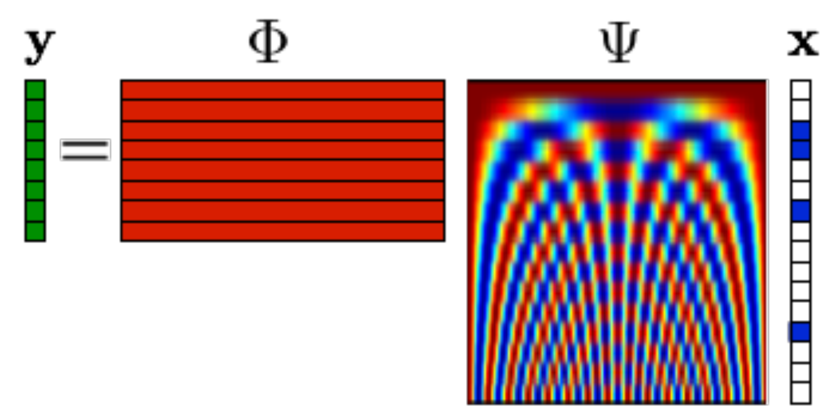
Background

- Sparse signal: Compact representation in an appropriate basis or frame.
- $\mathbf{x} \in \mathbb{C}^N$ and $\|\mathbf{x}\|_0 \leq K \ll N$, $\mathbf{A} \in \mathbb{C}^{M \times N}$, $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{C}^M$ where $\mathbf{A} = \Phi\Psi$ and $M \ll N$.



- Iterative hard/soft thresholding: set $\mathbf{x}^0 = \mathbf{0}$; at iteration t,

$$\mathbf{x}^{t+1} = \eta_K(A^H\mathbf{z}^t + \mathbf{x}^t),$$
$$\mathbf{z}^t = \mathbf{y} - A\mathbf{x}^t.$$

Approximate Message Passing:

- Start from $\mathbf{x}^0 = \mathbf{0}$ and $\mathbf{z}^0 = \mathbf{y}$,

$$\mathbf{x}^{t+1} = \eta_K(A^H\mathbf{z}^t + \mathbf{x}^t),$$
$$\mathbf{z}^t = \mathbf{y} - A\mathbf{x}^t + \frac{1}{\delta}\langle \eta_K'(A^H\mathbf{z}^{t-1} + \mathbf{x}^{t-1})\rangle.$$

- $\mathbf{x}^t$ and $\mathbf{z}^t$ are signal estimate and residual of estimation at iteration t.
- Adding an "Onsager" correction term to the algorithm, which improves the sparsity-undersampling tradeoff.

Denoiser-Based Approximate Message Passing:

- "Onsager" correction term shapes the distribution of the signal estimate: $\mathbf{x}^t + A^H\mathbf{z}^t \approx \mathbf{x} + \mathbf{n}$, $\mathbf{n} \sim \mathcal{N}(0, \sigma_t^2\mathbf{I})$.
- AMP succeeds because denoiser for signal class is applied on this estimate (e.g., soft thresholding for sparse signals).
  $\Longrightarrow$ Replace thresholding function $\eta_K(x)$ with arbitrary **denoiser** $\mathbb{T}_l(x, K)$ to extend AMP applicability to arbitrary signal classes [1].
- Use *Monte Carlo* simulations to estimate the Onsager correction term [2].

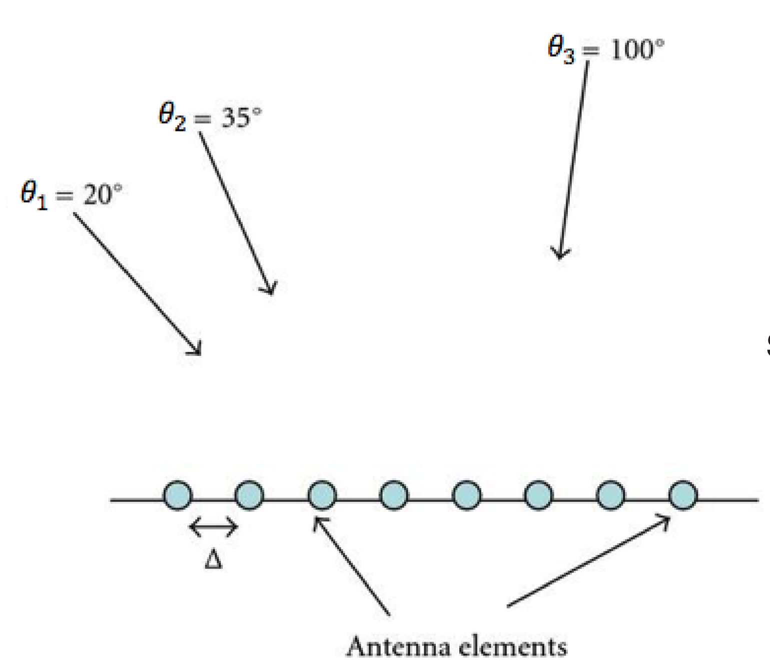
Group ℓ_1 Minimization:

- Consider multiple signals with the same sparse support, $\mathbf{x}_i$ stacked as columns of a matrix $\mathbf{X}$, which will be row-sparse.
- Search for signals $\mathbf{x}_i$, leading to a row-sparse matrix X,

$$\hat{\mathbf{X}} = \underset{\tilde{\mathbf{X}}}{\operatorname{argmin}} \|\tilde{\mathbf{X}}\|_{2,1}, \text{ s.t. } \mathbf{Y} = \mathbf{A}\tilde{\mathbf{X}}$$

where $\|\mathbf{X}\|_{p,q} = \left(\sum_i \|\mathbf{x}_i\|_p^q\right)^{\frac{1}{q}}$ with $\mathbf{x}_i$ being the ith row of matrix $\mathbf{X}$.
- Also known as Multiple Measurement Vector (MMV).

Direction of Arrival Estimation:



K transmitters, P antennas, Q snapshots

$$\mathbf{x}(q) = \mathbf{S}(\theta)\mathbf{a}(q) + \mathbf{n}(q), \; q = 1, \ldots, Q$$
$$\mathbf{S}(\theta) = [\mathbf{s}(\theta_1), \; \ldots, \; \mathbf{s}(\theta_K)]$$
$$\mathbf{s}(\theta_i) = \begin{bmatrix} \exp\{-j\frac{P-1}{2}\frac{2\pi}{\lambda}\Delta\sin\theta_i\} \\ \exp\{-j\frac{P-3}{2}\frac{2\pi}{\lambda}\Delta\sin\theta_i\} \\ \vdots \\ \exp\{j\frac{P-1}{2}\frac{2\pi}{\lambda}\Delta\sin\theta_i\} \end{bmatrix}$$

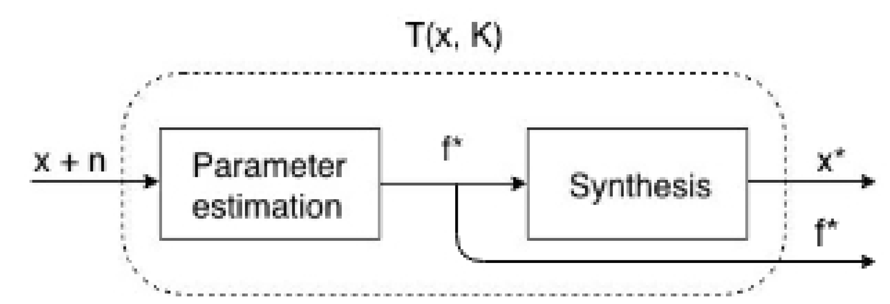
$\theta = [\theta_1, \ldots, \theta_K]$
$\mathbf{S}(\theta)$: Fourier matrix with frequencies $f_k = \frac{\Delta}{\lambda}\sin\theta_k$ for $k = 1, \ldots, K$.

AMP with Parametric Denoisers

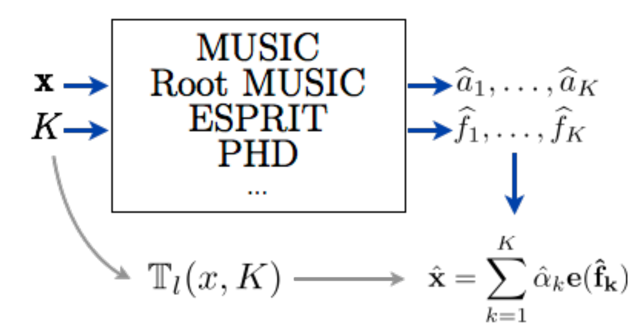
- $\mathbf{x}$ obeys a low-dimensional parametric model, e.g., array snapshots in DOA:

$$\mathbf{x}[p] = \sum_{k=1}^{K} a_k e^{-j2\pi f_k p/N}, \; p = 1, \ldots, P,$$

where $f_k = \frac{\Delta}{\lambda}\sin\theta_k$ for $k = 1, \ldots, K$.
- Can pair statistical parameter estimation algorithms with generative signal models to provide parametric denoisers .



- Many algorithms suggested for estimating the frequencies from the signal, know as line-spectral estimation.



Finally, $\hat{\theta}_k = \sin^{-1}(\frac{\lambda}{\Delta}\hat{f}_k)$ for $k = 1, \ldots, K$.

MMV-Based DOA Estimation

- $\mathbf{x}_p$: snapshots gathered by receiver p.
- Each antenna takes measurements independent of the others, $\mathbf{y}_p = \Phi\mathbf{x}_p$.
- Gather $\mathbf{x}_p$'s as columns of matrix X.
- $\mathbf{Y} = \Phi\mathbf{X}^T + \mathbf{N} = \Phi\mathbf{A}^T\mathbf{S}^T + \mathbf{N}$.
- $\mathbf{A}$ is row-sparse, since all the columns have the same locations for the nonzero elements.
- Even if bearings are off the grid, the MMV returns accurate enough estimate to be used by Root MUSIC.
- $$\mathbf{Y} \xrightarrow{\text{group } \ell_1 \text{ min.}} \hat{\mathbf{X}} \xrightarrow{\text{Root MUSIC}} \{f_k\}_{k=1}^K \rightarrow \{\theta_k\}_{k=1}^K$$

Prior Work on DOA Estimation [3]

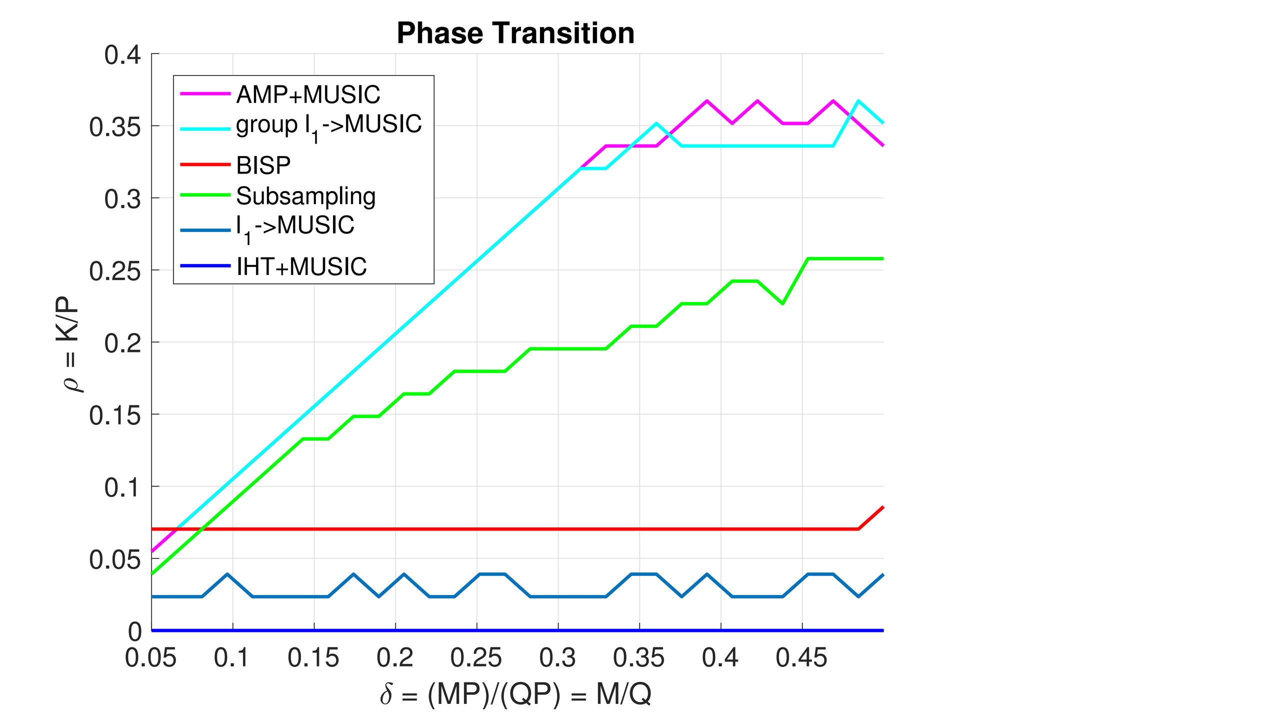
On-Grid: (e.g., CS with dictionary $\mathbf{S}(\theta)$ [4])
- The DOAs are assumed to lie on a prescribed grid.
- Requires grid selection resulting in mismatches between the adopted discrete grid points and the true continuous DOAs.

Off-Grid: (e.g., Group $\ell_1 \rightarrow$ MUSIC)
- A grid is still required to perform sparse estimation but the resulted DOA estimates can go outside of this grid.
- Most of the presented algorithms involve nonconvexoptimization; and thus only local convergence can be guaranteed.

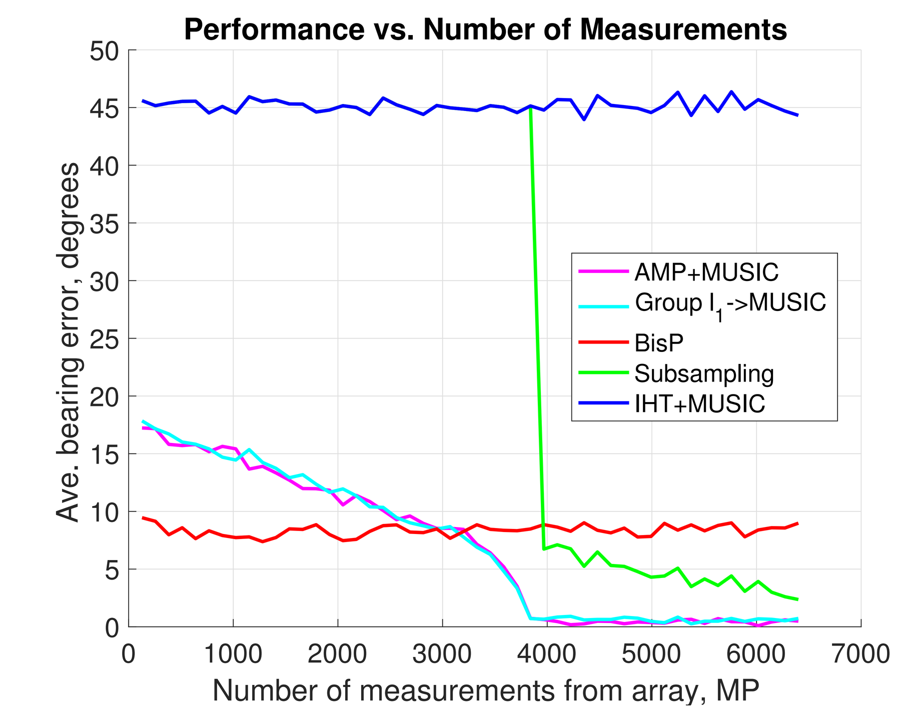
Gridless: (e.g., AMP+MUSIC)
- These methods directly operate in the continuous domain.
- No parameter grid required; no mismatch problem present.
- Existing gridless methods can only be applied to uniform or sparse linear arrays.

Numerical Experiments

- Consider uniform linear array of $P = 128$ antennas, receiving signals of length $Q = 128$ from K transmitters.
- DOA estimation error: cost of the Hungarian matching between the vectors containing the true parameter values and their estimates.



- The maximum value of the normalized sparsity $\rho = K/P$, as a function of the normalized measurement rate $\delta = M/Q$ for successful DOA recovery.
- Successful recovery: up to 1° of average DOA estimation error over K tramsmitters.
- At least 50% successful recovery over 100 signal trials.
- Phase transition plot is usually interpreted as showing the division between the (δ, ρ) region:
  - Area below the curve: the probability of success goes to one as $QP \rightarrow \infty$.
  - Area above the curve: the probability of success goes to zero as $QP \rightarrow \infty$.



- Fixed sparsity level, $K = 30$; fixed number of antennas, $P = 128$.
- Average DOA estimation error as a function of total number of measurements MP over 100 trials.

Acknowledgments and References

[1]. D. L. Donoho, A. Maleki, and A. Montanari, "Message passing algorithms for compressed sensing", Proc. Nat. Acad. Sci., vol. 106, no. 45, pp. 18914–18919, Nov. 2009.
[2]. C. A. Metzler, A. Maleki, and R. G. Baraniuk, "From denoising to compressed sensing", *IEEE Transactions on Information Theory*, vol. 62, no. 9, pp. 5117–5144, April 2016.
[3] Z. Yang, J. Li, P. Stoica, and L. Xie, "Sparse Methods for Direction-of-Arrival Estimation". Available at `arXiv:1609.09596`
[4]. M. F. Duarte and R. G. Baraniuk, "Spectral compressive sensing", *Applied and Computational Harmonic Analysis*, vol. 35, pp. 111–129, July 2013.

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